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# Explicitly Invertible Approximations of the Gaussian Q-Function: A Survey

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**ABSTRACT** Communications and information theory use the Gaussian $Q$-function, a positive and decreasing function, across the literature. Its approximations were created to simplify mathematical study of the Gaussian $Q$-function expressions. This is important since the $Q$-function cannot be represented in closed-form terms of elementary functions. In a noise model with the Gaussian distribution function and various digital modulation schemes, closed-form approximations of the Gaussian $Q$-function are used to predict a digital communications system's symbol error probability (SEP) or bit error probability (BEP). Another significant scenario pertains to fading channels, whereby it is important to accurately determine, through a closed-form expression, the precise evaluations of complex integrals involved in the computations of SEP or BEP. In addition to the aforementioned scenarios, it is imperative for a communications system designer to ascertain the requisite operational signal-to-noise ratio for the specific application, based on the target SEP (or BEP). In this scenario, the crucial role of the *explicit invertibility* of the Gaussian $Q$-function approximation is of significant importance in achieving this objective. In this paper we propose a survey of the approximations of the Gaussian $Q$-function found in the literature, reviewing also the approximations originally given for the 4 classical *special functions* related to it, restricting the analysis to the *explicitly invertible* ones, and classifying them on the basis of their accuracy (on the significant range), simplicity, and easiness of inversion, also distinguishing the bounds from approximations. We also list the inverses of some of them, already published or newly found in this research.

**INDEX TERMS** Approximations, bit error probability (BEP), *Explicit invertibility*, Gaussian noise, Gaussian Q-function, normal cumulative distribution function, normal quantiles, symbol error probability (SEP), telecommunications channels.

## I. INTRODUCTION

THE $Q$-FUNCTION and the other 4 related *special functions* – $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and Mills' ratio $m(x)$ – widely discussed in this survey, and all equivalent by formulas of Table 1, are mathematical functions of significant importance broadly used in various disciplines, such as probability theory, statistics, signal processing, and communications engineering. There are several reasons why these functions hold significance:

1) Probability calculations: Probability calculations often use Gaussian distributions and make use of various *special functions*, including the $Q$-function and the 4 additional related functions. The $Q$-function, in particular, provides the likelihood that a random variable with a normal distribution will exceed a specific value.
2) Signal processing: The $Q$-function and $\mathrm{erfc}(x)$ function are commonly employed in signal processing to evaluate the likelihood of bit error in digital communication systems. The significance of this lies in the development of communication systems capable of effectively transmitting data across channels with high levels of noise.

3) Statistical analysis: The $Q$-function is employed in statistical analysis for the purpose of data modeling and parameter estimation. In the context of hypothesis testing, the $Q$-function is a useful tool for the computation of $p$-values.
4) Mathematical modeling: The $Q$-function and the error function, $\mathrm{erf}(x)$, frequently appear in mathematical models that describe a wide range of events. An illustration of the utilization of the error function, $\mathrm{erf}(x)$, can be observed in the heat equation within the context of physics, as well as in the Black-Scholes equation in the field of finance.
5) Computational efficiency: In certain situations, the utilization of the $Q$-function and the complementary error function, $\mathrm{erfc}(x)$, offers a more efficient and precise approach for computing specific probabilities and integrals compared to alternative methodologies.

### A. BACKGROUND

The Gaussian $Q$-function, denoted as $Q(x)$ or equivalently as the complementary error function, $\mathrm{erfc}(x)$, is extensively utilized in the field of communications and information theory. Its widespread usage can be attributed to its significant contribution in the performance analysis of several systems. Given the various equivalent definitions of the function $Q(x)$, or, equivalently, of the complementary error function $\mathrm{erfc}(x)$, as presented in Section II-A, it is evident that these definitions pose mathematical challenges. Furthermore, it is widely acknowledged that the $Q$-function cannot be expressed using elementary functions in a closed-form manner. Consequently, the existing literature has proposed various approximations and bounds for the Gaussian $Q$-function, which have been extensively examined in this survey. Closed-form expressions are of great significance in the evaluation of communication systems' performance, since they facilitate mathematical analysis. These expressions are typically used to quantify the symbol error probability (SEP) or bit error probability (BEP) in such systems. In addition, it is important to have closed-form formulations of the system performance in order to facilitate system optimization.

