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Magnetic Landau Quantization Effects on the Magnetic Moment and Specific Heat of a T-3 Dice Lattice

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ABSTRACT In this work we analyze the statistical thermodynamic functions and magnetic moment of a Dice lattice subject to a normal quantizing magnetic field. Our analysis addresses the Grand Potential and Helmholtz Free Energy, as well as the magnetic moment, entropy and specific heat at constant volume, explicitly determining their magnetic field dependencies in the degenerate statistical regime, replete with de Haas-van Alphen oscillatory phenomenology (and other magnetic field dependence); and also determining their temperature dependencies jointly with magnetic field features in the approach to the zero temperature limit. Furthermore, we evaluate the Grand Potential exactly, for arbitrary temperature and density. Our results are obtained with consideration of the presence of heat and particle baths with fixed chemical potential and they are discussed in relation to other pertinent work on the subject.

INDEX TERMS Dirac materials, magnetic Landau quantization, magnetic moment, specific heat, thermodynamics, T-3 dice lattice.

I. INTRODUCTION

This article is primarily concerned with the determination of the role of an impressed, normal magnetic field B_0 and Landau quantization in the statistical thermodynamic functions of the T-3 Dice lattice [1], [2]. The Dice lattice is a recent addition to the group of "Dirac materials," which have low energy spectra and Hamiltonians linearly proportional to momentum: They include the Group VI Dichalcogenides [3], Topological Insulators [4], and Silicene [5]. Such materials came under universal scrutiny when the first one, Graphene [6], [7], [8], [9], [10], [11], was found to have exceptional electrical conduction and sensing properties that gave promise of a new era of electronic devices and computers. For their ground-breaking work on Graphene, the 2010 Nobel prize was awarded to Geim and Novoselov. Carrier dynamics in the Dirac materials is analogous to that of relativistic electrons and positrons, further focusing interest in them.

For the T-3 pseudospin 1 Dice lattice in a magnetic field B_0, the low energy Dirac Hamiltonian, H, has the 3×3 matrix form [1], [2] ($\hbar \to 1$; α is a speed parameter of the T-3 lattice)

$$H = \alpha \begin{bmatrix} 0 & \pi_- & 0 \\ \pi_+ & 0 & \pi_- \\ 0 & \pi_+ & 0 \end{bmatrix}, \tag{1}$$

where, in position representation ($\mathbf{x} = (x, y)$),

$$\pi_+ = \pi_x + i\pi_y \quad ; \quad \pi_- = \pi_x - i\pi_y, \tag{2}$$

and the magnetic field B_0, whose importance as a probe of the properties of matter and as an agent of modifying them (especially through Landau quantization of their spectra) [12], [13], is introduced through the 2D canonical momentum components as (e is the charge of an electron in the field of the T-3 Dice lattice and $\pi = (\pi_x, \pi_y)$ is its canonical momentum),

$$\pi_x = \frac{1}{i}\frac{\partial}{\partial x} + \frac{eB_0}{2}y \quad ; \quad \pi_y = \frac{1}{i}\frac{\partial}{\partial y} - \frac{eB_0}{2}x. \tag{3}$$

Section II addresses the Landau quantized Dice lattice in regard to formulation of the Grand Potential Ω in terms of the ordinary partition function $\hat{Z}(\beta)$ [14], [15], [16], [17], which is explicitly determined in terms of the Dice lattice carrier energy spectrum using the appropriate retarded Green's function [18]. Following this, the Grand Potential, Helmholtz Free Energy, magnetic moment, entropy and specific heat are explicitly determined in Section III in the degenerate statistical regime; and we also carry out an exact evaluation of the Grand Potential for arbitrary temperature and density (in any statistical regime) in the Appendix. These determinations all exhibit the magnetic field dependence, replete with de Haas-van Alphen oscillatory phenomenology in the zero temperature degenerate limit, along with temperature dependence in the approach to this limit. Section IV presents a summary and discussion relating to other recent pertinent work on the subject, particularly in regard to the calculation of magnetic moment.

