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On the Relation Between Fourier Frequency and Period for Discrete Signals, and Series of Discrete Periodic Complex Exponentials

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ABSTRACT Discrete complex exponentials are almost periodic signals, not always periodic; when periodic, the frequency determines the period, but not viceversa, the period being a chaotic function of the frequency, expressible in terms of *Thomae's function*. The absolute value of the frequency is an increasing function of the subadditive functional of *average variation*. For discrete signals that are either sums or series of periodic complex exponentials, the decomposition into their periodic, additive components allows for their *filtering according to period*. Likewise, their *period-frequency spectrum* makes predictable the effects on period of convolution filtering. Ramanujan-Fourier series are a particular case of the signal class of *series of periodic complex exponentials*, a broad class of signals on which a transform, discrete both in time and in frequency, called the *DFDT Transform*, is defined.

INDEX TERMS Almost periodic sequence, Ramanujan sums, Ramanujan-Fourier series, Thomae's function, variation, period-frequency relation.

I. INTRODUCTION

Discrete complex exponentials behave differently from continuous complex exponentials in that the relationship between period and Fourier frequency is not as straightforward as it is for continuous complex exponentials: $|\omega|T = 2\pi$, where T stands for the period, and ω for the frequency. Discrete complex exponentials $e^{jn\theta}$ are *uniformly almost periodic* [13]–[16], but not always periodic. When periodic, the period is a positive integer N; remarkably, $|\theta|$ does not uniquely determine N, and N depends univocally on θ in a chaotic way, closely related to *Thomae's function*; see Section III-B. The relation between period and frequency for discrete complex exponentials is made explicit with the *period-frequency matrix*, defined in Section IV-C. The frequency is well related to the *average variation* of the complex exponential, as in Fig. 2 of Section I-B.

Ramanujan sums have been considered for application in Signal Processing, e.g. in [11] and [22]. In particular, they have been used to study the periodicity structure of streaming data [10], and also how the periodicity of a sampled continuous signal does not imply the periodicity of the continuous signal [9]. We are interested in periodicity, as it results from the addition of periodic signals (see Section V). Also, in Section VII, we consider the relation between the periodicity of a sampled continuous complex exponential and the period of the resulting discrete complex exponential. The integrality of Ramanujan sums is usually proved using Möbius inversion; we include a deduction that uses the pin train signal in Section III-C. The usefulness of Möbius inversion in the computation of Fourier transforms has been considered e.g in [12] and in [21]. Ramanujan Sine Sums, also integer valued, have not been considered as much in Signal Processing; we briefly review them in Section III-D. Series of Ramanujan sums have been considered in Number Theory but not so much in Signal Processing; more generally, we claim that series of periodic complex exponentials are an important source of signals for Signal Processing. It seems to us that discrete almost periodic signals are underrepresented in Signal Processing; exceptions being [8], [7] and [5]; we stress the almost periodicity of complex exponentials of frequencies that are not rational multiples of 2π. Series of discrete periodic complex exponentials are also a source of almost periodic discrete signals. The functional of average variation, which measures the roughness of a signal, is not used often in Signal Processing, although the related Total Variation functional [6] is commonly used in Image Processing since 1994; we find that the average variation (defined in Section I-B) usefully and intuitively discriminates among the discrete complex exponentials of a given period, and also that its application to Ramanujan Sums, in Section III-E, is insightful.

A few remarks on terminology: a *continuous function* is continuous in the sense of calculus, while a *continuous-time*, *continuous* or *analog signal* is a function whose domain is a *continuum*, e.g. an interval of the real line. A *discrete signal* is a function whose domain is a countable set, e.g. the integers or the rationals. Being the variables of the discrete-time and continuous-time Fourier transforms, we call *Fourier frequencies* the angular frequencies θ and ω of the discrete complex exponentials $e^{jn\theta}$, $n \in \mathbb{Z}$, and the continuous complex exponentials $e^{jt\omega}$, $t \in \mathbb{R}$.[1] From this point onwards, unless otherwise specified, whenever the terms *signal* or *complex exponential* are used, the discrete case will be understood.

Section II characterizes the space of the Fourier frequencies as a circle, and classifies the periodicity of complex exponentials into periodic and almost periodic. Fourier frequency looses its intuitive meaning in the discrete case, and the functional of *average variation* (Section I-B) supplies some of the lost intuition by assigning a degree of smoothness to each complex exponential; thus, for each period, we give the frequencies of the smoothest and the roughest complex exponentials. We review the definition of almost periodicity for discrete signals (sequences). The separate concepts dealt with in Section II are not novel but the approach as a whole is. We think there is a tendency to extrapolate from the continuous to the discrete case that does not always hold good.

Section III reviews a few relevant concepts from Number Theory, including both Ramanujan cosine and sine sums. We study the average variation of Ramanujan (cosine) sums and we show the role of Thomae's function relating period and Fourier frequency. Also, since for the computation of series of periodic complex exponentials (considered in Sections IV and VI) you use series over the rational numbers, we give the ordering (called Farey ordering) in which such series are to be summed up. Sections III-A, III-B and III-E present contributions of the paper, novel to Signal Processing, notwithstanding the probable fact that mathematicians may have considered them previously.

Under the proviso that, for signals in practice that are either periodic or series of periodic signals, each additive component of a given period comes from a different source, the decomposition of such signals into sums of signals of different periods becomes an important tool. Section IV introduces the concepts of the *period-frequency matrix*, the *Discrete-Frequency, Discrete-Time Transform*, and the notion of filtering periodic signals *according to period*, rather than frequency. These are tools that deal with signals in the combined domain of period and frequency. The consideration of the *period-frequency support* of periodic signals leads to the consideration in Section IV-D of signals that are (infinite) series of periodic complex exponentials (see also Section VI), which are signals that are not necessarily Ramanujan-Fourier Series (see Section III-F), opening the way to the consideration of nonperiodic signals without a DTFT. These signals are "bigger" than signals with a DTFT, and we define a norm for them in Section VI.

Section V introduces the notions of *strong* and *weak periodicity*; a signal of period N is said to be weakly periodic if it can be expressed as a sum of signals of periods strictly less than N. Otherwise, it is said to be strongly periodic. Sawtooth signals are shown to be strongly periodic. Also, the periodicity of sums of periodic signals is characterized on the basis of Fourier frequency. We consider particularly the case when the period of a sum of periodic signals is less than the least common multiple of the periods of the summands, a topic treated in [25] somewhat differently.

The use of Ramanujan sums in the Discrete Fourier Transform has proven useful in Signal Processing, see e.g. [21]. Likewise, the use of series of Ramanujan sums; see e.g. [1]. In a sense, an inverse *Ramanujan Fourier transform* expresses a signal as a series combination of weighted Ramanujan sums. We generalize this idea by allowing the possibility of assigning a different weight to each periodic complex exponential in a series of all the periodic complex exponentials. This allows for example the possibility of separate series of the smooth and rough periodic complex exponentials. Thus, Section VI formally defines the *discrete-frequency, discrete-time transform DFDT*, introduced in Section IV, by characterizing certain spaces of signals on which *Carmichael's inner product* can be defined, used in turn for the computation of the DFDT transform.

Applications to continuous-time signals are given in Section VII.

The paper is written in a lemma-proof style and several examples are given.

A. INITIAL REMARKS ON PERIODICITY

To use the term *period* in an unambiguous sense, we define a *repetition time* of a signal $s : \mathbb{Z} \to \mathbb{C}$ to be any integer k such that $\forall n \in \mathbb{Z},\ s_{n+k} = s_n$. Thus, the signal is *periodic* if it has a positive repetition time, and its *period* is given by the minimal positive repetition time. If the period is N, the signal is said to be N*-periodic*. Lemmas 1–3 make rigorous certain results using this terminology.

Lemma 1: If L is a repetition time of the N-periodic signal s, then L is a multiple of N: $N|L$.

Proof: By the division algorithm, there are numbers c and r, $0 \le r < N$, such that $L = cN + r$. If $N \nmid L$ then $r > 0$ but then $\forall n\ s_n = s_{n+L} = s_{cN+r} = s_{n+r}$ gives a contradiction. ■

[1] $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of the natural numbers, the integers, the rationals, the reals and the complex numbers, respectively. $\mathbb{Z}^+$ is the set of the positive integers.

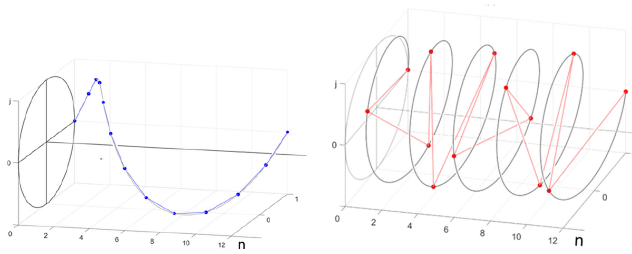

FIGURE 1. The 12-periodic complex exponentials $e^{j\frac{2\pi}{12}n}$, left, and $e^{j\frac{2\pi}{12}5n}$, right. The lengths of the projections on the vertical plane of the line segments are the corresponding (constant) pointwise variations.

For each frequency $\theta \in [0, 2\pi)$, the complex exponential $e^{j\theta n}$ is periodic if and only if its frequency θ is a rational multiple $2\pi q$ of 2π, otherwise it is only almost periodic. In the first case, for $\theta = 2\pi\frac{M}{N}$, with M and N relatively prime, the period is N. Thus, for each $N \in \mathbb{Z}^+$, there are precisely $\varphi(N)$ complex exponentials of period N, where φ stands for Euler's *totient function*; see Section III.

A signal is periodic if and only if it is a linear combination of periodic complex exponentials, i.e. if its spectrum support is finite and consists of frequencies of the form $2\pi q$, $q \in \hat{\mathbb{Q}} := \mathbb{Q} \cap [0, 1)$. The period of a periodic signal is the least common multiple of the periods of the complex exponentials it is a linear combination of, as shown in Lemma 9 of Section V.

Almost periodicity for discrete signals is defined in Section II-C. Almost periodicity was initially defined [13], [15], [14] for functions of a continuous argument and soon afterwards [17], [16] for sequences. Periodicity is a structural property of signals which, since the times of Euler, the Bernoulli's and Fourier, has the trigonometric-series aspect as well; likewise, almost periodicity has both aspects [15]. The structural aspect is intuitive and the trigonometrical-series provides a tool to handle the signals.

A linear combination of complex exponentials, as well as a series of complex exponentials, such as a Ramanujan-Fourier series, can be non-periodic, almost periodic. The consideration of almost periodic inputs to linear systems [18] is of interest, theoretical and practical [16], and a discrete, autonomous nonlinear system can generate almost periodic signals.

B. AVERAGE VARIATION

The *average variation* of a signal $s = \{s_n\}$, is given by

$$\overline{\mathrm{var}}(s) := \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} |s_n - s_{n+1}| \tag{1}$$

and provides a measure of what Tukey might have called its *roughness* [36]. The pointwise variation (i.e. the absolute difference of each two consecutive values) of a complex exponential $e^{j\theta n}$, periodic or not, is constant (see Fig. 1) and its average variation is

$$\begin{aligned}\overline{\mathrm{var}}(\{e^{j\theta n}\}) &= |e^{j\theta n} - e^{j\theta(n+1)}| = |1 - e^{j\theta}| \\ &= 2|\sin(\frac{\theta}{2})|; \end{aligned} \tag{2}$$

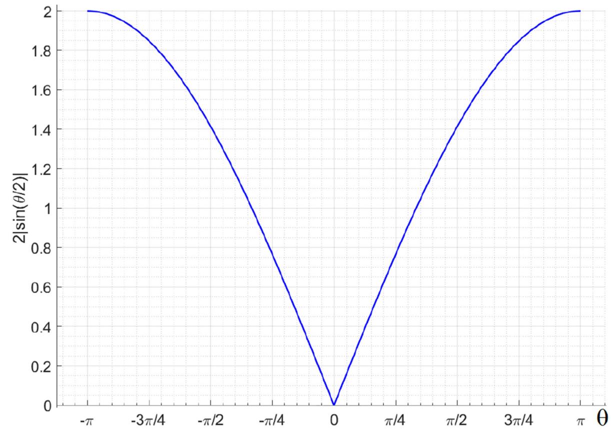

FIGURE 2. The average variation of $e^{j\theta n}$ vs. the Fourier frequency θ.

see Fig. 2. It is close to $|\theta|$, for small $|\theta|$.

For an N-periodic signal s, the average variation is given by the average variation over one period:

$$\overline{\mathrm{var}}(\{s_n\}) := \frac{1}{N}\left(\sum_{n=0}^{N-2} |s_n - s_{n+1}| + |s_{N-1} - s_0|\right).$$

Average variation is a measure of roughness; defining the *smoothness* of a complex exponential as two minus its average variation, the roughness is high for frequencies θ with $|\theta| \approx \pi$, while for $|\theta| \approx 0 = 2\pi$ (i.e. equivalent as frequencies, on the basis of Equation (3), below) the smoothness is high. Complex exponentials of the same period but of different Fourier frequencies have different average variations.

C. SERIES OF PERIODIC COMPLEX EXPONENTIALS

Of recent interest in Signal Processing are both Ramanujan Sums (which are periodic) and Ramanujan-Fourier Series [27], [22], [26]. The complex exponentials of a given period in a Ramanujan-Fourier series are given the same weight; we generalize this by considering series of periodic complex exponentials where each periodic complex exponential in a series of periodic complex exponentials may have a different weight, in what we have termed the DFDT; see Sections IV-D and VI. For a convergent series of periodic complex exponentials, the pairing of the sequence of the coefficients of the series, and the resulting signal, is a pairing of signals that are discrete both in the time and in the frequency domains.

The countable, dense set $\{2\pi q : q \in [0, 1) \cap \mathbb{Q}\}$ gives the set of the frequencies of the periodic complex exponentials. For the spectral analysis and filter design of series of periodic complex exponentials, we group the Fourier frequencies according to period, on the *period-frequency matrix*, as explained in Section IV.

II. COMPLEX EXPONENTIALS

Denote as $\boldsymbol{\Theta}$ the set $[0, 2\pi)$ of the Fourier frequencies. The similarity of two Fourier frequencies θ_1 and θ_2 is inherited from the similarity of the corresponding complex exponentials

$\{e^{j\theta_1 n}\}$ and $\{e^{j\theta_2 n}\}$, by using the metric

$$d(\theta_1, \theta_2) := \sup\{|e^{j\theta_1 n} - e^{j\theta_2 n}| : n \in \mathbb{Z}\}, \tag{3}$$

on the set $\boldsymbol{\Theta}$ of the Fourier frequencies. With this metric, the space of the Fourier frequencies is a topological circle, and it is *cyclically ordered* [30]. Other subsets of $\mathbb{R}$ can be used as Fourier-frequency sets; namely, any interval of length 2π, half closed, half open, such as $[0, 2\pi)$ or $(-\pi, \pi]$; the second case allows for the consideration of both positive and negative frequencies. The endpoints of any interval of length 2π are equivalent: as $\theta \to 2\pi$, $e^{j\theta n} \to e^{j0n}$ since, for each $\epsilon > 0$ there is a $\delta > 0$ such that $\forall n \in \mathbb{Z}$, $|e^{j\delta n} - e^{j(2\pi-\delta)n}| = 2|\sin(\delta n)| < \epsilon$.

A. PERIODICITY OF COMPLEX EXPONENTIALS

The signal e^{jn} is not periodic, it is only almost periodic; in fact, a complex exponential $e^{j\theta n}$ is periodic if and only if its frequency θ is a rational multiple of 2π:

Lemma 2: The complex exponential $e^{jn\theta}$ is periodic if and only if θ is a rational multiple of 2π.