There exist three fundamental scenarios in which the necessity for highly efficient closed-form approximations of the Gaussian $Q$-function emerges, including cases – described in Scenario 1) – where approximations to integer powers of the Gaussian $Q$-function are also required:

1) In the additive white Gaussian noise (AWGN) channel scenario, it is of great interest to have highly efficient closed-form approximations of the Gaussian $Q$-function in order to determine the error probability for many different digital modulation schemes [1]. In fact, starting from the simple binary ones, as the binary amplitude modulation (2-AM) and the binary phase shift keying (BPSK), the BEP involves the Gaussian $Q$-function (see, e.g., [1, Formula 8.18]). Considering more complicated higher order constellations, it is of great interest to have highly efficient closed-form approximations to integer powers of the Gaussian $Q$-function, too, since the SEP involves the Gaussian $Q$-function and its square for the quadrature phase shift keying (QPSK, see, e.g., [1, Formula 8.19]), or the integer powers of the Gaussian $Q$-function up to $Q^4(x)$ for the differentially encoded QPSK modulation (see, e.g., [1, Formula 8.38]), or even up to $Q^6(x)$ for the triangular quadrature amplitude modulation (TQAM) with maximum ratio combining (MRC) [2]. Bounds or approximations to integer powers of the Gaussian $Q$-function, needed for evaluating the SEP for the more complicated higher order constellations, have been addressed in [3], [4], [5], [6].
2) A second important scenario is represented by communications environments in which the channel signal-to-noise ratio probability density function (pdf) follows a fading distribution: over these channels it is extremely important to work with approximations of the Gaussian $Q$-function allowing exact evaluations of complex integrals involved in the error probabilities computations. The approximations of the $Q$-function found in [4] and [7], [8], [9], [10], [11], [12], [13] were all meant to easily compute the average SEP (ASEP) over fading channels, finding in some cases closed form expressions, too.
3) Besides these two scenarios, another important need of a communications system designer is to derive, given the target symbol error or bit error probability of a communications system, the operating signal-to-noise ratio needed by the considered application. To this purpose, the simple and *explicit invertibility* of the considered approximation of the $Q$-function assumes a very important role. See Section IV-F for this issue, and already here we quote from classical Cooper and McGillem book [14, p. 70]

> "Often of equal importance are the inverses of these functions that are needed to find the parameters that lead to observed or specified probabilities of events."

and from the paper [15] of Polyanskiy, Poor, and Verdú

> "For general classes of channels new achievability and converse bounds are given, which are tighter than existing bounds for wide ranges of parameters of interest, and lead to tight approximations of the maximal achievable rate for blocklengths $n$ as short as 100. $(\cdots)$ the maximal rate achievable with error probability $\varepsilon$ is closely approximated by $C-\sqrt{\frac{V}{n}}\,Q^{-1}(\varepsilon)$ where $C$ is the capacity, [and] $V$ is a characteristic of the channel referred to as *channel dispersion*."

Already many papers and books as [6], [9], [11], [16], [17], [18], [19], [20], [21], [22], [23], [24], [24], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39] explicitly addressed the inversions of

**TABLE 1.** The 20 mutual relations among the 5 functions $Q(x)$, $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and $m(x)$.

| | $Q(x)$ | $\Phi(x)$ | $\mathrm{erf}(x)$ | $\mathrm{erfc}(x)$ | $m(x)$ |
|---|---|---|---|---|---|
| $Q(x)$ | | $1-\Phi(x)$ | $\frac{1}{2}\left(1-\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$ | $\frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$ | $m(x)\,\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}$ |
| $\Phi(x)$ | $1-Q(x)$ | | $\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$ | $1-\frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$ | $1-m(x)\,\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}$ |
| $\mathrm{erf}(x)$ | $1-2\,Q(x\,\sqrt{2})$ | $2\,\Phi(x\,\sqrt{2})-1$ | | $1-\mathrm{erfc}(x)$ | $1-m(x\,\sqrt{2})\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-x^2}$ |
| $\mathrm{erfc}(x)$ | $2\,Q(x\,\sqrt{2})$ | $2\left(1-\Phi(x\,\sqrt{2})\right)$ | $1-\mathrm{erf}(x)$ | | $m(x\,\sqrt{2})\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-x^2}$ |
| $m(x)$ | $Q(x)\sqrt{2\,\pi}\,\mathrm{e}^{\frac{x^2}{2}}$ | $(1-\Phi(x))\,\sqrt{2\,\pi}\,\mathrm{e}^{\frac{x^2}{2}}$ | $\left(1-\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{\frac{x^2}{2}}$ | $\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{\frac{x^2}{2}}$ | |