II. GRAND POTENTIAL FOR THE DICE LATTICE IN A MAGNETIC FIELD

Our analysis of the statistical thermodynamic functions of a Landau-quantized T-3 Dice lattice proceeds from the Sondheimer-Wilson formulation of the Grand Potential Ω in terms of the ordinary (not "Grand") partition function $\hat{Z}(\beta)$ [β is inverse thermal energy, $\beta = \frac{1}{\kappa_B T'}$, where κ_B is the Boltzmann constant and T' is Kelvin temperature] [14], [15], [16], [17]:

$$\begin{aligned}\Omega &= F - \mu N \\ &= -\frac{\beta}{4}\int_c \frac{ds}{2\pi i}\frac{\hat{Z}(s)}{s^2}\int_0^\infty dE e^{Es} sech^2\left(\frac{[E-\mu]\beta}{2}\right). \quad (4)\end{aligned}$$

Here, F is the Helmholtz Free Energy, μ represents the chemical potential, N is number, and c represents the inverse Laplace transform integration contour. The ordinary partition function $\hat{Z}(\beta)$ may be expressed in terms of the trace (Tr) of the associated retarded Green's function $G^{ret}_{T>0}(\mathbf{x}, \mathbf{x}'; T = t - t')$ in the positive time order ($T > 0$) as [19]

$$\hat{Z}(\beta) = \int d^2x \;\; Tr \;\; \left(iG^{ret}_{T>0}(\mathbf{x}, \mathbf{x}; T \to -i\beta)\right), \quad (5a)$$

and, for the spatially uniform system at hand, $G^{ret}_{T>0}(\mathbf{x}, \mathbf{x}; T) = G^{ret}_{T>0}(\mathbf{0}, \mathbf{0}; T)$ so the 2D integral is $\int d^2x =$ area, whence

$$\hat{Z}(\beta) = (\text{area}) Tr\left(iG^{ret}_{T>0}(\mathbf{0}, \mathbf{0}; T \to -i\beta)\right), \quad (5b)$$

with T replaced by $-i\beta$. Since the partition function $\hat{Z}(\beta)$ is proportional to area, all the extensive thermodynamic functions to be derived from it will also be proportional to area, including the Grand potential Ω, Helmholtz Free Energy F, magnetic moment M, entropy S, and specific heat C_v. The pseudospin 1 retarded Green's function $G^{ret}_{T>0}$ for the Dice lattice has been explicitly determined subject to Landau quantization in a normal magnetic field [18], and its diagonal elements are given in position/frequency representation as follows ($\mathbf{B}_0$ is the uniform, constant magnetic field; $\hbar \to 1$):

$$G^{ret}(\mathbf{x}, \mathbf{x}'; \omega) = e^{i\left(\frac{e}{2}\mathbf{x}\cdot\mathbf{B}\times\mathbf{x}'\right)} G'(\mathbf{x} - \mathbf{x}'; \omega) \quad (6)$$

and ($\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$; $\delta^2(\mathbf{R}) = \delta(x - x')\delta(y - y')$ is a 2D Dirac delta function)

$$G'_{11}(\mathbf{R}, \omega) = G'_{33}(\mathbf{R}, \omega) = \frac{1}{2}G'_{22}(\mathbf{R}, \omega) + \frac{1}{2\omega}\delta^2(\mathbf{R}), \quad (7)$$

where (L_n are the Laguerre polynomials, $\omega \to \omega + i0^+$ for retardation; α is a characteristic speed of the Dice lattice),

$$\begin{aligned}G'_{22}(\mathbf{R}, \omega) = &\frac{eB_0}{2\pi}\omega e^{-eB_0R^2/4} \\ &\times \sum_{n=0}^{\infty} L_n\left(\frac{eB_0R^2}{2}\right)\frac{1}{\omega^2 - 2(2n+1)\alpha^2 eB_0}. \quad (8)\end{aligned}$$