Proof: Clearly, the condition is sufficient. Also, if $e^{j\theta n}$ is periodic, there is a positive integer r such that $e^{j\theta(n+r)} = e^{j\theta n}e^{j\theta r} = e^{j\theta n}$; then, since an exponential never takes the value 0, $e^{j\theta r} = 1$. Then, for some integer k, $\theta r = 2\pi k$ and the condition is necessary as well. ■

Lemma 3: The period of $e^{j2\pi \frac{M}{N} n}$, with $M < N$ and M and N relatively prime, is N.

Proof: Clearly, N is a repetition time of $e^{j2\pi \frac{M}{N} n}$. Let $\gcd(N, M) = 1$ and assume that there is another positive repetition time N' less than N; then, for some integer k, $\frac{M}{N} N' = k$, but then, being $\frac{M}{N}$ equal to $\frac{k}{N'}$, it is reducible, which is a contradiction. ■

B. SMOOTHEST AND ROUGHEST PERIODIC COMPLEX EXPONENTIALS

Let $\hat{\boldsymbol{\Theta}} := 2\pi\hat{\mathbb{Q}} = \{\theta = 2\pi q : q \in \mathbb{Q} \cap [0, 1)\}$ denote the set of the Fourier frequencies of periodic complex exponentials.

The closer to 0 the frequency of a complex exponential is, the smaller its variation is, and the closer to π the frequency is, the larger the variation. For $N > 2$, among the $\varphi(N)$ complex exponentials of period N, two are *smoothest* (see Lemma 4), and two are *roughest* (see Lemma 5). For small values of N, i.e. for $N = 2, 3, 4$ and 6, they coincide; see Fig. 11, below. We present the formulae for the frequencies of the smoothest and roughest complex exponentials of given period, in the following two lemmas and corollary.

Lemma 4: The smoothest complex exponentials of period N are those of frequencies $2\pi \frac{1}{N}$ and $2\pi \frac{N-1}{N}$.

Proof: The lemma follows from the fact that 1 and N, as well as $N - 1$ and N are relatively prime for all N, and from the variation of complex exponentials, visualized in Fig. 2. ■

For example, the smoothest complex exponentials of period 6 are those of frequencies $\frac{2\pi}{6}$ and $2\pi \frac{5}{6}$. Regarding the frequencies of the roughest complex exponentials, you have,

Lemma 5: The roughest N-periodic complex exponentials, writing N as $N = 4m + k$, with $k = -1$, $k = 0$, $k = 1$ or $k = 2$, have frequencies $2\pi \frac{2m-1}{4m+k}$ and $2\pi \frac{2m+k+1}{4m+k}$, for $k = -1, 0, 2$, and $2\pi \frac{2\,m}{4m+1} = 2\pi \frac{N-1}{2\,N}$ and $2\pi \frac{2m+1}{4m+1} = 2\pi \frac{N+1}{2\,N}$, for $k = 1$.

Proof: We are interested in frequencies $2\pi \frac{M}{N}$ closest to π, i.e. $\frac{M}{N}$ closest to $\frac{1}{2}$, with $\gcd(M, N) = 1$.

We show first that $N = 4m + k$ and $2m - 1$ are relatively prime for $k = -1, 0, 2$. If not, a prime p divides both $2m - 1$ and $4m + k$, for $k \in \{-1, 0, 2\}$, then, for some prime p, $\exists r, s$, $4m + k = rp$ and $2m - 1 = sp$; then $k + 2 = p(r - 2\,s)$, that is, p divides $k + 2$, $k = -1, 0$ or 2. If p divides 1 then $p = 1$, a contradiction; if p divides 2, then $p = 2$, but p divides the odd number $2m - 1$ as well, a contradiction; if p divides 4, then $p = 2$ but p divides the odd number $2m - 1$ as well, also a contradiction. We now show that $N = 4m + 1$ and $2\,m$ are relatively prime. Otherwise, a prime p divides both $4m + 1$ and $2\,m$ i.e. $\exists r, s$, $4m + 1 = rp$ and $2m = sp$ then $1 = p(r - 2\,s)$ then $p = 1$, a contradiction.

By Lemma 8 below, $2m + k + 1$ and $4m + k$, for $k = -1, 0, 2$, are also relatively prime, and, likewise, $2m + 1$ and $4m + k$ are relatively prime.

It remains to notice that, for each m, with fixed denominator, $\frac{2m-1}{4m-1}$, $\frac{2m-1}{4\,m}$ and $\frac{2m-1}{4m+2}$, as well as $\frac{2\,m}{4m+1}$, are (irreducible) fractions, closest to $\frac{1}{2}$. ■

Corollary 1: For odd periods N, the roughest complex exponentials have frequencies $2\pi(\frac{1}{2} \pm \frac{1}{2\,N})$; for even N, of the form $4\,m$ (i.e. a multiple of four), the roughest complex exponentials have frequencies $2\pi(\frac{1}{2} \pm \frac{1}{N})$. Lastly, for N even, of the form $4m + 2$, the roughest complex exponentials have frequencies $2\pi(\frac{1}{2} \pm \frac{2}{N})$.

Interestingly enough, the sum of a smoothest and a roughest complex exponentials of period 12, using Ramanujan notation for periodized segments,

$$\begin{aligned} e^{j2\pi \frac{1}{12} n} + e^{j2\pi \frac{5}{12} n} &= 2j^n \cos\left(\frac{2\pi}{6} n\right) \\ &= \overline{2, j, 1, 2j, -1, j, -2, -j, -1, -2j, 1, -j} \end{aligned}$$

which takes values only on the axes of the complex plane.

C. NON-PERIODIC COMPLEX EXPONENTIALS

We denote the *integer interval* $\{n \in \mathbb{Z} : M \leq n \leq N\}$ as $/M, N/$.

It follows from Lemma 2 that, for the frequencies

$$\theta \in [0, 2\pi) - \hat{\boldsymbol{\Theta}} = \boldsymbol{\Theta} - \hat{\boldsymbol{\Theta}}$$

that are non-rational multiples of 2π, $e^{j\theta n}$ is not a periodic signal. A discrete signal $\{s_n\}$ is said to be *almost periodic* [17], [16], if for each real positive ϵ there is a length $L \in \mathbb{Z}^+$ such that each integer interval $/k, k + L/$, $k \in \mathbb{Z}$, of length L, contains a number R such that

$$\forall n \in \mathbb{Z} \quad |s_n - s_{n+R}| < \epsilon.$$

The following two lemmas give known results whose proofs are not easy to find in the Signal Processing literature.

Lemma 6: Each complex exponential $\mathrm{e}^{\mathrm{j}\theta n}$ is almost periodic.

Proof: Given $\epsilon > 0$, partition the circle into L arcs $(\mathrm{e}^{\mathrm{j}\frac{2\pi}{L}m}, \mathrm{e}^{\mathrm{j}\frac{2\pi}{L}(m+1)})$, $m \in /0, L-1/$, such that $|\mathrm{e}^{\mathrm{j}\frac{2\pi}{L}} - 1| < \epsilon$. By the pigeonhole principle, for each k, out of the $(L+1)$-point set $\{\mathrm{e}^{\mathrm{j}\theta n} : n \in /kL, (k+1)L/\}$, there are two integers, say n_1 and n_2, such that $\mathrm{e}^{\mathrm{j}\theta n_1}$ and $\mathrm{e}^{\mathrm{j}\theta n_2}$ are in one of the arcs and therefore $|\mathrm{e}^{\mathrm{j}\theta n_1} - \mathrm{e}^{\mathrm{j}\theta n_2}| < \epsilon$. If $\mathrm{e}^{\mathrm{j}\theta n_1} = \mathrm{e}^{\mathrm{j}\theta n_2}$ the signal is periodic; otherwise, with $R = n_2 - n_1$, $|\mathrm{e}^{\mathrm{j}\theta(n+R)} - \mathrm{e}^{\mathrm{j}\theta n}| = |\mathrm{e}^{\mathrm{j}\theta R} - 1| < \epsilon$. ∎

Complex exponentials meet a condition stronger than almost periodicity:

Lemma 7: For each $\epsilon > 0$ there is a positive integer R such that

$$\forall n \in \mathbb{Z} \ \ |\mathrm{e}^{\mathrm{j}\theta n} - \mathrm{e}^{\mathrm{j}\theta(n+R)}| < \epsilon.$$

Proof: If the exponential is N-periodic, take $R = N$. Otherwise, if $\mathrm{e}^{\mathrm{j}\theta n}$ is not periodic, then for a given ϵ there is a positive integer R such that

$$\left|1 - \mathrm{e}^{\mathrm{j}\theta R}\right| < \epsilon \ ,$$

since, as proven by H. Weyl [31], the sequence $\{(nx)\}$, for[2] irrational x, is uniformly distributed in $[0, 1)$; correspondingly, for θ not a rational multiple of 2π, the sequence $\{n\theta\}$, mod-2π, is uniformly distributed in $[0, 2\pi)$, and the lemma follows. ∎

III. RAMANUJAN SUMS AND SOME NUMBER THEORY

Let the (unique) prime factorization of the positive integer N into (distinct) prime numbers be

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_R^{\alpha_R} \tag{4}$$

where $\alpha_i \geq 1$, for each $i \in /1, R/$; we refer to the numbers $p_i^{\alpha_i}$ as the *largest-power-of-a-prime factors* (abbreviated as *lpp factor*) of N. Euclid's *algorithm of division* says that, for any integers N and M, with $M > 0$, there are integers c and r, with $0 \leq r < M$, such that $N = cM + r$. If $r = 0$, one says that M *divides* N and write $M|N$; otherwise, write $M \nmid N$. The number of divisors $d(N)$ of N is given by $\prod_{i=1}^{R}(\alpha_i + 1)$, and the sum of the divisors $\sigma(N)$ of N is $\sigma(N) = \sum_{a|N} a$. $\varphi(N)$ denotes the number of relatively prime numbers to N, not larger than N; it is known as Euler's *totient function*.

The following result is well known but seldom made explicit.

Lemma 8: If $M \in /1, N/$ is relatively prime to N, that is if $\gcd(M, N) = 1$, then so is $N - M$.

Proof: We prove the contrapositive. If N and $N - M$ are not relatively prime, there is a prime p that divides both $N - M$ and N, then $\exists r, s, N - M = rp$ and $N = sp$, then $M = p(s - r)$ and p divides M; then N and M are not relatively prime. ∎

Alternatively to the notation of Equation (4), letting $\{p_i\}$ be the ascending *count* (i.e. a one-to-one correspondence with the natural numbers) of (all) the prime numbers, given numbers m and n, you may write

$$n = \prod_{i=1}^{\infty} p_i^{\alpha_i}, \quad m = \prod_{i=1}^{\infty} p_i^{\beta_i} \tag{5}$$

where only a finite number of the α_i's, and of the β_i's, are positive. The least common multiple and the greatest common divisor of m and n are then given by $\mathrm{lcm}(m, n) = \prod_{i=1}^{\infty} p_i^{\max(\alpha_i, \beta_i)}$ and $\gcd(m, n) = \prod_{i=1}^{\infty} p_i^{\min(\alpha_i, \beta_i)}$; note that each lpp factor of $\mathrm{lcm}(m, n)$ is an lpp factor of at least one m and n. Also, $\mathrm{lcm}(m, n)\gcd(m, n) = mn$. The max and the min operators being associative, if A and B are sets, then $\mathrm{lcm}(A \cup B) = \mathrm{lcm}(\mathrm{lcm}(A), \mathrm{lcm}(B))$ and $\gcd(A \cup B) = \gcd(\gcd(A), \gcd(B))$.

A. FAREY SEQUENCES AND SERIES OVER $\hat{\mathbb{Q}}$

The Farey sequences induce an order for series that are summed over the rationals[3] in $[0, 1)$. The *Farey sequence* [29] F_n of order n is the ascending (finite) sequence of the irreducible fractions between 0 and 1 whose denominators do not exceed n. Thus, for example, $F_6 = \{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\}$. The set $\hat{\mathbb{Q}}$ of the rationals in the interval $[0, 1)$ is countable and it can be given the count, called here the *Farey ordering* $\{q_k\} = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9} \ldots\}$, where the irreducible fractions are given the dictionary order, with denominators in ascending order, and the numerators in ascending order.

The ordering of the terms of a series matters and the order in which the terms are to be summed must be specified.[4] For a function $f : \hat{\mathbb{Q}} \to \mathbb{C}$, e.g. $f(q) = c_q \mathrm{e}^{\mathrm{j}2\pi qn}$, we assume the Farey ordering of the terms in the series $\sum_{q\in\hat{\mathbb{Q}}} f(q)$:

$$\sum_{q\in\hat{\mathbb{Q}}} f(q) := \lim_{N\to\infty} \sum_{q\in F_N - \{1\}} f(q). \tag{6}$$

B. THOMAE'S FUNCTION AND THE RELATION PERIOD-FREQUENCY

Let $\eta(q)$, $q \in \hat{\mathbb{Q}} - \{0\}$, give the denominator of the irreducible fraction equivalent to q; thus, for $\frac{m}{n}$ any fraction equivalent to q,

$$\eta\left(\frac{m}{n}\right) := \frac{n}{\gcd(m, n)}. \tag{7}$$

Call η the *denominator function*, and extend it, with $\eta(0) = 1$. Restricted to $\hat{\mathbb{Q}}$, *Thomae's function* is given by

$$\tau(q) = \frac{1}{\eta(q)};$$

[2]Here, (nx) denotes the decimal part of nx. Incidentally, Walther's [17] interest in almost periodic sequences stemmed from Hecke's study of the analytic continuation of the series $\sum_{n=1}^{\infty}(nx)z^n$, $|z| < 1$, with x irrational.

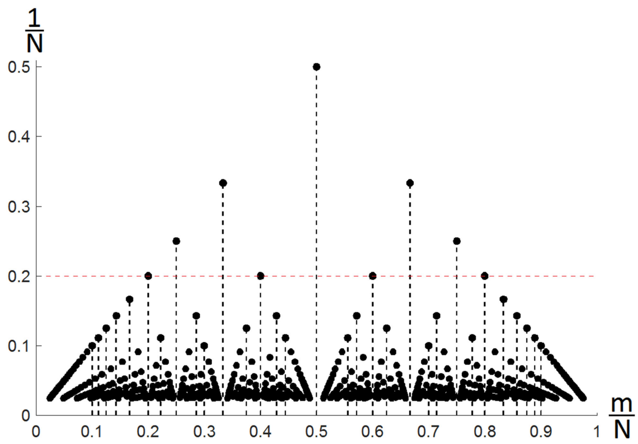

FIGURE 3. **Stem plot of Thomae's function $\tau(\frac{m}{N}) = \frac{1}{N}$, for irreducible rational argument. Plotted for $0 < m < N \leq 40$. To avoid clutter, the stem is not always drawn.**

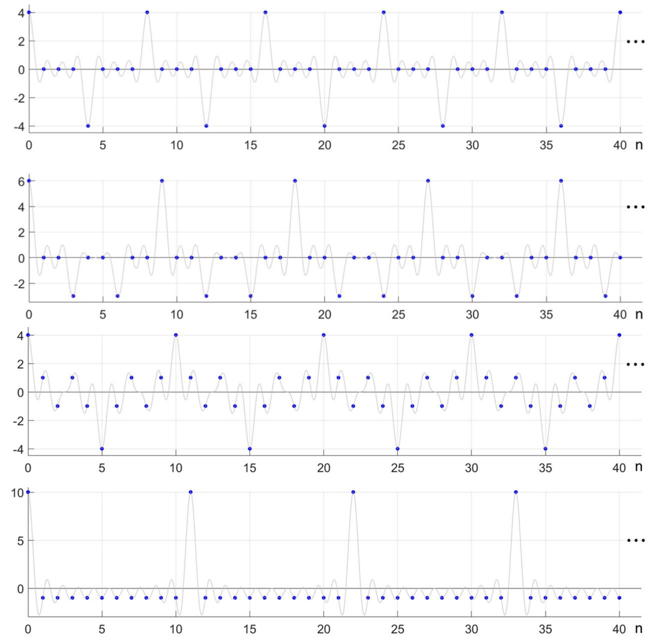

FIGURE 4. **Ramanujan sums $c_8(n)$, $c_9(n)$, $c_{10}(n)$ and $c_{11}(n)$.**

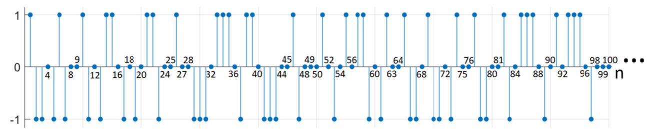

FIGURE 5. **Möbius function $\mu(n)$.**

see Fig. 3. In general, for rational argument $x = \frac{m}{n}$, with m and n coprime, $\tau(x) = n$ and, for irrational x, $\tau(x) = 0$.