**TABLE 2.** The 12 mutual relations among the 4 inverse functions $Q^{-1}(y)$, $\Phi^{-1}(y)$, $\mathrm{erf}^{-1}(y)$, and $\mathrm{erfc}^{-1}(y)$.

| | $Q^{-1}(y)$ | $\Phi^{-1}(y)$ | $\mathrm{erf}^{-1}(y)$ | $\mathrm{erfc}^{-1}(y)$ |
|---|---|---|---|---|
| $Q^{-1}(y)$ | | $\Phi^{-1}(1-y)$ | $\sqrt{2}\,\mathrm{erf}^{-1}(1-2\,y)$ | $\sqrt{2}\,\mathrm{erfc}^{-1}(2\,y)$ |
| $\Phi^{-1}(y)$ | $Q^{-1}(1-y)$ | | $\sqrt{2}\,\mathrm{erf}^{-1}(2\,y-1)$ | $\sqrt{2}\,\mathrm{erfc}^{-1}(2(1-y))$ |
| $\mathrm{erf}^{-1}(y)$ | $\frac{1}{\sqrt{2}}\,Q^{-1}\left(\frac{1-y}{2}\right)$ | $\frac{1}{\sqrt{2}}\,\Phi^{-1}\left(\frac{1+y}{2}\right)$ | | $\mathrm{erfc}^{-1}(1-y)$ |
| $\mathrm{erfc}^{-1}(y)$ | $\frac{1}{\sqrt{2}}\,Q^{-1}\left(\frac{y}{2}\right)$ | $\frac{1}{\sqrt{2}}\,\Phi^{-1}\left(1-\frac{y}{2}\right)$ | $\mathrm{erf}^{-1}(1-y)$ | |

the approximations of $Q(x)$ or $\Phi(x)$ or $\mathrm{erf}(x)$ or $\mathrm{erfc}(x)$ – which are all equivalent by formulas of Table 2 – and, in particular, [6], [9], [11], [22], and [29] considered the importance of the *explicit inversion* in the environment of communications theory.

## B. ORIGINAL CONTRIBUTION

This paper proposes a survey of the approximations of the Gaussian $Q$-function found in the literature, reviewing also the approximations originally given for the 4 classical *special functions* related to it (see Section II-C), focusing in particular on the *explicitly invertible* (see Definition in Section V) ones. More precisely, we review published approximations of $Q(x)$ with these characteristics:

1) published for $Q(x)$ or the other 4 related functions $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and Mills' ratio $m(x)$ (and $\mathrm{erfc}\sqrt{x}\cdots$), from which $Q(x)$ may be immediately obtained (see Table 1);
2) holding at least for any $x > 0$;
3) defined by a single expression, that is to say not piecewise defined;
4) defined in closed form by means of elementary functions with standard names used in mathematics (for the issue of standard names used in mathematics see Remarks 1 and 2 in Section V);
5) being *explicitly invertible*.

In Table 2 we list the 12 mutual relations among the 4 inverse functions $Q^{-1}$, $\Phi^{-1}$, $\mathrm{erf}^{-1}$, and $\mathrm{erfc}^{-1}$.

In Tables 3–13 we list many *explicitly invertible* (see Section V) approximations, classifying them according to their *Types* (see Section VI) indicating also:

1) their accuracy (see Section VII);
2) their complexity, measured in several ways (see Section VIII);
3) their easiness of invertibility (see Section IX);
4) the fact that they are or not a bound (see Section X).

The mathematical analysis of the approximations presented in Tables 3–13 has been thoroughly conducted in Section VI. This analysis includes a comprehensive examination of the classification of these approximations into different *Types*, which can contribute to a better comprehension of their mathematical characteristics.

Furthermore, a thorough analysis of the categorization of approximations according to their levels of invertibility (*InvLev*) has been published in Section IX. This classification aids in understanding the degree of ease with which certain approximations can be inverted. In order to enhance the clarity of this classification, we have included in Tables 14–18 both newly computed and previously published explicit inverses of the aforementioned approximations of $Q(x)$ at the most advanced levels of invertibility: *InvLev 4*, *InvLev 5*, *InvLev 6*, *InvLev 6.5*, and *InvLev 7*, respectively.