The trace is given by

$$\begin{aligned}TrG'(\mathbf{R} = 0, \omega) &= 2TrG'_{22}(\mathbf{R} = 0, \omega) \\ &= (area)\frac{eB_0}{2\pi}\sum_{n=0}^{\infty}\sum_{\pm}\frac{1}{\omega \pm \epsilon_n} + \frac{\delta^2(0)}{\omega}, \quad (9)\end{aligned}$$

where the Landau quantized energy spectrum is given by

$$\epsilon_n = \sqrt{2(2n+1)\alpha^2 eB_0}. \quad (10)$$

It should be noted that the results in (6)–(9) describe the diagonal elements of the full time-ordered Green's function (time-ordered in regard to the order of its internal creation and annihlation operators). This involves the $\delta^2(\mathbf{R})/\omega$-term on the right of (9), which arises from the $\delta^2(\mathbf{R})\delta(T)$-driving term of the inhomogenous time-ordered Green's function equation in position-time representation. However, from (5) it is clear that we need the part of the Green's function having the particular time order $T = t - t' > 0$ (to the exclusion of $T \leq 0$); and this obeys the homogeneous counterpart of the Green's function equation, excluding the driving term $\delta^2(\mathbf{R})\delta(T)$ in positon-time representation, thus excluding $\delta^2(\mathbf{R})$-terms in position-frequency representation. Apart from this exclusion, the equations for the elements of the retarded Green's function for the particular time order $T > 0$ are the same as those of the time-ordered one with the omission of the $\delta^2(\mathbf{R})$-driving term, which is therefore omitted henceforth.

The n-series of (8), (9) must be understood to terminate at $n_{\max}$ corresponding to the maximum Landau eigen-energy for which the approximation of "Dirac" linearity of the Hamiltonian (energy) as a function of momentum is valid, past which curvature of the underlying energy bands invalidates the approximation. Fourier transforming (9) to direct time representation, we have

$$\begin{aligned}TrG(\mathbf{x}, \mathbf{x}; T) &= TrG'(\mathbf{R} = 0; T) \\ &= -i\frac{eB_0}{2\pi}\sum_{n=0}^{n_{\max}}\sum_{\pm} e^{\pm i\epsilon_n T}. \quad (11)\end{aligned}$$

Consequently, (5) yields the ordinary partition function as

$$\hat{Z}(\beta) = \frac{eB_0}{2\pi}\sum_{n=0}^{n_{\max}}\sum_{\pm} e^{\pm\epsilon_n\beta} = \frac{eB_0}{\pi}\sum_{n=0}^{n_{\max}} cosh(\epsilon_n\beta), \quad (12)$$

where we now understand $\hat{Z}(\beta)$ and all extensive thermodynamic variables on a "per-unit area" basis $\hat{Z}(\beta)/area \leftrightarrow \hat{Z}(\beta)$.

III. THERMODYNAMIC FUNCTIONS AND MAGNETIC MOMENT OF THE T-3 DICE LATTICE IN THE DEGENERATE REGIME

To study the degenerate regime, we examine the E-integral of (4) with the change of variable $z = [E-\mu]\beta/2$, so this integral becomes

$$\int_0^\infty dE e^{Es} sech^2\left(\frac{[E-\mu]\beta}{2}\right) = \frac{2}{\beta}e^{s\mu}\int_{-\mu\beta/2\to-\infty}^{\infty} dz e^{2sz/\beta} sech^2 z. \quad (13)$$

To examine the degenerate regime, we put $-\mu\beta/2 \to -\infty$ in the lower limit of the z-integral, so the E-integral is [20]

$$\int_0^\infty dE\cdots = \frac{4\pi}{\beta^2}\frac{se^{s\mu}}{\sin\left(\frac{\pi s}{\beta}\right)}, \quad (14)$$

and in the degenerate regime Ω takes the form

$$\Omega = F - \mu N = -\frac{\pi}{\beta}\int_c \frac{ds}{2\pi i}\frac{e^{s\mu}\hat{Z}(s)}{s\sin\left(\frac{\pi s}{\beta}\right)}. \quad (15)$$