Let *Tomae's signal* $t : \mathbb{N} \to \hat{\mathbb{Q}}$ be given by $t_0 = 0$ and $t_k = \tau(q_k)$ for $k \geq 1$, where $\{q_k\}$ is the Farey ordering of the rationals in $[0, 1)$.

Let the function $N : \hat{\boldsymbol{\Theta}} \to \mathbb{Z}^+$ be given by $N(\theta) = \eta(\frac{\theta}{2\pi})$; it gives the period of the periodic exponential $\mathrm{e}^{\mathrm{j}\theta n}$. It is a chaotic function: a change of θ, no matter how small, is likely to produce a large change of N; in fact, any positive-length interval $(\theta_0, \theta_0 + \epsilon)$ contains frequencies of complex exponentials of arbitrarily large periods. Also, for any period n, and any positive-length interval $(\theta_0, \theta_0 + \epsilon) \subset [0, 2\pi)$, the corresponding complex exponentials of period less than n are finite in number, possibly zero, while the set of corresponding complex exponentials of period larger than N is infinite. See Fig. 10, below.

C. RAMANUJAN (COSINE) SUMS

We briefly review the definition of Ramanujan sums, in order to set the notation. For each positive integer N, the *Ramanujan sum* [20] of order N is the function $\mathrm{c}_N : \mathbb{Z}^+ \to \mathbb{C}$ given by

$$\mathrm{c}_N(n) := \sum_{k \in \mathbb{Z}^*(N)} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn} = \sum_{k \in \mathbb{Z}^*(N)} \cos\left(\frac{2\pi}{N}kn\right) \qquad (8)$$

where $\mathbb{Z}^*(N) := \{k \in /1, N/ : \gcd(k, N) = 1\}$.[5] Each Ramanujan sum $\mathrm{c}_N(n)$ is precisely the sum of the complex exponentials of period N and is therefore N-periodic (see Section II.) Each Ramanujan sum $\mathrm{c}_N(n)$ is an even function with respect to $n = \frac{N}{2}$, that is

$$\mathrm{c}_N\left(\frac{N}{2} - n\right) = \sum_{k \in \mathbb{Z}^*(N)} (-1)^k \cos\left(\frac{2\pi}{N}kn\right) = \mathrm{c}_N\left(\frac{N}{2} + n\right) \qquad (9)$$

Ramanujan sums are integer valued; in fact, you can write the *discrete comb signal*[6] as

$$\mathrm{p}_N(n) := \sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{j}2\pi n\frac{k}{N}} = \sum_{d|N} \mathrm{c}_d(n),$$

and, using *Möbius inversion formula* [29], Ramanujan concludes that

$$\mathrm{c}_N(n) = \sum_{d|N} \mu\left(\frac{N}{d}\right) \mathrm{p}_d(n) = \sum_{d|n, d|N} \mu\left(\frac{N}{d}\right) d \qquad (10)$$

where μ denotes the *Möbius function*; see Fig. 5. Vaidyanathan *et al.* [22], have given an alternative proof of the integrality of Ramanujan sums. Clearly, for each N, $|\mathrm{c}_N(n)| \leq |\mathrm{c}_N(N)| = \varphi(N) \leq N - 1$; see Fig. 4. The max of $\mathrm{c}_N(n)$ is not

[3]No count $c : \mathbb{N} \to \mathbb{Q}$ preserves the standard (inherited from the reals), linear order of $\mathbb{Q}$.

[4]The order is immaterial when the series is absolutely summable.

[5]Actually, $\mathbb{Z}^*(N)$ is the multiplicative group [28] whose elements are the elements of the additive group $\mathbb{Z}(N) = \{0, 1, \ldots N-1\}$ that are relatively prime to N.

[6]The discrete comb signal, or pin train, of period N and average 1, is given by $\mathrm{p}_N(kN) = N$ whenever its argument is an integer multiple of N, and $\mathrm{p}(n) = 0$ otherwise.

a monotonic function of N but, overall, it increases with N; in particular, for N prime, the max of $c_N(n)$ is $N-1$.

D. RAMANUJAN SINE SUMS

Ramanujan also introduced [20] the *sine sums*

$$s_N(n) = \sum_{k\in\mathbb{Z}^*(N)} j^{k-1} \sin\frac{2\pi kn}{N} \tag{11}$$

$$= \frac{1}{2}\sum_{k\in\mathbb{Z}^*(N)} ((-1)^k j^N - 1) j^k e^{j\frac{2\pi}{N}kn}. \tag{12}$$

For even N, the coprime k's are odd; then, j^{k-1} in Equation (11) is real, and so is $s_N(n)$. For N of the form $4m+2$, $m\in\mathbb{N}$, $s_{4m+2} = \mathbf{0}(n)$ is the zero signal since, then,

$$s_N(n) = \sum_{k\in\mathbb{Z}^*(N)} j^{k-1}\sin\frac{2\pi kn}{N}$$

$$= \sum_{k\in\mathbb{Z}^*(N), k<\frac{N}{2}} j^{k-1}\sin\frac{2\pi kn}{N} + (-1)^k j^{k-1}\sin\frac{2\pi kn}{N}$$

is zero (the k's are odd.) The case Ramanujan was interested is $N = 4\,m$; then

$$s_{4\,m}(n) = \sum_{k\in\mathbb{Z}^*(4\,m)} (j^{-k} - j^k) j^{\frac{n}{m}k},$$

is integer-valued for $n\in\mathbb{Z}$, since we know that for even N, $s_{4\,m}(n)$ is real and, in addition, $|(j^{-k}-j^k)j^{\frac{n}{m}k}|$ takes only the values 0 and 2. In particular,

$$s_4(n) = \overline{0, 2, 0, -2}\,,$$

$$s_8(n) = \overline{0, 0, 4, 0, 0, 0, -4, 0}\,,$$

$$s_{12}(n) = \overline{0, 2, 0, 4, 0, 2, 0, -2, 0, -4, 0, -2} \text{ and,}$$

$$s_{32}(n) = \overline{0, 0, 0, 0, 0, 0, 0, 0, 16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,}$$
$$\overline{0, 0, 0, 0, 0, -16, 0, 0, 0, 0, 0, 0, 0}\,;$$

see Fig. 6.

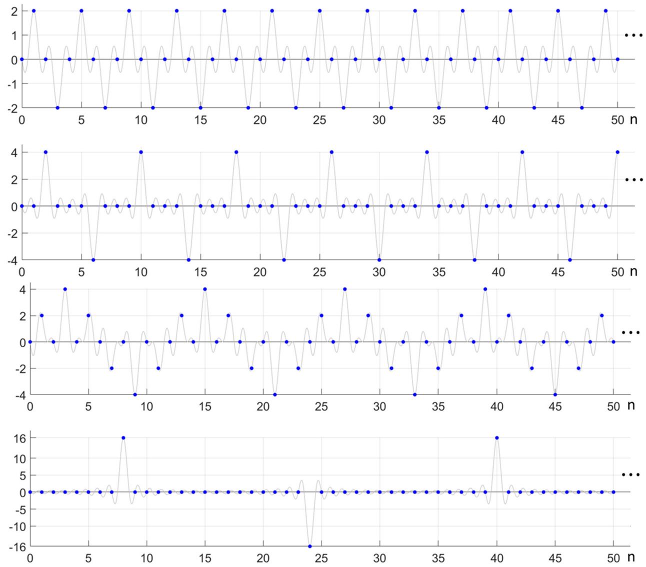

FIGURE 6. Ramanujan's s-sums $s_4(n)$, $s_8(n)$, $s_{12}(n)$ and $s_{32}(n)$.

E. AVERAGE VARIATION OF RAMANUJAN SUMS

We present a few results regarding the average variation of Ramanujan sums, together with their deductions. Being a sum of zero-average signals, for $N>1$, the average of each Ramanujan sum c_N is zero; or, from Equation (10), see [29], and the linearity of the average operator,

$$\text{av}[c_N(n)] = \sum_{d|N}\mu\left(\frac{N}{d}\right)\text{av}[p_d(n)] = \sum_{d|N}\mu(d) = 0;$$

A computation of the average variations of the first 10 Ramanujan sums gives 0, 4/2, 6/3, 8/4, 10/5, 8/6, 14/7, 16/8, 24/9 and 24/10. As a function of N, $\overline{\text{var}}(c_N(n))$ is a chaotic function, yet, it has some patterns that we explain below; see Fig. 7 and Table 1. From Equation (10), with N as in equation

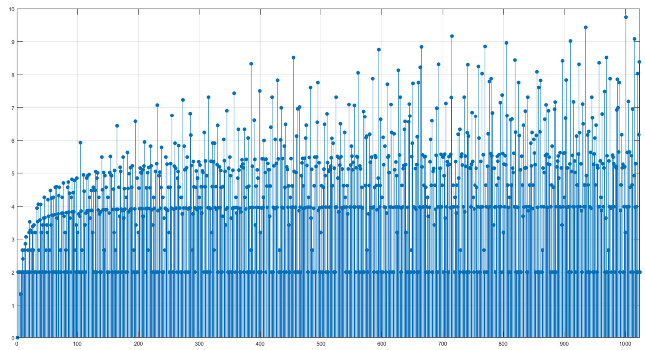

FIGURE 7. Stem plot of the average variation of Ramanujan sums c_N, for $N \in /0, 1000/$.

TABLE I Average Variation of Ramanujan Sums $c_N(n)$, According to Several Possible Factorizations of N

N	$\forall i\ \alpha_i = 1$	$\exists i\ \alpha_i > 1$
2^{α_1}	2	2
3^{α_1}	2	$8/3 \approx 2.67$
5^{α_1}	2	$16/5 = 3.2$
7^{α_1}	2	$24/7 \approx 3.43$
$2^{\alpha_1}3^{\alpha_2}$	6/3=2	8/3≈ 2.67
$2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}$	96/30=3.2	$64/15 \approx 4.27$
$2^{\alpha_1}5^{\alpha_2}$	24/10=2.4	16/5=3.2
$3^{\alpha_1}5^{\alpha_2}$	46/15≈ 3.07	64/15≈ 4.27
$2^{\alpha_1}5^{\alpha_2}7^{\alpha_3}$	320/70≈ 4.57	192/35≈ 5.48
$3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}$	622/105≈ 5.92	2304/315≈ 7.31
$2^{\alpha_1}7^{\alpha_2}$	40/14≈ 2.86	24/7≈ 3.43
$3^{\alpha_1}7^{\alpha_2}$	74/21≈ 3.52	32/7≈ 4.57
$5^{\alpha_1}7^{\alpha_2}$	142/35≈ 4.06	192/35≈ 5.48
$2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}7^{\alpha_4}$	1248/210≈ 5.94	256/35≈ 7.31

Observe the constant subsequences of $\overline{\text{var}}(c_N)$, in Fig. 7

(4), you have

$$\mathrm{c}_N(n) = \mathrm{p}_N(n) - \sum_{i \in /1,R/} \mathrm{p}_{\frac{N}{p_i}}(n) + \sum_{i,j \in /1,R/, i \neq j} \mathrm{p}_{\frac{N}{p_i p_j}}(n) + \sum_{i,j,k \in /1,R/, i\neq j, i \neq k, j \neq k} \mathrm{p}_{\frac{N}{p_i p_j p_k}}(n) + \cdots + (-1)^R \mathrm{p}_1(n) \tag{13}$$

where $R = \omega(N)$ is the number of primes that divide N. From definitions,

$$\overline{\mathrm{var}}(\mathrm{c}_N) = \frac{1}{N}\sum_{n=1}^{N} \Big| \sum_{k \in \mathbb{Z}^*(N)} e^{j\frac{2\pi}{N}k(n+1)} - \sum_{k\in\mathbb{Z}^*(N)} e^{j\frac{2\pi}{N}kn} \Big| = \frac{1}{N}\sum_{n=1}^{N} \Big| \sum_{k\in\mathbb{Z}^*(N)} (e^{j\frac{2\pi}{N}k} - 1) e^{j\frac{2\pi}{N}kn} \Big|. \tag{14}$$

The average variation of c_1 is zero. For p prime,

$$\mathrm{c}_p(n) = \mathrm{p}_p(n) - \mathrm{p}_1(n) = \overline{p-1, -1, \ldots -1}$$

and thus $\overline{\mathrm{var}}(\mathrm{c}_p) = \frac{2p}{p} = 2$.

We say that a signal $\{s_n\}$ is *sparse* whenever every non-zero term is surrounded by zero terms, i.e. $\forall n,\ s_n s_{n-1} = 0$. The *interpolation operator* [23] is useful for the computation of the average variation Ramanujan sums. The degree-L interpolation operator Υ_L, $L \in \mathbb{Z}^+$, when applied to a signal x, produces the signal

$$y_n := [\Upsilon_L(x)]_n = L\, x_{\frac{n}{L}}$$

whenever $\frac{n}{L} \in \mathbb{Z}$, and $y_n = 0$ otherwise. The interpolation operator preserves the average of the signal and, as we show below, in some cases, the average variation of the signal as well. Also, for $L > 1$, the interpolated version of a signal is always sparse. Lastly, note that the average variation of a sparse signal and of an interpolated version is the same.

If the prime p divides N then, $p^2 | pN$, $\mu(p^2) = 0$ and

$$\mathrm{c}_{pN}(n) = [\Upsilon_p(\mathrm{c}_N)](n); \tag{15}$$

thus, for example,

$$\mathrm{c}_{12}(n) = \overline{4, 0, 2, 0, -2, 0, -4, 0, -2, 0, 2, 0} = [\Upsilon_2(\mathrm{c}_6)](n) = [\Upsilon_2(\overline{2, 1, -1, -2, -1, 1})](n)$$

If a Ramanujan sum is so interpolated twice, the second time, its average variation does not change. In particular, for $N = 2^\alpha$, the average variation is 2, since

$$\mathrm{c}_{2^\alpha}(n) = \mathrm{p}_{2^\alpha}(n) - \mathrm{p}_{2^{\alpha-1}}(n) = \overline{2^{\alpha-1}, 0, \ldots 0, -2^{\alpha-1}, 0, \ldots, 0}.$$

Similarly, for $N = p^\alpha$ a power of the prime p,

$$\mathrm{c}_{p^\alpha}(n) = \mathrm{p}_{p^\alpha}(n) - \mathrm{p}_{p^{\alpha-1}}(n) = \overline{p^\alpha - p^{\alpha-1}, 0, \ldots 0, -p^{\alpha-1}, 0, \ldots 0, -p^{\alpha-1}, 0, \ldots, 0}$$

and the average variation is

$$\overline{\mathrm{var}}(\mathrm{c}_{p^\alpha}) = \frac{1}{p^\alpha}(2(p^\alpha - p^{\alpha-1}) + 2(p-1)p^{\alpha-1}) = 4(1 - p^{-1});$$

for example, the number 3 being prime, $\mathrm{c}_3(n)$ has average variation 2; the average variation of $\mathrm{c}_{3^2}(n)$ is $\frac{8}{3}$; therefore, for $\alpha \geq 2$, the average variation of $\mathrm{c}_{3^\alpha}(n)$ is $\frac{8}{3}$; see Table 1.

More generally, with $p_1 < \ldots < p_R$ a fixed collection of primes, for all N's of the form $N = p_1^{\alpha_1} \ldots p_R^{\alpha_R}$, with $\alpha_i \geq 1$, and at least one $\alpha_i \geq 2$, the average variation of the corresponding Ramanujan sums $c_N(n)$ is the same; the reasons being that firstly, the original sum was "sparse" and secondly, that when you multiply one such N times p_i, $i \in /1, R/$, the resulting sum $\mathrm{c}_{p_i N}(n)$ is the interpolated version $[\Upsilon_{p_i}(\mathrm{c}_N)](n)$ $\mathrm{c}_N(n)$. of interpolated to $[\Upsilon_{p_i}(\mathrm{c}_N)](n)$, $i \in /1, R/$. For example, $\overline{\mathrm{var}}(\mathrm{c}_{12}) = \overline{\mathrm{var}}(\mathrm{c}_{18}) = \overline{\mathrm{var}}(\mathrm{c}_{36}) = \ldots$; see Table 1.

F. SERIES OF RAMANUJAN SUMS

Equations (6.1) and (7.2) of Ramanujan's paper [20] read

$$\sum_{N=1}^{\infty} \frac{1}{N^{s+1}} \mathrm{c}_N(n) = \frac{\sigma_s(n)}{n^s \zeta(s+1)} = \frac{\sigma_{-s}(n)}{\zeta(s+1)},\ s \geq 0 \tag{16}$$

where $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is Riemann's zeta function, and $\sigma_s(n)$ is the sum $\sum_{d|n} d^s$ of the s^{th} power of the divisors of n.[7] Thus, for $s = 1$, he deduces that, for $n \geq 1$,

$$\sum_{N=1}^{\infty} \frac{1}{N^2} \mathrm{c}_N(n) = \frac{6}{\pi^2} \frac{\sigma(n)}{n}; \tag{17}$$

the signal in this equation is a series of periodic complex exponentials that is neither almost periodic nor bounded.[8] For each N prime, $|\frac{1}{N}\mathrm{c}_N(n)|$ periodically takes values arbitrarily close to 1; still, since $\frac{1}{\zeta(s+1)} = \sum_{N=1}^{\infty} \frac{\mu(N)}{N^{s+1}}$ and for $s = 0$, $\sum_{N=1}^{\infty} \frac{\mu(N)}{N} = 0$, Ramanujan deduces from Equation (16) that

$$\sum_{N=1}^{\infty} \frac{1}{N} \mathrm{c}_N(n) = \mathbf{0}(n),\ n \geq 1, \tag{18}$$

is the zero signal $\mathbb{Z}^+ \to \mathbb{C}$; see Fig. 8 for a truncated version of the series, where it is apparent that the convergence is not uniform.

As a number theorist, Ramanujan considered the sums $\mathrm{c}_N(n)$ for $n \in \mathbb{Z}^+$; notice that, for $n = 0$, you have

$$\sum_{k=1}^{\infty} \frac{1}{k} \mathrm{c}_k(0) = \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty;$$

thus, as a signal $\mathbb{Z} \to \mathbb{C}^*$,

$$\sum_{k=1}^{\infty} \frac{1}{k} \mathrm{c}_k(n) = \infty \delta(n),$$

[7] It is not hard to show that $\sigma_{-s}(n) = \frac{\sigma_s(n)}{n^s}$.

[8] $\rho(n) = \frac{\sigma(n)}{n}$ is known as the *relative sum-of-divisors function*. The Riemann Hypothesis is equivalent to $\frac{\sigma(n)}{n} < e^\gamma \log\log(n)$, for $n > 5040$ [35].

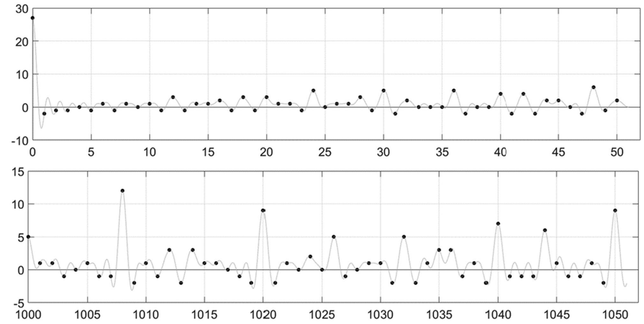

FIGURE 8. **Truncated Ramanujan-Fourier series $\sum_{N=1}^{30} \frac{1}{N} c_N(n)$, plotted for $n \in /0, 50/$, and $n \in /1000, 1050/$.**

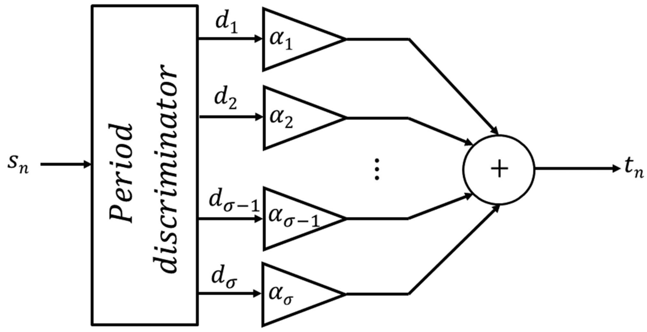

FIGURE 9. **The N-periodic signal s is *filtered by period*. For each of the σ divisors d of N, the i-th output of the *period discriminator* is the d-periodic signal s^d (see Equation (21)). Each period component is multiplied by a coefficient α_d, before addition.**

where δ is Kronecker's delta sequence.

Ramanujan sums can be extended in several ways; for example, in Equation (8), you might allow for $n \in \mathbb{N}$, or $n \in \mathbb{Z}$; if $n \in \mathbb{R}$ then you get a continuous signal [27]; $n \in \mathbb{C}$, if you are interested in analytic functions of the complex variable.

IV. PERIOD-FREQUENCY SPECTRUM AND FILTERING

With the exception of Subsection IV-D and IV-E, this Section, as well as the following one, deal with periodic signals. Section VI deals with (infinite) series of periodic signals. We introduce the notion of the (unique) representation of an N-periodic signal as the superposition of its d-periodic components ($d|N$); if the N-periodic component is not null, we say that the signal is strongly periodic, otherwise it is said to be weakly periodic. The decomposition according to period allows for the filtering of the signal according to period; these notions are extended to the case of signals that are series of periodic signals. Under the assumption that the each d-periodic component of an N-periodic signal arises from a different source, the filtering according to period of the signal can either enhance or deplete the contribution of each source. A period-frequency matrix allows for the explicit representation of periodic signals, as well as series of periodic signals, in the period-frequency domain.

If s is N-periodic and $X = \mathrm{DFT}[s_0, \ldots s_{N-1}]$, with $X_k = \frac{1}{N}\sum_{n=0}^{N-1} s_n e^{-j\frac{2\pi}{N}nk}$, for $k \in /0, N-1/$, then

$$s_n = \sum_{k=0}^{N-1} X_k e^{j2\pi\frac{k}{N}n} = \sum_{k=1}^{N} X_k e^{j2\pi\frac{k}{N}n},\ n \in \mathbb{Z} \tag{19}$$

uniquely expresses s as a linear combination of complex exponentials, of Fourier frequencies $\frac{2\pi}{N}k$. With $X_N := X_0$, it can be reorganized as

$$s_n = \sum_{d|N} \sum_{(k,N)=d} X_k e^{j2\pi\frac{k}{N}n} = \sum_{d|N} \sum_{(k,N)=d} X_k e^{j2\pi\frac{k/d}{N/d}n}, \tag{20}$$

where $(k, N) := \gcd(k, N)$. With $d' = \frac{N}{d}$, call each component

$$s_n^{d'} := \sum_{k:(k,N)=d} X_k e^{j2\pi\frac{k/d}{d'}n} \tag{21}$$

the *d'-periodic component* of s; you can write then,

$$s_n = \sum_{d|N} s_n^{d'} = \sum_{d|N} s_n^d \tag{22}$$

This decomposition into signals of period d, $d|N$, is unique, except for the order of the terms.

The average of s is $X_N = X_0$. For the remaining divisors d of N, the average power associated with the d-periodic component is

$$S_d^{\Pi} := \sum_{r<d\,:\,(r,d)=1} |X_{\frac{N}{d}r}|^2 = \sum_{r<d\,:\,(r,d)=1} |X_{d'r}|^2. \tag{23}$$

A. FILTERING PERIODIC SIGNALS ACCORDING TO PERIOD

If each of the periodic components

$$s_n^d = \sum_{r<d:(r,d)=1} X_{rd'}\, e^{j2\pi\frac{r}{d}n} \tag{24}$$

of signal s is multiplied times a given factor α_d, you get the *filtered-by-period* signal

$$t_n = \sum_{d|N} \alpha_d\, s_n^d = \sum_{d|N} \alpha_d \sum_{r:(r,d)=1} X_{rd'}\, e^{j2\pi\frac{r}{d}n}, \tag{25}$$

as in Fig. 9.

B. PERIOD SUPPORT OF A PERIODIC SIGNAL

Writing a periodic signal s as

$$s_n = \sum_{r=0}^{R} b_r e^{j2\pi q_r n},\ q_r \in \hat{\mathbb{Q}}, \tag{26}$$

its average power is given by

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} s_n s_n^* \\ &= \sum_{r=0}^{R}\sum_{s=0}^{R} b_r b_s^* \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^{j2\pi(q_r-q_s)n} \\ &= \sum_{r=0}^{R}\sum_{s=0}^{R} b_r b_s^* \delta_{r-s} = \sum_{r=0}^{R} b_r b_r^*. \end{aligned} \tag{27}$$

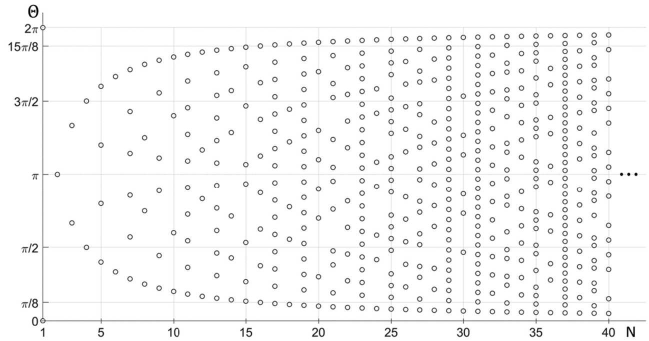

FIGURE 10. The period-frequency matrix; the vertical axis is for frequency and the horizontal axis is for period. To each $q \in \hat{\mathbb{Q}}$ there corresponds the point $(\eta(q), 2\pi q)$ of the matrix. The *Farey ordering* is given here by running over the points, from left to right, upwards on each column. The number of points on the N-th column is $\varphi(N)$.

Let the *period support* of the signal be the set of the periods of the complex exponentials a periodic signal is a linear combination of. For the signal in Equation (19), you have

$$\mathbf{\Pi} := \{d|N : S_d^{\mathbf{\Pi}} \neq 0\}. \tag{28}$$

(see Equation (23)) and, for the signal in Equation (26),

$$\mathbf{\Pi} := \{\eta(q_r) : b_r \neq 0, r \in /0, R/\}, \tag{29}$$

where η is the denominator function, of Equation (7).

As shown in Lemma 9 below, the period of a periodic signal is the least common multiple of its period support, i.e. $\mathrm{lcm}(\mathbf{\Pi})$.

C. PERIOD-FREQUENCY SUPPORT OF A SERIES OF PERIODIC SIGNALS

Consider series of periodic signals of the form

$$s_n = \sum_{q \in \hat{\mathbb{Q}}} b_q e^{j2\pi qn} = \lim_{k \to \infty} \sum_{q \in F_k^*} b_q e^{j2\pi qn} \tag{30}$$

where the order of the terms in the series is the Farey order of Equation (6), and $F_k^* := F_k - \{1\}$. The spectrum of signals that are either sums or series of periodic complex exponentials can be organized on the basis of a *period-frequency matrix*, so that the frequencies and periods of the periodic complex exponentials that compose a signal are organized in an explicit way. Let the *period-frequency matrix* be given by the set

$$\left\{\left(N, 2\pi \frac{M}{N}\right) : N, M \in \mathbb{Z}^+, M \leq N, \gcd(M, N) = 1\right\}. \tag{31}$$

The period-frequency matrix is a subset of the Cartesian product $\mathbb{Z}^+ \times \hat{\mathbf{\Theta}}$, as shown in Fig. 10. The period-frequency matrix is the 90°-rotated graph of the inverse of the $\mathbb{Q}$-restricted Thomae's function τ of Fig. 3. It is an infinite version of the *Farey array* considered in [24]. For each component $b_q e^{j2\pi qn}$ of a periodic signal, or a series of periodic signals, if the limit exchange of Equation (32) below is valid, there corresponds the average power $|b_q|^2$ and the point $(\eta(q), 2\pi q)$ of the period-frequency matrix. The distribution of the average power in the combined domain of period and Fourier frequency is useful information; notice how any convolution filter with transfer function $H(\theta)$ which is nonzero over an interval, will let pass complex exponentials of arbitrarily large periods. With a low-pass filter you may block out exponentials of small periods; with a sharp cutoff frequency θ_c, you block out periods smaller than or equal to $\lfloor 2\pi\theta_c^{-1} \rfloor$.

The average power $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} s_n s_n^*$ of the signal in Equation (30) cannot be computed always with a Parseval relation, unless the exchange of limits

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lim_{k\to\infty} \sum_{q_r \in F_k^*} b_{q_r} e^{j2\pi q_r n} \sum_{q_s \in F_k^*} b_{q_s}^* e^{-j2\pi q_s n} \\ &= \lim_{k\to\infty} \sum_{q_r \in F_k^*} \sum_{q_s \in F_k^*} b_{q_r} b_{q_s}^* \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^{j2\pi(q_r - q_s)n} \end{aligned} \tag{32}$$

is valid, in which case, the average power is given by $\lim_{k\to\infty} \sum_{q_r \in F_k^*} |b_{q_r}|^2$. This is made more precise with *Carmichael's inner product*, in Section VI-C.

Let the *period-frequency spectrum* of signals of the form in Equation (30) (with either finite or infinite non-null terms or not) be defined on the period-frequency matrix, and be given by

$$\Gamma(N, \theta) := \Gamma(\eta(q), 2\pi q) = |b_q|^2.$$

Correspondingly, let the *period-frequency spectrum support* be given by

$$\Xi := \{(\eta(q), 2\pi q) : b_q \neq 0\};$$

it is a subset of the period-frequency matrix. The period-frequency spectrum support of each Ramanujan sum $c_N(n)$ is precisely the column of abscissa N of the period-frequency matrix.

D. THE DFDT: AN INTRODUCTION

Equation (30) can also be written as

$$s_n = \sum_{k=0}^{\infty} b_{q_k} e^{j2\pi q_k n} = \sum_{k=0}^{\infty} b'_k e^{j2\pi q_k n} \tag{33}$$

where $\{q_k\} = \{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5} \ldots\}$ is the Farey ordering of $\hat{\mathbb{Q}}$, and $b'_k := b_{q_k}$. For s periodic, b'_k eventually becomes 0. In Section VI-D, $\{b_q : q \in \hat{\mathbb{Q}}\}$, if the context makes it clear, $\{b'_k, k \in \mathbb{N}\}$, is termed the *discrete-frequency, discrete-time* (DFDT) Fourier transform of the signal s. The signals of Equations (16), (17) and (18) are signals of this type (33) and thus, the signals of Equations (17) and (18) are DFDT inverses of sequences $\{b_q\}$, with $b_0 = 0$ and, for $q > 0$, $b_q = \eta^{-1}(q)$, and $b_q = \eta^{-2}(q)$, respectively. Since $\sum_{k=1}^{\infty} \frac{1}{k} c_k(n) = \sum_{q \in \hat{\mathbb{Q}} - \{0\}} \tau(q) e^{j2\pi qn}$, the inverse DFDT of Thomae's signal is the signal $\infty\delta(n)$. You may also get almost periodic signals with Equation (30).