Noting that the $\int d^2x\ldots$ integral of $\hat{Z}(\beta)$ in (5) just produces an *area* factor, we can put Ω, F, N, on *a per-unit-area basis* $(n = N/area; \Omega \to \Omega/area; F \to F/area$; also $M \to M/area$; $S \to S/area$; $C_v \to C_v/area)$ as

$$\Omega = F - \mu n = \frac{-eB_0}{2\beta}\sum_{n=0}^{n_{\max}}\sum_{\pm}\int_c \frac{ds}{2\pi i}\frac{e^{s(\mu\pm\epsilon_n)}}{s\sin\left(\frac{\pi s}{\beta}\right)}. \quad (16)$$

To evaluate the s-integral of (16) we exponentiate the integrand factor as $\frac{1}{s} = \int_0^\infty dxe^{-sx}$, so that $(z \equiv \pi s/\beta)$

$$\int \frac{ds}{2\pi i}\frac{e^{s(\mu\pm\epsilon_n)}}{s\sin\left(\frac{\pi s}{\beta}\right)} = \beta\int_0^\infty dx\int_c \frac{dz}{2\pi i}\frac{e^{z\beta([\mu\pm\epsilon_n]/\pi - x)}}{\sin z}. \quad (17)$$

Noting that the contour of z-integration along c is a straight line from $z = -i\infty + 0^+$ to $+i\infty + 0^+$, we consider closing the contour with a parallel line (c') from $i\infty - \pi^+$ to $-i\infty - \pi^+$, on which $dz_{c'} = -dz_c$ and $\sin(z') = -\sin(z)$ [21]. Moreover, the closed contour integrand of $\oint = \int_c + \int_{c'}$ has the residue "1" at $z = 0$, so that

$$\oint \frac{dz}{2\pi i}\cdots = \int_c \frac{dz_c}{2\pi i}\cdots + \int_{c'}\frac{dz_{c'}}{2\pi i}\cdots$$
$$= \left(1 + e^{-\pi\beta[[\mu\pm\epsilon_n]/\pi - x]}\right) \times \int_c \frac{dz}{2\pi i}\frac{e^{z\beta[[\mu\pm\epsilon_n]/\pi - x]}}{\sin(z)} = 1. \quad (18)$$

Consequently, the x-integration of (17) is given by [22]

$$\int_0^\infty dx \frac{1}{1 + e^{-\pi\beta[[\mu\pm\epsilon_n]/\pi - x]}} = \frac{1}{\pi\beta}\ln\left(1 + e^{\beta[\mu\pm\epsilon_n]}\right) \quad (19)$$

and, for the degenerate regime under consideration, we obtain the Grand Potential Ω (*per-unit-area*) as

$$\Omega = F - \mu n = \frac{-eB_0}{2\pi}\sum_{n=0}^{n_{\max}}\sum_{\pm}\frac{1}{\beta}\ln\left(1 + e^{[\beta(\mu\pm\epsilon_n)]}\right). \quad (20)$$

For low temperature such that $\beta(\mu\pm\epsilon_n) >> 1$, we have

$$\Omega = \frac{-eB_0}{2\pi}\sum_{n=0}^{n_{\max}}\sum_{\pm}\left\{\eta_+(\mu\pm\epsilon_n)|\mu\pm\epsilon_n| + \frac{1}{\beta}e^{-\beta|\mu\pm\epsilon_n|}\right\}, \quad (21)$$

where $\eta_+(x) = 1$ for $x > 0$; 0 for $x < 0$. The first term on the right is the zero temperature degenerate limit and the last term on the right describes the temperature dependence in the approach to the zero temperature limit. Another exact evaluation of Ω for arbitrary temperature and density is presented in the Appendix.