The DFDT is asymmetric in the sense that, in the time domain, you have two-sided signals $\{s_n : n \in \mathbb{Z}\}$ while, in the frequency domain, you have one-sided signals $\{b'_k : k \in \mathbb{N}\}$, unless you consider negative frequencies and the rationals in $(-\frac{1}{2}, \frac{1}{2}]$ instead of $\hat{\mathbb{Q}}$, and their corresponding ordering $\mathbb{N} \to \mathbb{Q} \cap (-\frac{1}{2}, \frac{1}{2}]$, as in $\{q'_0, q'_1, q'_2, q'_{-1}, q'_3, q'_{-2} \ldots\} = \{0, \frac{1}{2}, \frac{1}{3}, -(1-\frac{2}{3}), \frac{1}{4}, -(1-\frac{3}{4}), \frac{1}{5} \ldots\}$, resulting from the Farey ordering, and with $q' = -(1-q)$, whenever $q \in (\frac{1}{2}, 1)$.

E. FILTERING BY PERIOD SERIES OF PERIODIC COMPLEX EXPONENTIALS

Equation (20) expresses a periodic signal in terms of its d-periodic components, $d|N$. In the limit, as $N \to \infty$, you get that the signals of Equations (30) and (33) can be reorganized as series of d-periodic signals, $d \in \mathbb{Z}^+$

$$s_n = \sum_{d=1}^{\infty} \sum_{r<d:(r,d)=1} B_{r,d}\, \mathrm{e}^{\mathrm{j}2\pi \frac{r}{d} n}, \tag{34}$$

and, given a sequence $\{\alpha_d : d \in \mathbb{Z}^+\}$ of *coefficients*, you get the *filtered-by-period* signal

$$t_n = \sum_{d=1}^{\infty} \alpha_d \sum_{r<d:(r,d)=1} B_{r,d}\, \mathrm{e}^{\mathrm{j}2\pi \frac{r}{d} n}; \tag{35}$$

provided there is convergence. Thus, for example, with $\alpha_d = d$, the filtering of a signal of the form in Equation (16), with $s > 1$, gives the signal $\frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)}$.

V. PERIOD OF SUMS OF PERIODIC SIGNALS

The period of a sum of periodic signals may be smaller than the least common multiple of the periods of the summands (it can be as small as the least common multiple divided by the greatest common divisor), when they are weakly periodic. The sawtooth signals, as well as the pulse trains, are shown to be strongly periodic.

If the frequencies in of the spectrum support of a periodic signal are known with full accuracy, the period can be computed, as the following lemma indicates.

Lemma 9: The linear combination

$$s_n = \sum_{i=0}^{R-1} A_i \mathrm{e}^{\mathrm{j}2\pi \frac{M_i}{N_i} n}, n \in \mathbf{Z}, \tag{36}$$

of independent ($\frac{M_i}{N_i} \neq \frac{M_j}{N_j}$), periodic, complex exponentials, with $\gcd(M_i, N_i) = 1$, is periodic and its period is given by the least common multiple of $\{N_0, \ldots N_{R-1}\}$.

Proof: The addition of a constant signal to a periodic signal does not modify the period, therefore the Fourier frequency 0 is not considered. Let L be the least common multiple of the set of denominators $\{N_i : i \in /0, R-1/\}$, thus L is a repetition time of the linear combination $\{s_n\}$. Assume the period T is less than L, then

$\sum_{i=0}^{R-1} A_i e^{j2\pi \frac{M_i}{N_i}(n+T)} = \sum_{i=0}^{R-1} A_i e^{j2\pi \frac{M_i}{N_i} n}$; then $\sum_{i=0}^{R-1} A_i e^{j2\pi \frac{M_i}{N_i} T} e^{j2\pi \frac{M_i}{N_i} n} = \sum_{i=0}^{R-1} A_i e^{j2\pi \frac{M_i}{N_i} n}$, then $\sum_{i=0}^{R-1} A_i (e^{j2\pi \frac{M_i}{N_i} T} - 1) e^{j2\pi \frac{M_i}{N_i} n} = 0$;

not all coefficients $A_i(e^{j2\pi \frac{M_i}{N_i} T} - 1)$ in the linear combination above are zero, since L is less than the least common multiple of the N_i's; then the orthogonal set of signals $\{e^{j2\pi \frac{M_i}{N_i} n}\}$ is linearly dependent, a contradiction. ■

Corollary 2: The period of a periodic signal is given by the least common multiple $\mathrm{lcm}(\Pi)$ of its period support.

Corollary 3: The period-frequency support of a p^α-periodic signal, where p is a prime number and α is a positive integer, includes a point $(p^\alpha, 2\pi q)$ with first coordinate p^α.

A. SUMS OF PERIODIC SIGNALS

The sum of two periodic signals is periodic since the least common multiple of their periods is a positive repetition time of the sum.

Corollary 4: The period of a sum of periodic signals is given by the least common multiple of the periods of the complex exponentials in the summands that do not cancel out.

Proof: It follows from Lemma 9 and from the fact that the expansion of a periodic signal into a linear combination of complex exponentials is unique. ■

The period of a sum of periodic signals is a divisor of the least common multiple of the periods of the summands.

Corollary 5: If the period of the sum of two periodic signals is smaller than the least common multiple of the periods of the signals being added then the periods share at least one common lpp factor.

Proof: We prove the contrapositive. Assume the periods n and m of the signals being added do not share a common lpp factor; that is, with $n = \Pi p_i^{\alpha_i}$ and $m = \Pi p_i^{\beta_i}$, as in Equation (5), assume that whenever $\alpha_i \neq 0$ or $\beta_i \neq 0$, then $\alpha_i \neq \beta_i$. Then, since for each $\alpha_i > 0$, at least one complex exponential of period $p_i^{\alpha_i}$ composes one of the signals, and likewise, for each $\beta_i > 0$, at least one complex exponential of period $p_i^{\beta_i}$ is a term of the other signal, no cancellation of these complex exponentials of lpp factor periods is possible. Then the period of the sum is the least common multiple of m and n. ■

For example, since $12 = 2^2 \times 3$ and $18 = 2 \times 3^2$ have no lpp factor in common, the period of the sum of signals of periods 12 and 18 is always 36. On the other hand, the period of the sum of signals of periods $10 = 2 \times 5$ and $15 = 3 \times 5$ can be $6 = 2 \times 3$.

Corollary 6: [25] - If the signals r and s are periodic, of periods R and S, respectively, and if T is the period of $r + s$ then $\mathrm{lcm}(S, R) = \mathrm{lcm}(S, T) = \mathrm{lcm}(T, R)$.

Proof: $\mathrm{lcm}(S, R)$ is the product of the common lpp factors of R and S, times the lpp factors of R that are not lpp factors of S, times the lpp factors of S that are not lpp factors of R. If there are no cancellations of complex exponentials of periods that are common factors of R and S, then the corollary follows easily. If there are cancellations of complex exponentials of periods that are common lpp factors of S and R, and the period of $r+s$ is less than $\mathrm{lcm}(R,S)$, then, since the lpp factors missing in T are present in both R and S, the corollary is still true. ■

B. STRONG AND WEAK PERIODICITIES

For each frequency $2\pi\frac{k}{N}$ in the spectrum support Σ of an N-periodic signal, either $\gcd(N,k)=1$, or $\gcd(N,k)>1$; in the first case, the corresponding complex exponential $e^{\mathrm{j}\frac{2\pi}{N}kn}$ has period N and, in the second case, a period that is a proper divisor of N. Accordingly, we partition the spectrum support Σ of the N-periodic signal into two subsets,

$$\Sigma = \Sigma_S \cup \Sigma_W, \tag{37}$$

with $\Sigma_S := \{\frac{2\pi}{N}k : \gcd(N,k)=1\}$ and $\Sigma_W := \{\frac{2\pi}{N}k : \gcd(N,k)>1\}$, and call Σ_S the *strong support* of the signal and Σ_W the *weak support*. If $\Sigma_S \neq \emptyset$, the signal is said to be *strongly N-periodic*; otherwise, it is said to be *weakly N-periodic*, in which case, the least common multiple of the denominators of the fractions $\frac{k}{N}$ corresponding to the frequencies $\frac{2\pi}{N}k \in \Sigma_W$, when reduced, is N. Conversely, if $\Sigma_W=\emptyset$, or if the least common multiple of the denominators of the fractions $\frac{k}{N}$ corresponding to the frequencies in Σ_W, when reduced, is less than N, then the signal is strongly periodic: $\Sigma_S \neq \emptyset$.

The *periodization* $s=\overline{x}=\{\ldots x|x|x|\ldots\}$, of a length-$N$ segment x, has period N if and only if x itself is not a repeated concatenation. As in Equation (20), s can be written as

$$s_n = s_n^N + s_n'' \tag{38}$$

where

$$s_n^N := \sum_{k:\gcd(k,N)=1} X_k e^{j\frac{2\pi}{N}kn}, \tag{39}$$

$$s_n'' = \sum_{d|N}\ \sum_{k:\gcd(k,N)=d>1} X_k e^{j\frac{2\pi}{N}kn}. \tag{40}$$

s^N and s'' are said to be the *strongly periodic* and *weakly periodic components* of s, respectively.

The signal s can be weakly periodic only if at least two primes divide N; thus, if N is a power of a prime, $N=p^\alpha$, with p prime, s is strongly periodic. Such is the case of the 8-periodic signal $\cos n\pi + \cos n\frac{\pi}{2} + \cos n\frac{\pi}{4}$. All periods in the period support of a signal whose period is the power p^α of a prime p are powers of p.

Lemma 10: The period of a sum of strongly periodic signals, of periods N and M with $N \neq M$, is the least common multiple of N and M.

Proof: The complex exponentials of periods N and M present in the Fourier decomposition of the signals, do not cancel out. (For $N=M$, the strongly periodic components could cancel out.) ■

Lemma 11 below is used to prove Lemma 12.

Lemma 11: $\sum_{n=1}^{N-1} n\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}n} = \frac{N}{2}(-1+\mathrm{j}\cot\frac{2\pi}{2N})$

Proof: Note that the set of vertices

$$v_k = \sum_{n=0}^{k} \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k},\ k \in /0, N-1/ \tag{41}$$

determines a regular N-gon in the upper complex plane, $\Re z \geq 0$, with a vertex at the origin. It is not hard to show, by scaling and rotating the standard polygon with vertices $\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}$, $k \in /0, N-1/$, centered at the origin, that the polygon in (41) has center at

$$\frac{\mathrm{e}^{\mathrm{j}(\frac{\pi}{2}-\frac{2\pi}{2N})}}{|\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}}-1|}.$$

The average of the vertices gives the center of the polygon, therefore

$$\frac{1}{N}\sum_{k=0}^{N-1} v_k = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{k}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k} = \frac{\mathrm{e}^{\mathrm{j}(\frac{\pi}{2}-\frac{2\pi}{2N})}}{|\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}}-1|};$$

that is,

$$\frac{1}{N}\left(1+\sum_{n=1}^{N} n\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n}\right) = \frac{\mathrm{e}^{\mathrm{j}(\frac{\pi}{2}-\frac{2\pi}{2N})}}{|\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}}-1|};$$

thus,

$$\sum_{n=1}^{N-1} n\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n} = N\frac{\mathrm{e}^{\mathrm{j}(\frac{\pi}{2}-\frac{2\pi}{2N})}}{|\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}}-1|} - N$$

or

$$\begin{aligned}
\sum_{n=1}^{N-1} n\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n} &= N\left(\frac{\mathrm{j}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{2N}}}{\sqrt{(\cos\frac{2\pi}{N}-1)^2+\sin^2(\frac{2\pi}{N})}} - 1\right)\\
&= N\left(\frac{\mathrm{j}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{2N}}}{\sqrt{2}\sqrt{1-\cos\frac{2\pi}{5}}} - 1\right)\\
&= N\left(\frac{\sin\frac{2\pi}{2N}}{2|\sin(\frac{2\pi}{2N})|} - 1 + \mathrm{j}\frac{\cos\frac{2\pi}{2N}}{2|\sin(\frac{2\pi}{2N})|}\right)\\
&= N\left(-\frac{1}{2}+\frac{\mathrm{j}}{2}\cot\frac{2\pi}{2N}\right)
\end{aligned}$$

which is the lemma.[9] ■

You have, therefore,

Lemma 12: The signal $\overline{0, 1, \ldots N-1}$ is strongly periodic.

Proof: Since it holds in general that

$$\gcd(1,N) = \gcd(N-1,N) = 1,$$

[9]Incidentally, for $\omega \neq 0$, mod-2π, the series $\sum_{n=1}^{\infty} n\mathrm{e}^{\mathrm{j}\omega n}$ is *Abel summable* to $\frac{1}{2}\frac{1}{\cos\omega-1}$. In fact, since $\sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}$, $|z|<1$, the *Abel means* of the series are $A(r)=\sum_{n=1}^{\infty} n(r\mathrm{e}^{\mathrm{j}\omega})^n = \frac{r\mathrm{e}^{\mathrm{j}\omega}}{(1-r\mathrm{e}^{\mathrm{j}\omega})^2}$, for $r<1$, and, in the limit, as $r\to 1^-$, you get $A(1)=\frac{1}{2\cos\omega-2}=\frac{-1}{4}\sin^{-2}\frac{\omega}{2}$.

it is sufficient to prove that the coefficient

$$X_1 = \frac{1}{N}\sum_{n=0}^{N-1} n\, e^{-j\frac{2\pi}{N}n}$$

of the exponential $e^{j\frac{2\pi}{N}n}$, in the expansion $\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}nk}$ of the signal, is not null. In fact,

$$\Re(X_1) = \Re\left(\frac{1}{N}\sum_{n=0}^{N-1} n e^{-j\frac{2\pi}{N}n}\right) = \frac{1}{N}\sum_{n=0}^{N-1} n\cos\left(\frac{2\pi}{N}n\right) = -\frac{1}{2},$$

as the previous lemma says. ■

For example, the strongly periodic signal $\cos(\frac{2\pi}{12}n)n$ of period 12 plus the weakly periodic signal $\cos(\frac{2\pi}{5}n) - \cos(\frac{2\pi}{12})n$ of period 60, have a sum of period 5. Also, the signal $\overline{0,1,2} + \overline{0,1}$, of period 6, plus the signal $\overline{0,1,2,3,4} - \overline{0,1}$, of period 10, give the signal $\overline{0,1,2,3,4} + \overline{0,1,2}$, of period 15.

The method of the proof of Lemma 12 can be used in other cases, for example,

Lemma 13: The signals $\overline{0,\ldots 0,1}$ and, in general, the signals of the form $\overline{0,0,\ldots 0,1,\ldots 1}$ (i.e. *the pulse trains*) are strongly periodic.

Proof: In the first case, $X_1 = e^{j\frac{2\pi}{N}(N-1)} \neq 0$ and, in the second case, $X_1 = \sum_{n=M>0}^{N-1} e^{j\frac{2\pi}{N}n} \neq 0$. ■

C. THE CASE WHEN THE PERIOD OF A SUM OF PERIODIC SIGNALS IS LESS THAN THE LEAST COMMON MULTIPLE OF THE PERIODS OF THE SUMMANDS

Let s and t be periodic signals with period supports $\mathbf{\Pi}_s$ and $\mathbf{\Pi}_t$, respectively. Let $N = \mathrm{LCM}(\mathbf{\Pi}_s)$ and $M = \mathrm{LCM}(\mathbf{\Pi}_t)$ be the corresponding periods. Let $s' = s^N$ and $t' = t^M$ be the strongly periodic components, and $s'' = \sum_{d|N,d<N} s^d$ and $t'' = \sum_{d|M,d<M} t^d$ be the weakly periodic components of the signals, as in Equations (39) and (40). The period support $\mathbf{\Pi}_{s+t}$ of $s+t$ is not necessarily $\mathbf{\Pi}_s \cup \mathbf{\Pi}_t$, due to possible cancellation of complex exponentials; the smallest possible period support of $s+t$ is the symmetric difference $(\mathbf{\Pi}_s \cup \mathbf{\Pi}_t) - (\mathbf{\Pi}_s \cap \mathbf{\Pi}_t)$, since $\mathbf{\Pi}_s \cap \mathbf{\Pi}_t$ is the set of the periods of the complex exponentials, common to s and t. We are interested in the case $\mathrm{LCM}(\mathbf{\Pi}_{s+t}) < \mathrm{LCM}\{\mathrm{LCM}(\mathbf{\Pi}_s), \mathrm{LCM}(\mathbf{\Pi}_t)\}$.