The density may be evaluated as $n = -\partial\Omega/\partial\mu$, so (20) yields

$$n = (eB_0/2\pi)\sum_{n=0}^{n_{\max}}\sum_{\pm}\left(1/\left[1 + e^{-\beta(\mu\pm\epsilon_n)}\right]\right),$$

and in the degenerate limit of zero temperature, we have

$$n = (eB_0/2\pi)\sum_{n=0}^{n_{\max}}\sum_{\pm}\eta_+(\mu\pm\epsilon_n).$$

The magnetic moment M (*per-unit-area*/volume) is given by [15], [16], [17]

$$M = -\left(\frac{\partial F}{\partial B_0}\right)_{T,V,N} = -\left(\frac{\partial\Omega}{\partial B_0}\right)_{T,V,N}, \quad (22)$$

and using (21) we have magnetization as

$$M = \frac{e}{2\pi}\sum_{n=0}^{n_{\max}}\sum_{\pm}\left\{\eta_+(\mu\pm\epsilon_n)\left(\left[\mu\pm\frac{\epsilon_n}{2}\right] + \frac{\epsilon_n}{2}\right) + \frac{1}{\beta}e^{-\beta|\mu\pm\epsilon_n|}\left(1 - \frac{\beta\epsilon_n}{2}\right)\right\}. \quad (23)$$

(The $\partial/\partial B_0$ - differentiation in (22) is carried out holding μ fixed in consideration of the presence of a particle bath.)

The first term on the right of (23) in the curly brackets of the magnetic moment M is the zero temperature degenerate limit, which exhibits de Haas-van Alphen oscillatory phenomenology as variation of the magnetic field gives rise to successive vanishings of $\mu - \epsilon_n \to 0$ in $\eta_+(\mu\pm\epsilon_n)$, as successive Landau levels ϵ_n cross the Fermi level μ, inducing abrupt population/de-population of the levels. The second term on the right of (23) provides finite temperature corrections in the degenerate statistical regime, which are relatively small contributions describing temperature dependence in the approach to zero temperature, in the exponentially small "tail" of the Fermi function. The dominance of the first term in the magnetic moment M, which is manifestly positive, assures the positivity of the magnetic moment M, thus identifying the Dice lattice as a paramagnetic material.

The entropy, S, is determined by a variation of the Helmholtz Free Energy, F, in the thermodynamic relation [23], [24]

$$dF = -pdV - SdT' + \mu dN \tag{24}$$

(p is pressure (lineal in 2D), μ is chemical potential, T' is Kelvin temperature, N is number and V is volume (area in 2D)). Holding N and V constant, the entropy (per unit area) may be identified as

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T'}\right)_{N,V,\mu} \\ &= -\left(\frac{\partial \Omega}{\partial T'}\right)_{N,V,\mu} \\ &= \kappa_B \beta^2 \left(\frac{\partial \Omega}{\partial \beta}\right)_{\mu}. \end{aligned} \tag{25}$$

In considering $(\partial F/\partial T')_{N,V,\mu}$, it should be noted that only the explicit dependence of $F(\beta,\mu)$ on β contributes, to the exclusion of implicit dependence on β through μ (as determined by the expression for density $n(\beta,\mu)$), since such a contribution would be of the form $\partial F/\partial\mu \times \partial\mu/\partial\beta$ which vanishes identically because [15], [16], [17]

$$\partial F/\partial \mu \equiv 0, \tag{26}$$

by definition of the density n. In this context the μ-dependence of F cannot contribute to $\partial F/\partial T'$, so the term $-n\partial\mu/\partial T'$ in (25) must be understood to vanish in combination with any implicit dependence of $-\partial\Omega/\partial T'$ on β through μ.