Lemma 14: The smallest possible period of $s+t$ is $\frac{\mathrm{LCM}(N,M)}{\mathrm{GCD}(N,M)}$

Proof: When the periodic components of common periods cancel out. ■

Corollary 7: If $\mathrm{GCD}(M,N) = 1$, then the period of $s+t$ is $\mathrm{LCM}(M,N)$.

Proof: No cancellation of complex exponentials is possible and $\mathbf{\Pi}_{s+t} = \mathbf{\Pi}_s \cup \mathbf{\Pi}_t$. ■

Lemma 15: If $N = M$ and the period of $s+t$ is less than N then $s' = -t'$. This includes the case $s' = t' = 0$.

Proof: Otherwise, the sum is strongly periodic of period N. ■

Lemma 16: If $N \neq M$ and the period $\mathrm{LCM}(\mathbf{\Pi}_{s+t})$ of $s+t$ is less than $\mathrm{LCM}(N,M)$ then $s' = t' = \mathbf{0}$. (Equivalently, If $N \neq M$, and either s or t is strongly periodic, then the period of $s+t$ is the LCM of N and M.)

Proof: Without loss of generality, suppose $N > M$. If $s' \neq \mathbf{0}$ then $N \in \Pi_s$, and a complex exponential of period N composes s, which will not cancel out when added to t. Thus, no cancellation of common lpp factors is possible, and each lpp factor of N divides the period of the sum. Of course, no cancellation of non-common lpp factors is possible either. Each lpp factor of s and t divides the period of $s+t$, which is then the LCM of N and M. If $t' \neq \mathbf{0}$, and M-periodic components of s cancel out t', then $M|N$ and the period of the sum is still N. If t' is not cancelled out, the period of the sum is the LCM of N and M. ■

Corollary 8: If M is a proper divisor of N, and s is strongly periodic, then the period of $s+t$ is N.

If all summands in a finite sum of periodic signals of different periods are strongly periodic, the period of the sum is the least common multiple of the periods of the summands. The period of the sum of two strongly periodic signals of the same period may be smaller if, in addition to the strongly periodic components cancelling out, the weakly periodic components are not full or, by further cancellation of complex exponentials of equal frequencies, the least common multiple of the resulting period support is smaller.

Lemma 17: If a sum of two periodic signals has a period smaller than the least common multiple of the periods of the signals being added then it must be the case that the periods have at least one common lpp factor p^α

Proof: If signals s and r of periods $R = \prod p_i^{\alpha_i}$ and $S = \prod p_i^{\beta_i}$, respectively, have sum t of period $T = \prod p_i^{\gamma_i}$, where the α_i's, β_i's and γ_i's are nonnegative integers and the p_i's are the prime numbers in ascending order. By Corollary 6,

$$\prod p_i^{\max(\alpha_i,\beta_i)} = \prod p_i^{\max(\alpha_i,\gamma_i)} = \prod p_i^{\max(\beta_i,\gamma_i)}.$$

If

$$T = \prod p_i^{\gamma_i} < \prod p_i^{\max(\alpha_i,\beta_i)} = \mathrm{lcm}(R,S)$$

then, for some i, $\gamma_i < \max(\alpha_i,\beta_i)$ then, since $\max(\alpha_i,\beta_i) = \max(\alpha_i,\gamma_i) = \max(\beta_i,\gamma_i)$, it must be the case that $\gamma_i \le \alpha_i$ and $\gamma_i \le \beta_i$ and then $\alpha_i = \beta_i$. ■

D. EXAMPLES

The N-set $\{\overline{1,0,\ldots,0}, \overline{1,1,0,\ldots,0}, \overline{1,\ldots,1,0}, \overline{1}\}$ of N-periodic signals is linearly independent. Their sum is $\overline{N,1,2,\ldots N-1}$; this signal, as well as the signal $\overline{1,2,\ldots,N}$, are strongly periodic, as Lemma 18 and Corollary 9 show.

Lemma 18: The DFT of $y = [N, N-1, \ldots 1]$ is

$$Y_0 = \frac{N+1}{2};\ Y_k = \frac{1}{1 - e^{-j\frac{2\pi}{N}k}},\ k \in /1, N-1/.$$

Proof: The DFT of the N-point signal $[1,\ldots 1,0,\ldots 0]$, having M consecutive ones followed by $N-M$ consecutive zeros, is given by

$$X_0 = \frac{M}{N},$$

$$X_k = \frac{1}{N}\sum_{n=0}^{M-1} \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} = \frac{1}{N}\frac{1-\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}Mk}}{1-\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k}},$$

with $k \in /1, N-1/$ and $M \in /1, N/$. Thus, by the linearity of the DFT,

$$Y_0 = \sum_{M=1}^{N} \frac{M}{N} = \frac{N+1}{2},$$

and

$$Y_k = \sum_{M=1}^{N} X_k = \frac{1}{N}\sum_{M=1}^{N} \frac{1-\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}Mk}}{1-\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k}}$$

$$= \frac{1}{N}\frac{N-\sum_{M=1}^{N}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}Mk}}{1-\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k}}.$$

Now, since

$$\sum_{M=1}^{N} \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}Mk} = \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k}\sum_{M=0}^{N-1}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kM}$$

$$= \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k}\frac{1-\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}Nk}}{1-\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k}} = 0$$

then,

$$Y_k = \frac{1}{1-\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k}}.$$

■

The following lemma is a well known property of the DFT, whose proof we include for completeness.

Lemma 19: If the DFT of $[x_0, x_1, \dots x_{N-1}]$ is X_k, $k \in /0, N-1/$, then the DFT of $[x_N, x_{N-1}, \dots x_1]$ is $[X_N, X_{N-1}, \dots X_1]$, with the proviso that $x_N := x_0$ and $X_N := X_0$.

Proof: Since $X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}$, then

$$X_{N-k} = \frac{1}{N}\sum_{n=0}^{N-1} x_n \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{n=0}^{N-1} x_n \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(N-n)k}$$

or, changing the order of the terms,

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_{N-m}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}mk}.$$

■

Corollary 9: The DFT of $y = [1, 2, \dots N]$ is

$$Y_0 = \frac{N+1}{2},$$

$$Y_k = \frac{\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}}{1-\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}} = \frac{1}{\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k}-1}, \quad k \in /1, N-1/.$$

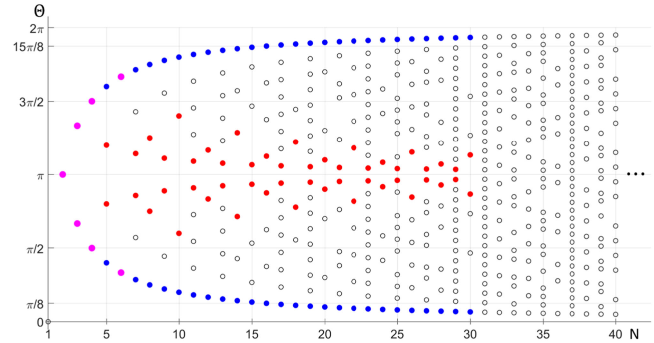

FIGURE 11. For periods $N \in /2, 30/$, the frequencies of the smoothest (blue), and the roughest (red), complex exponentials are indicated, in the period-frequency matrix. For $N = 2, 3, 4$, the roughest and smoothest complex exponentials coincide (purple).

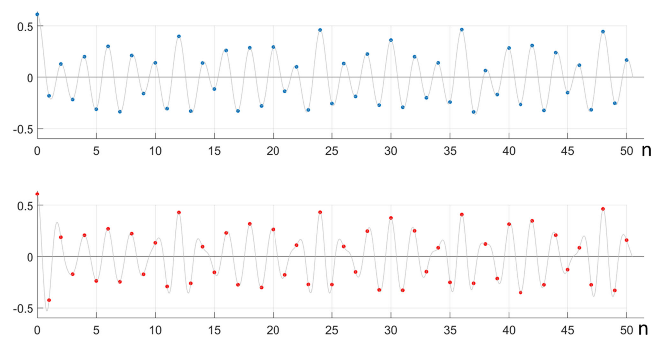

FIGURE 12. Time plots of the linear combinations, with weights $\frac{1}{N^2}$, of the smoothest (above), and roughest (below), cosines of periods $N \in /2, 30/$ (see Fig. 11). The averaged variations of the segments shown (over the interval $n \in /0, 50/$) are 0.5131, and 0.5136, respectively. For the interval /1000, 1050/ (not shown), the averaged variations are 0.5132 and 0.5113.

E. SEPARATE SUPERPOSITIONS OF ROUGH AND SMOOTH COMPLEX EXPONENTIALS

Each Ramanujan sum $c_N(n)$ adds the complex exponentials of period N, of both large and small average variation. The average variation is a subadditive functional; we consider briefly the signals that result from the separate addition of complex exponentials of small and large variation, as the period ranges as $N \in /N_1, N_2/$, with, say $N_1 = 2$ and $N_2 = 30$. Consider thus sums

$$s_n = \sum_{N=2}^{30} a_N \cos\left(2\pi\frac{k(N)}{N}n\right)$$

with $k(N)$ being either the integer relatively prime to N that makes the corresponding cosine smoothest (with $k(N) = 1$ or $k(N) = N-1$), or roughest ($k(N) \approx \frac{N}{2}$). The period-frequency support of $\{s_n\}$ is either *peripheral* (in the smooth case) or *central* (in the rough case); see Figs. 11, 12 and 13.

VI. THE DFDT TRANSFORM: AN EXTENSION OF THE RFT

Series of periodic complex exponentials, of which the Ramanujan-Fourier series are a particular case, provide the Signal Processing and the Time Series Communities with a theoretical source of nonperiodic signals that are "bigger" than those with a DTFT. Based on the notion of average, we

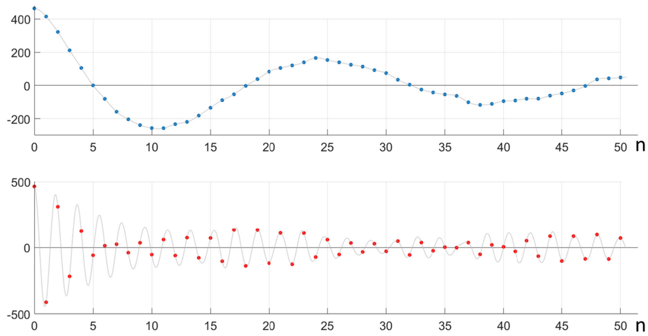

FIGURE 13. Linear combinations of the smoothest (above), and roughest (below) cosines of periods *N* from 2 through 30, with weights given by *N*. The averaged variations in the interval /0, 50/ are 32.497 and 168.664. For the interval /1000, 1050/ (not shown), the averaged variations are 15.969 and 54.636.

present a formalism and a corresponding space of signals, analogous to that used by Carmichael in the context of the Ramanujan-Fourier series.

Ramanujan [20] considered series

$$s_n = \sum_{N=1}^{\infty} \alpha_N \mathrm{c}_N(n) \tag{42}$$

of Ramanujan sums. Each such *Ramanujan series*, also known as a *Ramanujan-Fourier series*, is a series of periodic complex exponentials where the weight of the complex exponentials of each period is the same. The period of $\mathrm{c}_N(n)$ is N, yet, a large period is not a synonym of smooth, as the average variation can be large; see Fig. 7. Ramanujan series were initially proposed as *arithmetic functions*, i.e. meaningful sequences for number theory [26]. In signal processing, it is advantageous to consider $n \in \mathbb{Z}$, $n \in \mathbb{R}$ and $n \in \mathbb{C}$ as well. Planat [27] has considered Ramanujan series for the modelling of low-frequency, $1/f$ noise.

A. THE RAMANUJAN FOURIER TRANSFORM

With Equation (42), the set $\{\mathrm{c}_N(n) : N \geq 1\}$ of the Ramanujan sums becomes a basis for a space of signals, in what is known as the *Ramanujan Fourier Transform* (RFT), $\{s_n\} \mapsto \{\alpha_N\}$, [32], [22]. The RFT assigns to the signal s of Equation (42), the sequence $\{\alpha_N\}$ of the corresponding coefficients; Equation (42) gives the *inverse Ramanujan Fourier transform* of the signal $\{\alpha_N\}$, $n \geq 1$.

Both the DFDT and the RFT assign (one-sided) discrete signals to (two-sided) discrete signals. Thus, for example, the inverse RFT of the signal $\{\frac{1}{N^2} : n \geq 1\}$ is the signal $\frac{6}{\pi^2}\frac{\sigma(n)}{n}$, and the inverse RFT of the signal $\{\frac{1}{N} : N \geq 1\}$ is the zero signal $\mathbf{0}(n)$.

Given the signal $\{s_n\}$, the left hand side of Equation (42), the matter of how to compute its RFT, the sequence of coefficients $\{\alpha_N\}$, arises. This is achieved with large generality using a formula proposed by Carmichael in 1932 [32]. The formula is the limit of an average and can be interpreted as an *inner product* of sorts. We use a similar approach for the computation of the DFDT.

In Carmichael's deduction there is a limit exchange that is not always valid, as he points out, and the formula fails to cover certain cases of Ramanujan-Fourier series, such as the case of $\alpha_N = \frac{1}{N}$. More study is necessary regarding the domain and the kernel of the inverse RFT.

B. SIGNALS OF BOUNDED AVERAGE, OF BOUNDED AVERAGE MAGNITUDE, AND OF BOUNDED AVERAGE SQUARE MAGNITUDE

For brevity, in this Section, we consider sequences $\mathbb{N} \to \mathbb{C}$ that start with the subindex 0. Let the magnitude $|s|$ of a signal $s = \{s_n\}$ be given its componentwise magnitude, $|s| = \{|s_n|\}$, and, similarly, let the square of the signal be given by its componentwise square: $s^2 = \{s_n^2\}$.

The *average* (or arithmetic mean) of a one-sided sequence $\{s_n, n \in \mathbb{N}\}$ is given by

$$\overline{s} = \lim_{m\to\infty} \frac{1}{m} \sum_{n=0}^{m-1} s_n.$$

Note that, if $s_n \to L$, then $\overline{s} = L$; also, if $s \in l_\infty$, then $|\overline{s}| \leq ||s||_\infty$. The average square magnitude of the signal s is denoted

$$\overline{|s^2|} = \lim_{m\to\infty} \frac{1}{m} \sum_{n=0}^{m-1} |s_n^2|.$$

There are bounded signals with no average. Consider for example the signal $\{\underline{1}, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, \ldots\}$; see [34]. [19] is a good source for spaces of discrete signals.

Denote as $\mathbf{a}$ the linear space of the signals with a finite average, the space of the signals with a finite average magnitude as $\mathbf{b}_1$, and the space of the signals with a finite average square magnitude as $\mathbf{b}_2$. Also, denote the space of the signals having zero average as $\mathbf{a}_0$, the space of the signals having zero average magnitude as $\mathbf{b}_{10}$, and the space of the signals having zero average square magnitude as $\mathbf{b}_{20}$. Note that each summable signal has a zero average; also, that each signal summable in magnitude has zero average magnitude, i.e. $l_1 \subset \mathbf{b}_{10}$, since, for $s \in l_1$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=0}^{m-1} |s_n| \leq \lim_{m\to\infty} \frac{1}{m} \sum_{n=0}^{\infty} |s_n| = 0;$$

and, likewise, that $l_2 \subset \mathbf{b}_{20}$.