The fact that the usual properties of entropy (positivity $S > 0$ and $S \to 0$ when $T' \to 0$) persist in cases where there is an unbounded negative component of the energy spectrum (in addition to the positive energy component) has been verified quite generally in separate work [25]. For the degenerate regime of the Dice lattice we employ (21) for Ω, with the result

$$S = \kappa_B \frac{eB_0}{2\pi} \sum_{n=0}^{n_{\max}} \sum_{\pm} e^{-\beta|\mu \pm \epsilon_n|}(1 + \beta|\mu \pm \epsilon_n|). \tag{27}$$

Furthermore, the specific heat at constant volume is given by

$$\begin{aligned} C_v &= T' \frac{\partial S}{\partial T'} = -\beta \frac{\partial S}{\partial \beta} \\ &= \kappa_B \beta^2 \frac{eB_0}{2\pi} \sum_{n=0}^{n_{\max}} \sum_{\pm} |\mu \pm \epsilon_n|^2 e^{-\beta|\mu \pm \epsilon_n|}. \end{aligned} \tag{28}$$

IV. SUMMARY, DISCUSSION

We have employed a Green's function formulation jointly with the Sondheimer-Wilson approach to the determination of the Grand Potential Ω of T-3 Dice lattice carriers in a normal, quantizing magnetic field. The result for Ω, presented in (20) and (21), describes Ω in the zero temperature degenerate limit and also provides the leading temperature corrections in the approach to zero temperature. The magnetic moment (*per-unit-area*/volume) is also determined in this degenerate regime, including temperature corrections in the approach to the zero temperature limit, and is presented in (23). Both the Grand Potential and the magnetization exhibit de Haas-van Alphen oscillatory features in the form of Heaviside cut-off-functions $\eta_+(\mu - \epsilon_n)$, which describe abrupt population changes as the magnetic field varies. Such $dHvA$ oscillations in the Dice Lattice were also found earlier for Ω and M (in a different representation) by Raoux et al. [26]. However, in that work (and in its reference 9) the magnetic moment seems to have been calculated as the (negative) magnetic field derivative of the Grand Potential holding density fixed (with μ as a function of density and field), in contrast to our derivative of the Helmholtz Free Energy holding μ fixed (in the presence of a particle bath). Although our calculation differs from that of Raoux et al. due to the differing physical conditions of the calculations, both indicate that the Dice lattice is paramagnetic ($M > 0$). These paramagnetic results for the pseudospin 1 Dice lattice stand in sharp contrast to the diamagnetism of pseudospin 1/2 graphene.

We have also analyzed the Entropy and presented it in the degenerate regime in (27). It is devoid of $dHvA$ oscillatory behavior since the only terms of Ω responsible for $dHvA$ oscillations are those of the zero temperature degenerate limit, in which the Entropy vanishes, even in materials having an unbound negative energy component in their spectra.

Finally, we have also presented the specific heat at constant volume for Dice Lattice carriers in a normal magnetic field (28), exhibiting its temperature dependence. C_v is also devoid of $dHvA$ oscillations, but it does depend on magnetic field strength, as seen in (28) along with its temperature dependence. Our Appendix also presents an exact evaluation of the Grand Potential Ω for arbitrary temperature and density.

The importance of the specific heat as a standard characterization technique to understand the underlying physics of the materials was pointed out by Stewart [27] and Geballe's group [28], [29], [30], [31], [32]; and our specific heat results in (3.18) can be employed to interpret relevant data for this purpose. Of course, our results for de Haas-van Alphen oscillations in the magnetic moment can also be employed to characterize the material parameters.

Notwithstanding the fact that this statistical thermodynamic analysis addresses a "relativistic" material in a magnetic field, it should be noted that it is subject to the Bohr-van Leeuven theorem [33], [34], [35]: In the classical limit ($\hbar \to 0$), the discrete Landau levels experience a kind of phase-averaging such that their low field limit is devoid of de Haas-van

Alphen oscillations in the magnetic moment and other statistical thermodynamic functions. This arises from the role of the magnetic field as it is introduced by the Hamiltonian replacement $\mathbf{p} \to \boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}/c$, from which the magnetic field can be completely eliminated by transformation in classical statistical functions.