Note that $l_1 \neq \mathbf{b}_{10}$ since the signal $\{\frac{1}{n}, n \geq 1\}$ of Example VI-B1 belongs to $\mathbf{b}_{10} - l_1$. Also, remember that $l_1 \subset l_2$[10] since, for $s \in l_1$,

$$\begin{aligned}\sum_{n=0}^{\infty} |s_n|^2 &= \sum_{n:|s_n|\geq 1} |s_n|^2 + \sum_{n:|s_n|<1} |s_n|^2 \\ &\leq \sum_{n:|s_n|\geq 1} |s_n|^2 + \sum_{n:|s_n|<1} |s_n|,\end{aligned}$$

[10] In general, whenever $0 < p_1 \leq p_2$, $l_{p_1} \subset l_{p_2}$.

the first sum being finite, and the second being smaller than its magnitude sum. The signal of Example VI-B1 is in $l_2 - l_1$, and in $\mathbf{b}_{20}$.

1) EXAMPLE

The z-transform of $\frac{1}{n}\mathrm{u}_{n-1}$ is $-\log(1-z^{-1}),\ |z|>1$; also, its average is zero since

$$\frac{1}{N}\sum_{n=2}^{N}\frac{1}{n} \leq \frac{1}{N-1}\int_1^N \frac{1}{x}\mathrm{d}x = \frac{1}{N-1}\ln(N)$$

tends to zero, as N tends to infinity, and so,

$$\frac{1}{N}\sum_{n=1}^{N}\frac{1}{n} = \frac{1}{N} + \frac{1}{N}\sum_{n=2}^{N}\frac{1}{n}$$

tends to zero, as well. This is also an example of a signal not in l_1 but having a DTFT (except at $\theta = 0$):

$$S(\theta) = \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{e}^{-\mathrm{j}\theta n} = -\log(1-\mathrm{e}^{-\mathrm{j}\theta}),\ \theta \neq 0;$$

$|S(\theta)|$ is an unbound, convex function for $\theta \in (0, 2\pi)$, with a minimum at $\theta = \pi$ of $|S(\pi)| = \ln(2)$.

2) EXAMPLE

In [33], Hecke mentions the formula

$$\lim_{r\to 1^-}(1-r)\sum_{m=1}^{\infty}c_m r^m = \lim_{t\to\infty}\frac{1}{t}\sum_{m=1}^{t}c_m$$

for the arithmetic mean of the coefficients of a power series; we provide a proof of this. From a hint for an exercise in [28], note that

$$(1-r)\sum_{m=1}^{N}c_m r^m = \sum_{m=1}^{N}a_m r^m - c_{N+1}r^{N+1}$$

where $c_m = \sum_{k=1}^{m}a_k$, with $a_1 = c_1$, and $a_m = c_k - c_{k-1}$ for $m \geq 2$. Then, provided that $\lim_{r\to 1^-}(\lim_{N\to\infty}\{c_N r^N\}) = 0$,

$$\lim_{r\to 1^-}(1-r)\sum_{m=1}^{\infty}c_m r_m = \lim_{r\to 1^-}\sum_{m=1}^{\infty}a_m r^m = \sum_{m=1}^{\infty}a_m,$$

provided that $\{a_m\}$ is summable. Now, since

$$\sigma_N = \frac{1}{N}\sum_{n=1}^{N}c_n,$$

as $N\to\infty$, is the Cesaro sumvv of the series $\sum_{m=1}^{\infty}a_m$, in this sense,

$$\lim_{r\to 1^-}(1-r)\sum_{m=1}^{\infty}c_m r_m = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}c_n$$

3) EXAMPLE

Let $\{a_n\}$, $n \geq 0$ be a sequence with z-transform $A(z)$, $z \in \mathrm{ROC_a}$. The z-transform of $\{\mathrm{u}_n\} * \{a_n\}$ is $\frac{z}{z-1}A(z)$, on the intersection of the regions of convergence, and the z-transform of $\frac{1}{n+1}\mathrm{u}_{n+1}$ is

$$-z\log(1-z^{-1}),\ |z|>1.$$

Note that the n-th element of

$$\{b_n\} := \left\{\frac{\mathrm{u}_n}{n+1}\right\}.\{\mathrm{u}_n * a_n\}$$

is

$$b_n = \frac{1}{n+1}\sum_{k=0}^{n}a_k$$

and that, if $B(z)$ is the z-transform of b, by a property of the z-transform,

$$\overline{a} = \lim_{n\to\infty}b_n = \lim_{z\to 1}(z-1)B(z).$$

C. CARMICHAEL'S INNER PRODUCT

Consider the bilinear map $\mathbf{b}_2 \times \mathbf{b}_2 \to \mathbb{C}$ given by

$$\langle x, y\rangle := \lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1}x_n y_n^*; \tag{43}$$

for signals $t \in \mathbf{b}_2$, with $\langle t, t\rangle \neq 0$, you have Cauchy-Schwartz inequality since, for $s \in \mathbf{b}_2$,

$$0 \leq \langle s - \frac{\langle s,t\rangle}{\langle t,t\rangle}t, s - \frac{\langle s,t\rangle}{\langle t,t\rangle}t\rangle = \langle s,s\rangle - \frac{|\langle s,t\rangle|^2}{\langle t,t\rangle},$$

or,

$$|\langle s,t\rangle|^2 \leq \langle s,s\rangle\langle t,t\rangle. \tag{44}$$

This map $\langle\ ,\ \rangle$ is not quite an inner product due to the fact that there are nonzero signals $s \in \mathbf{b}_2$ for which $\langle s,s\rangle = 0$. By considering the space $\frac{\mathbf{b}_2}{\mathbf{b}_{20}}$ of equivalence classes $\overline{s}$ of signals in $\mathbf{b}_2$, modulo signals in $\mathbf{b}_{20}$, you get what we may term *Carmichel's inner product* given by, with so and to arbitrary signals in $\mathbf{b}_{20}$:

$$\begin{aligned}\left\langle\overline{s},\overline{t}\right\rangle &:= \langle s+so, t+to\rangle \\ &= \lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1}(s_n+so_n)(t_n+to_n)^* \\ &= \lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1}(s_nt_n^*+s_nto_n^*+so_nt_n^*+so_nto_n^*) \\ &= \langle s,t\rangle + \langle s,to\rangle + \langle so,t\rangle + \langle so,to\rangle \\ &= \langle s,t\rangle \end{aligned}\tag{45}$$

since, by Inequality 44,

$$|\langle s,to\rangle| \leq \sqrt{\langle s,s\rangle\langle to,to\rangle} = 0;$$

and, similarly, $\langle s,to\rangle = \langle so,t\rangle = \langle so,to\rangle = 0$.

Carmichael's inner product determines the norm $||\overline{s}||_C := \sqrt{\langle\overline{s},\overline{s}\rangle}$ on the space $\frac{\mathbf{b}_2}{\mathbf{b}_{20}}$. From now on, we drop the overlines, and equivalence classes are understood.

For $q_1, q_2 \in \hat{\mathbb{Q}}$, the corresponding inner product

$$\langle \{\mathrm{e}^{\mathrm{j}2\pi q_1 n}\}, \{\mathrm{e}^{\mathrm{j}2\pi q_2 n}\}\rangle = \lim_{m\to\infty} \frac{1}{m}\sum_{n=0}^{m-1} \mathrm{e}^{\mathrm{j}2\pi(q_1-q_2)n}$$

has been used by Vaidyanathan et al. for signal processing that uses the RFT.

Lemma 20: Complex exponentials are clearly bounded and have average given by

$$\lim_{m\to\infty} \frac{1}{m}\sum_{n=0}^{m-1} \mathrm{e}^{\mathrm{j}2\pi qn} = \begin{cases} 1, & \text{if } q \in \mathbb{Z} \\ 0, & \text{if } q \in \mathbb{Q}-\mathbb{Z} \end{cases} \tag{46}$$

Proof: If $q \in \mathbb{Z}$ then

$$\sum_{n=0}^{m-1} \mathrm{e}^{\mathrm{j}2\pi qn} = m$$

and, if $q \in \mathbb{Q}-\mathbb{Z}$, then

$$\frac{1}{m}\sum_{n=0}^{m-1} \mathrm{e}^{\mathrm{j}2\pi qn} = \frac{1}{m}\frac{1-\mathrm{e}^{\mathrm{j}2\pi qm}}{1-\mathrm{e}^{\mathrm{j}2\pi q}} \le \frac{1}{m}\frac{2}{|1-\mathrm{e}^{\mathrm{j}2\pi q}|} \xrightarrow{m\to\infty} 0.$$

■

Corollary 10: If x is N-periodic and y is M-periodic with DFT's $X = [X_0, \ldots X_{N-1}]$ and $Y = [Y_0 \ldots . Y_{M-1}]$, then

$$\langle x, y\rangle = \sum_{k=0}^{\min(M,N)-1} X_k Y_k^* \tag{47}$$

which is an identity of the Parseval type.

Proof: If x is N-periodic and y is M-periodic, you have

$$x_n = \sum_{k=0}^{N-1} X_k \mathrm{e}^{\mathrm{j}2\pi q_k n}, \text{ and, } y_n = \sum_{l=0}^{M-1} Y_l \mathrm{e}^{\mathrm{j}2\pi q_l n}$$

with $q_k = \frac{k}{N}$, $q_l = \frac{l}{M}$, and thus

$$\langle x, y\rangle = \lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1}\left(\sum_{k=0}^{N-1} X_k \mathrm{e}^{\mathrm{j}2\pi q_k n}\right)\left(\sum_{l=0}^{M-1} Y_l^* \mathrm{e}^{-\mathrm{j}2\pi q_l n}\right) \tag{48}$$

$$= \sum_{k=0}^{N-1}\sum_{l=0}^{M-1} X_k Y_l^* \lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1} \mathrm{e}^{\mathrm{j}2\pi(q_k-q_l)n}$$

$$= \sum_{k=0}^{N-1}\sum_{l=0}^{M-1} X_k Y_l^* \,\delta(k-l) \tag{49}$$

getting

$$\langle x, y\rangle = \sum_{k=0}^{\min(M,N)-1} X_k Y_k^*$$

■

Corollary 11: (Carmichael) For rationals q_k and q_l in [0, 1),

$$\langle \{\mathrm{e}^{\mathrm{j}2\pi q_k n}\}, \{\mathrm{e}^{\mathrm{j}2\pi q_l n}\}\rangle = 1,$$

if $k = l$, and, otherwise,

$$\langle \{\mathrm{e}^{\mathrm{j}2\pi q_k n}\}, \{\mathrm{e}^{\mathrm{j}2\pi q_l n}\}\rangle = 0.$$

Proof: It follows from Lemma 20. ■

Going back to (two-sided) signals $\mathbb{Z} \to \mathbb{C}$, for s and t as in Equation (30),

$$s_n = \sum_{q\in\hat{\mathbb{Q}}} b_q \mathrm{e}^{\mathrm{j}2\pi qn}, \quad t_n = \sum_{q\in\hat{\mathbb{Q}}} d_q \mathrm{e}^{\mathrm{j}2\pi qn}$$

let their correspondingly extended inner product be

$$\langle s, t\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} s_n t_n^*. \tag{50}$$

For example, if r is the signal $\{\ldots 0, \underline{0}, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ of example VI-B1; then, since $r \in \mathbf{b}_{20}$, $\langle r, \{\mathrm{e}^{\mathrm{j}2\pi q_l n}\}\rangle = 0$.

D. THE DFDT TRANSFORM

A natural extension of the RFT consists in allowing each complex exponential to have a different weight, while still using a series of the periodic complex exponentials, of frequencies in $\hat{\boldsymbol{\Theta}} = \{2\pi q : q \in \hat{\mathbb{Q}}\}$. The generalization, called the *Discrete-Frequency, Discrete-Time* (DFDT) Fourier transform, provides a tool that extends the current uses of the RFT for Signal Processing. The DFDT transform $s \mapsto b$ has inverse

$$s_n = \sum_{k=0}^{\infty} b_{q_k} \mathrm{e}^{\mathrm{j}2\pi q_k n}, \tag{51}$$

where $s : \mathbb{Z} \to \mathbb{C} \cup \{\infty\}$ is and $b : \mathbb{N} \to \mathbb{C}$. Each coefficient b_q, $q \in \hat{\mathbb{Q}}$ corresponds to the frequency $\theta_q = 2\pi q \in \hat{\theta}$. The transform signal $\{b_{q_k}\}$, or $\{b_k\}$ for short, can be often computed using Carmichael's inner product.

E. COMPUTATION OF THE DFDT TRANSFORM

In many cases, the DFDT $\{b_{q_k}\}$, abbreviated $\{b'_k\}$, of signal $s_n = \sum_{q\in\hat{\mathbb{Q}}} b_{q_k} \mathrm{e}^{\mathrm{j}2\pi qn}$ can be obtained with Carmichael's inner product of Equation (50). Consider

$$\langle s, \mathrm{e}^{\mathrm{j}2\pi q_l n}\rangle = \left\langle \sum_{k=0}^{\infty} b_k \mathrm{e}^{\mathrm{j}2\pi q_k n}, \mathrm{e}^{\mathrm{j}2\pi q_l n}\right\rangle$$

$$= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\sum_{k=0}^{\infty} b_k \mathrm{e}^{\mathrm{j}2\pi(q_k-q_l)n}; \tag{52}$$

if these limits commute, which depends on the b_k's, the coefficients are given by

$$b_k = \langle s, \mathrm{e}^{\mathrm{j}2\pi q_l n}\rangle \tag{53}$$

since, then, by corollary 11,

$$\langle s, \mathrm{e}^{\mathrm{j}2\pi q_l n}\rangle = \sum_{k=0}^{\infty} b_k \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} \mathrm{e}^{\mathrm{j}2\pi(q_k-q_l)n} = b_k,$$

and now

$$b_k = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} s_n \mathrm{e}^{-\mathrm{j}2\pi q_k n} \tag{54}$$

together with $s_n = \sum_{q\in\hat{\mathbb{Q}}} b_q \mathrm{e}^{\mathrm{j}2\pi qn}$ is a DFDT - DFDT^{-1} pair.

For signals $x_n = \sum_{k=0}^{\infty} X_k \mathrm{e}^{\mathrm{j}q_k n}$ and $y_n = \sum_{l=0}^{\infty} Y_l \mathrm{e}^{\mathrm{j}q_l n}$, if the limit exchange step from Equation (48) to Equation (49) is valid,

$$\langle x, y\rangle = \sum_{k=0}^{\infty} X_k Y_k^*. \tag{55}$$

Equation (54) computes the DFDT transform of x. Alas, the formula lacks generality, as the mentioned limit exchange is not always possible. In particular, for $s_n = \sum_{k=0}^{\infty} b_{q_k} \mathrm{e}^{\mathrm{j}2\pi q_k n}$, with $s \in \mathbf{b}_{20} - \{\mathbf{0}\}$,

$$\overline{|s^2|} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{k=0}^{\infty} b_{q_k} \mathrm{e}^{\mathrm{j}2\pi q_k n} \sum_{l=0}^{\infty} b_{q_l}^* \mathrm{e}^{-\mathrm{j}2\pi q_l n}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{q_k} b_{q_l}^* \mathrm{e}^{\mathrm{j}2\pi (q_k - q_l) n}$$

the limit exchange

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{q_k} b_{q_l}^* \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \mathrm{e}^{\mathrm{j}2\pi (q_k - q_l) n}$$

is not valid since, otherwise,

$$0 = \overline{|s^2|} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{q_k} b_{q_l}^* \delta_{k-l} = \sum_{k=0}^{\infty} |b_{q_k}|^2,$$

and $s = \mathbf{0}$.

F. DFDT TRANSFORM OF A TIME-INVERTED SIGNAL

Since

$$s_{-n} = \sum_{k=0}^{\infty} b_{q_k} \mathrm{e}^{-\mathrm{j}2\pi q_k n} = \sum_{k=0}^{\infty} b_{q_k} \mathrm{e}^{\mathrm{j}2\pi (1-q_k) n}, \tag{56}$$

a time inversion of the signal s corresponds to a reshuffling of the frequencies of its DFDT transform; if the transform of $\{s_n\}$ is $\{b_{q_k}\} = \{b_0, b_{\frac{1}{2}}, b_{\frac{1}{3}}, b_{\frac{2}{3}}, b_{\frac{1}{4}}, b_{\frac{3}{4}}, b_{\frac{1}{5}}, b_{\frac{2}{5}}, b_{\frac{3}{5}}, b_{\frac{4}{5}}, \ldots\}$, the transform of $\{s_{-n}\}$ is the *reshuffling* $\{b_{q_{f(k)}}\} = \{b_0, b_{\frac{1}{2}}, b_{\frac{2}{3}}, b_{\frac{1}{3}}, b_{\frac{3}{4}}, b_{\frac{1}{4}}, b_{\frac{4}{5}}, b_{\frac{3}{5}}, b_{\frac{2}{5}}, b_{\frac{1}{5}}, \ldots\}$; that is,

$$f(n_s + k) = n_{s+1} - k + 1,$$

for $k \in /1, n_{s+1} - n_s/$, where $n_s := \sum_{r=1}^{s} \varphi(r)$; vertically downshifting the portion $\theta \in (\pi, 2\pi)$ of the period-frequency matrix in Fig. 10. Alternatively, using negative frequencies, as in Section IV-D, with $\theta_q = 2\pi q$, if $S(\theta_q)$ is the DFDT transform of $\{s_n\}$, the transform of $\{s_{-n}\}$ is $S(-\theta_q)$. The DFDT is asymmetric in the sense that, in the time domain, you have two-sided signals $\{s_n : n \in \mathbb{Z}\}$ while, in the frequency domain, you have one-sided signals $\{b'_k : k \in \mathbb{N}\}$, unless you consider negative frequencies and the rationals in $(-\frac{1}{2}, \frac{1}{2}]$ instead of $\hat{\mathbb{Q}}$, and their corresponding ordering, mentioned in Section IV-D.

G. THE RFT AS A PARTICULAR CASE OF THE DFDT TRANSFORM

The inverse RFT of Equation (42) can be seen as a case of the inverse DFDT of Equation (51), with $b'_k = a_{\eta(q_k)}$, where $\eta(q) = 1/\tau(q)$ is the denominator function; that is,

$$\sum_{N=1}^{\infty} a_N \mathrm{c}_N(n) = \sum_{k=0}^{\infty} a_{\eta(q_k)} \mathrm{e}^{\mathrm{j}2\pi q_k n}.$$

For example, the inverse DFDT transform of Thomae's signal t_k of Section III-B, is

$$\sum_{q\in\hat{\mathbb{Q}}-\{0\}} \tau(q) \mathrm{e}^{\mathrm{j}2\pi qn} = \sum_{k=0}^{\infty} t_k \mathrm{e}^{\mathrm{j}2\pi qn} = \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{c}_k(n)$$

and it is zero for $n \neq 0$. In this case, the exchange of the limits in Equation (52) is not allowed. In this case, the signal in Equation (51), has DFDT transform $b' = \{b'_k\}$, with $b'_0 = 0$ and $b'_k = a_{\eta(q_k)} = \frac{1}{k}$, for $k \geq 1$, i.e.

$$b' = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \ldots\right\},$$

where each fraction $\frac{1}{l}$ repeats itself $\varphi(l)$ times. In this case,

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} b'_k \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{\mathrm{j}2\pi (q_k - q_l) n}$$

$$\neq \sum_{k=0}^{\infty} b'_k \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{\mathrm{j}2\pi (q_k - q_l) n}$$

$$= \sum_{k=1}^{\infty} b'_k \, \delta_l(k) = b_l = \frac{1}{r}$$

where $r := \min_t : l \leq \sum_{\nu=1}^{t} \varphi(\nu)$, since, as Ramanujan proved, for $n \geq 1$, $\sum_{k=0}^{\infty} b_k \mathrm{e}^{\mathrm{j}2\pi q_k n} = \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{c}_k(n) = \mathbf{0}(n)$.

In an inverse DFDT transform, the periods of the complex exponentials grow with the summation index. Since the average variation of the complex exponentials of a given period varies, you may consider separate series of rough and smooth periodic complex exponentials, as was done in Section V-E for finite sums. The N-periodic complex exponentials of smallest average variation are those of Fourier frequencies $2\pi \frac{1}{N}$ and $2\pi \frac{N-1}{N} = -2\pi \frac{1}{N}$. A real signal with smoothest Fourier components of all periodicities, is

$$s_n = \sum_{N=1}^{\infty} a_N \cos\left(2\pi \frac{1}{N} n\right)$$

For series of *roughest* complex exponentials, with frequencies closest to π,

$$r_n = a_2 \cos(n\pi) + \sum_{N=2}^{\infty} a_{2\,N} \cos\left(\left(\pi - 2\pi \frac{1}{2\,N}\right) n\right) + \sum_{N=1}^{\infty} a_{2N+1} \cos\left(\left(\pi - 2\pi \frac{1}{2(2N+1)}\right) n\right)$$

Truncated versions of these series are shown in Figs. 12 and 13.

VII. CONCLUSION

The distribution of the average power on the period-frequency matrix is a tool for filter design. Knowledge of the transfer function $H(\theta)$ of a discrete convolution filter, and of the period of a unit-amplitude periodic complex exponential to be filtered, is not enough information to predict the output of the filter, and any band-pass filter will let pass complex exponentials of arbitrarily large periods. Most discrete signals in signal processing are uniformly sampled versions of continuous signals, that respect Nyquist sampling rate. If the continuous complex exponential $e^{j\omega t} = e^{j\frac{2\pi}{\tau}t}$, of period τ, is sampled with sampling rate T^{-1}, the discrete complex exponential $e^{j\theta n}$, with $\theta = 2\pi\frac{T}{\tau}$ results (for $T = 1$, $\theta = \omega$, and the periods of the discrete and the continuous exponentials coincide). Now, if $\frac{T}{\tau}$ is rational, with corresponding irreducible fraction $\frac{M}{N}$, the period of the discrete complex exponential is N, of corresponding temporal duration NT, which can be very large. Notice as well that there are continuous complex exponentials of frequency arbitrarily close to ω that, when sampled with the sampling rate T^{-1}, give rise to discrete periodic complex exponentials of arbitrarily large periods.

Series of discrete, periodic complex exponentials constitute an important model of signals, some of them integer valued. The integrality of Ramanujan sums makes them suitable for applications in cryptography, as well as in fixed-point digital machines. On the other hand, unbound signals are not amenable for their computation with computers but are of theoretical interest, e.g. in Number Theory and in Time Series Analysis.

Given a signal $\{s_n\}$ that is a series of periodic complex exponentials, the sinc-interpolated signal

$$f(t) = \sum_{n=-\infty}^{\infty} s_n \mathrm{sinc}(\pi(t-n)) \tag{57}$$

is a nonperiodic continuous signal, bandlimited to $|\omega| \leq \pi$, with a countably infinite spectrum that includes frequencies ω arbitrarily close to zero. The fact that extremely low frequency signals (e.g planetary magnetic fields [3], [2]) are hard to observe [4] (among other things, the antenna would be inordinately large) has made difficult to observe such spectra.

If $s_n = \sum_k X_k e^{j\theta_k n}$ is a linear combination of complex exponentials, e.g. a Ramanujan sum, the continuous interpolated version according to Equation (57), is the continuous signal $f(t) = \sum_k X_k e^{j\theta_k t}$, and the filtering by period of $f(t)$ can be done in the discrete domain.

Signal Theory goes hand in hand with System Theory, as systems process signals. Mathematicians have considered often the case of almost periodic signals, as solutions of difference equations and their corresponding discrete systems. Almost periodic solutions of difference equations arise in practical situations; a subject that merits further research in Signal Processing [37].

The restriction of Thomae's function to the rationals in [0, 1] relates the period and Fourier frequency of the periodic complex exponentials, and its use is novel in Signal Processing. Also novel is the consideration of other signals defined on $\hat{\mathbb{Q}}$, such as the DFDT transforms (of series of periodic signals).

We have not considered amplitude-modulated signals, even though, in practice, signals are windowed, as they approximately exist over a finite interval only. Neither have we considered frequency-modulated signals. The subject of additive noise has not been considered either.

Whether a signal is periodic or not depends on knowing the whole signal. In practice, for a streaming signal, you can only know if, at a given point, the signal has repeated itself, or not. We are working on an algorithm that tells how many times (≥ 0) a signal being monitored has repeated itself so far.

REFERENCES

[1] H. G. Gadiyar and R. Padma, 'Ramanujan-Fourier series, the Wiener-Khintchine formula and the distribution of prime pairs," *Physica A: Stat. Mechanics Appl.*, vol. 269, no. 2–4, pp. 503–510, Jul. 1999.

[2] F. P. Zantis, J. Hribik and D. Durackova, 'Extremely low frequency signal acquisition, recording and analysis," *J. Elect. Eng.*, vol. 67, no. 3, pp. 185–191, 2016.

[3] B. J. Anderson, "Ultra-low-frequency magnetic pulsations in the earth magnetosphere," *Johns Hopkins APL Tech. Dig.*, vol. 11, no. 3/4, pp. 239–254, 1990.

[4] H. Burda, S. Begalla, J. Červený, J. Neefa, and P. Némec, 'Extremely low-frequency electromagnetic fields disrupt magnetic alignment of ruminants," *Proc. Nat. Acad. Sci.*, vol. 106, no. 14, pp. 5708–5713, 2009.

[5] S. He and B. Kedem, "Higher order crossings spectral analysis of an almost periodic random sequence in noise," *IEEE Trans. Inf. Theory*, vol. 35, no. 2, pp. 360–370, Mar. 1989.

[6] L. I. Rudin and S. Osher, 'Total variation based image restoration with free local constraints," in *Proc. 1st Int. Conf. Image Process.*, 1994, pp. 31–35.

[7] J. Kunisch and I. Wolff, 'Determination of sampling points for nearly DFT-equivalent almost-periodic Fourier transforms," in *Proc. 23rd Eur. Microw. Conf.*, 1993, pp. 710–713.

[8] D. Dehay, 'Discrete time observation of almost periodically correlated processes and jitter phenomena," in *Proc. 15th Eur. Signal Process. Conf.*, 2007, pp. 601–605.

[9] P. P. Vaidyanathan and S. Tenneti, "When does periodicity in discrete-time imply that in continuous-time?," in *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.*, 2018, pp. 4544–4548.

[10] P. Kulkarni and P. P. Vaidyanathan, "On the zeros of Ramanujan filters," *IEEE Signal Process. Lett.*, vol. 27, pp. 735–739, Apr. 2020.

[11] D. K. Yadav, G. Kuldeep, and S. D. Joshi, "Ramanujan sums as derivatives and applications," *IEEE Signal Process. Lett.*, vol. 25, no. 3, pp. 413–416, Mar. 2018.

[12] A. Wintner, *An Arithmetical Approach to Ordinary Fourier Series*. Baltimore, MD, USA: Waverly Press Inc., 1945.

[13] H. Bohr, *Almost Periodic Functions*. New York, NY, USA: Chelsea Publishing Company, 1947.

[14] W. Stepanoff, "Über einigen Verallgemeinnerungen der fastperiodischen Funktionen," *Math. Ann.*, vol. 97, pp. 338–356, 1926.

[15] A. S. Besicovitch, *Almost Periodic Functions*. Mineola, NY, USA: Dover Publishing Company, 1954.
[16] C. Corduneanu, “Almost periodic discrete processes,” *Libertas Mathmatica*, vol. 2, pp. 159–169, 1982.
[17] A. Walther, “Fastperiodische folgen und potenzreihen mit fastperiodischen koefficienten,” *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 6, pp. 217–234, 1928.
[18] Z. AlSharawi and J. Angelos, ‘Linear almost periodic difference equations,” *J. Comput. Math. Optim.*, vol. 4, no. 2, pp. 61–91, 2008.
[19] C. L. DeVito, *Functional Analysis*. New York, NY, USA: Academic Press, 1978.
[20] S. Ramanujan, ‘On certain trigonometrical sums and their applications in the theory of numbers,” *Trans. Cambridge Philos. Soc.*, vol. 22, no. 13, pp. 259–276, 1918.
[21] S. Samadi, M. O. Ahmad, and M. S. N. Swany, “Ramanujan sums and discrete Fourier transforms,” *IEEE Signal Process. Lett.*, vol. 12, no. 4, pp. 293–296, Mar. 2005.
[22] P. P. Vaidyanathan, “Ramanujan sums in the context of signal processing—Part I: Fundamentals,” *IEEE Trans. Signal Process.*, vol. 62, no. 16, pp. 4145–4157, Aug. 2014.
[23] O. Hinton, *EEE305 - Digital Signal Processing*. 2001. [Online]. Available: https://www.staff.ncl.ac.uk/oliver.hinton/eee305/
[24] P. P. Vaidyanathan and P. Pal, “The farey-dictionary for sparse representation of periodic signals,” in *Proc. IEEE Int. Conf. Acoustic Speech Signal Process.*, 2014, pp. 360–364.
[25] A. Restrepo and L. P. Chacón, ‘On the period of sums of discrete periodic signals,” *IEEE Signal Process. Lett.*, vol. 5, no. 7, pp. 164–166, 1998.
[26] M. Ram Murty, “Ramanujan series for arithmetical functions,” *Hardy-Ramanujan J.*, vol. 36, pp. 21–33, 2013.
[27] M. Planat, “Ramanujan sums for signal processing and low-frequency noise,” in *Proc. IEEE Int. Freq. Control Symp. PDA Exhib.*, 2002, pp. 715–720.
[28] E. M. Stein and R. Shakarchi, *Fourier Analysis—An Introduction, Princeton Lectures in Analysis*. Princeton, NJ, USA: Princeton Univ. Press, 2003.
[29] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed. Oxfod, U.K.: Oxford Univ. Press, 1979.
[30] N. Megiddo, “Partial and complete cyclic orders,” *Bull. Amer. Math. Soc.*, vol. 82, no. 2, pp. 274–276, Mar. 1976.
[31] H. Weyl, ‘Über die Gleichverteilung von Zahlen mod,” *Mathematische Annalen*, vol. 77, pp. 313–352, 1916.
[32] R. D. Carmichael, “Expansions of arithmetical functions in infinite series,” in *Proc. London Math. Soc.*, 1932, pp. 1–26.
[33] E. Hecke, ‘Über analytische Funktionen und die Verteilung von Zahlen mod. eins,” *Abhandlungen aus demMathematischen Seminar der Universität Hamburg*, vol. 1, pp. 54–76, 1922.
[34] H. Jeon and M. Hardy, “Bounded sequence with divergent Cesaro means,” 2016. [Online]. Available: https://mathstackexchange.com/questions/444889/bounded-sequence-with-divergent-cesaro-means
[35] K. Briggs, “Abundant numbers and the Riemann hypothesis,” *Exp. Math.*, vol. 15, no. 2, pp. 251–256, 2006.
[36] J. W. Tukey, *Exploratory Data Analysis*. Reading, MA, USA: Addison-Wesley, 1977.
[37] A. Walther, “Über lineare differenzengleichungen mit konstanten koeffizienten und fastperiodischer rechter seite,” *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, vol. 1927, pp. 196–216, 1927.

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