APPENDIX
EXACT EVALUATION OF THE GRAND THERMODYNAMIC POTENTIAL

In Ref. [14] it is shown that

$$\Omega = -\frac{\beta}{4}\int_0^\infty \frac{z(t)}{\cosh^2\left[\frac{\beta}{2}(t-\mu)\right]}dt, \tag{A.1}$$

where $z(t)$ is the inverse Laplace transform of $\beta^{-2}\tilde{Z}(\beta)$. From (12) we have

$$\frac{\tilde{Z}(s)}{s^2} = \frac{eB_0}{2\pi}\sum_{n=0}^{n_{\max}}\left[\frac{e^{\epsilon_n s}}{s^2} + \frac{e^{-\epsilon_n s}}{s^2}\right], \tag{A.2}$$

whose inverse Laplace transform is

$$z(t) = \frac{eB_0}{2\pi}\sum_{n=0}^{n_{\max}}\left[t + \epsilon_n + (t-\epsilon_n)\eta_+(t-\epsilon_n)\right]. \tag{A.3}$$

Thus, (A.1) may be written as (bear in mind $\eta_+(x) = 1 - \eta_+(-x)$)

$$\Omega = \frac{-\beta}{4}\frac{eB_0}{2\pi}\sum_{n=0}^{n_{\max}}(I_1 - I_2), \tag{A.4}$$

where

$$I_1 = \int_0^\infty dt \frac{2t}{\cosh^2\left(\frac{\beta}{2}[t-\mu]\right)} \quad \text{and}$$
$$I_2 = \int_0^{\epsilon_n} dt \frac{t-\epsilon_n}{\cosh^2\left(\frac{\beta}{2}[t-\mu]\right)}. \tag{A.5}$$

The integrations in (A.5) are elementary, yielding

$$\Omega = -\frac{eB_0}{4\pi\beta}\Big[\left(\beta\mu + 2\ln[2(1+e^{\beta\mu})]\right)(n_{\max}+1) + \beta\sigma\tanh\left(\frac{\beta\mu}{2}\right) + 2\sum_{n=0}^{n_{\max}}\ln\left(\cosh\left[\frac{1}{2}\beta(\mu-\epsilon_n)\right]\right)\Big], \tag{A.6}$$

where $\epsilon_{n_{\max}}$ is the highest Landau level in the linear "relativistic" regime and

$$\sigma = \sum_{n=0}^{n_{\max}}\epsilon_n = \frac{4}{3}\left(1 - \frac{1}{\sqrt{8}}\right)\alpha\sqrt{eB}n_{\max}^{3/2}\left[1 + O(n_{\max}^{-3/2})\right], \tag{A.7}$$

for large $n_{\max}$. Similarly, the final sum in (A.6) behaves asymptotically as

$$\sum_{n=0}^{n_{\max}}\ln\left(\cosh\left[\frac{1}{2}\beta(\mu-\epsilon_n)\right]\right) = 3n_{\max}\cosh\left[(\beta/2)\left(\mu - 2\alpha\sqrt{2eBn_{\max}}\right)\right]. \tag{A.8}$$

Thus, for $n_{\max}$ large, the grand thermodynamic potential is approximately given by

$$\Omega \approx -\frac{eB_0}{4\pi\beta}\Big[\left(\beta\mu + 2\ln\left[2\left(1+e^{\beta\mu}\right)\right]\right)(n_{\max}+1) + \frac{4}{3}\left(1-\frac{1}{\sqrt{8}}\right)\alpha\beta\sqrt{eB}n_{\max}^{3/2} + 6n_{\max}\ln\cosh\left[\frac{\beta}{2}\left(\mu - 2\alpha\sqrt{2eBn_{\max}}\right)\right]\Big]. \tag{A.9}$$

These results, which are essentially exact for $n_{\max}$ large, are consistent with the degenerate regime results described above in the main body of this article. In particular, for $kT_B/\mu << 1$ the asymptotic behavior of (A.1) is

$$\Omega = z(\mu) - \frac{\pi^2}{6}z''(\mu)\left(k_BT/\mu\right)^2 + O((k_BT/\mu)^4), \tag{A.10}$$

which is consistent with (21).

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