Table 19 compiles the most accurate approximations of $Q(x)$ for any given *Type* and *InvLev* on the interval $I_{\text{significant}} = [0.45, 4.5]$. This table provides a concise overview of the findings and highlights the most promising avenues for future research, as discussed in Section XIII.

## C. ORGANIZATION OF THE PAPER

The paper is organized as follows.

After the Introduction, in Section II we recall the basic definition of the $Q$-function and

- its several equivalent definitions (Section II-A) by
  - real integrals (Section II-A1),
  - a complex integral (Section II-A2),
  - limits (Section II-A3),
  - a differential equation (Section II-A4),
  - the function $\Phi(x)$ (Section II-A5),
  - a continuous fraction (Section II-A6),
  - power series (Section II-A7),
  - a class of function series (Section II-A8),
  - and random variables (Section II-A9),
- its different names (Section II-B),

- the 4 classical *special functions* related to the *special function* $Q(x)$ (Section II-C),
- the probabilistic meaning of the function $Q(x)$ and of the 4 classical related *special functions* (Section II-D),
- the general behavior of these mutually related 5 *special functions* (Section II-E),
- the symmetry formulas for these functions (Section II-F),
- and a historical note (Section II-G).

In Section III we recall

- the domain of practical interest of the function $Q(x)$ in information and communications theory (Section III-A)
- and some notable values, in the domain of interest, of $Q(x)$ and of the 4 classical *special functions* related to it (Section III-B).

In Section IV we treat the approximation of the function $Q(x)$, addressing

- the necessity of approximating $Q(x)$ (Section IV-A),
- some application examples of the function $Q(x)$ (Section IV-B),
- the most desirable attributes that an approximation should possess (Section IV-C),
- the issue of the names of the approximations adopted in the paper (Section IV-D),
- the valuable merits of an approximation of $Q(x)$, some of which will be deepened in Sections VII–X (Section IV-E),
- the utility for an approximation to be *explicitly invertible*, related to the issue of deriving the operating signal-to-noise ratio given the target SEP or BEP (Section IV-F),
- and the topic of approximating $Q(x)$ inverting an approximation of $Q^{-1}(y)$ (Section IV-G).

Section V is devoted to the explanation of the concept of *explicit invertibility* of an approximation.

In Section VI we produce a classification defining 7 classes (from *Type 0* to *Type 6*) of functions in which almost all the published approximations of $Q(x)$ fall, leaving in *miscellanea* (*Type 7*) the few remaining.

In Section VII we address the accuracy of the approximations, dealing with

- the issue of absolute and relative errors (Section VII-A),
- the tightness of the approximations of $Q(x)$ in telecommunications systems (Section VII-B),
- the properties of asymptoticity and asymptotic equivalence of an approximation of $Q(x)$, which are strictly related to its tightness (Section VII-C),
- the tightness of the inverse of an approximation of $Q(x)$ (Section VII-D).

Then, Section VIII treats the levels of complexity of the approximations from several points of view:

- the typographic complexity (Section VIII-A),
- the computational complexity (Section VIII-B),
- the decimal complexity (Section VIII-C),

all summarized by

- the total complexity (Section VIII-D).

Section IX gives a classification of the levels of easiness of *explicit invertibility*, adding

- some comments concerning types and invertibility levels (Section IX-A).

Finally, in Section X we treat the topic of bounds to $Q(x)$, treating separately

- upper bounds, related to the issue of the so-called *worst case* in performance analysis (Section X-A),
- and lower bounds (Section X-B).

Moreover we add some comments on

- bounds and inverses (Section X-C).

In Section XI we address the contents of Tables 3–13 listing 60 known approximations – found in this research – of $Q(x)$ which are *explicitly invertible* by means of

- elementary functions (even without standard names used in mathematics)
- and/or the Lambert $W$-function,

and in Section XII we address the contents of Tables 14–18 listing some published and new (computed in this research) inverses of these approximations of $Q(x)$ which have to be intended as approximations of the inverse of $Q(x)$.

In Section XIII we summarize the main results of the present research, making some final considerations about

- the precision (Section XIII-A),
- the types (Section XIII-B),
- and the complexity (Section XIII-C)

of the approximations, also in terms of prospective research directions.

Finally, Section XIV summarizes the conclusions.

The paper is completed by 2 appendices, addressing

- the concept of the hidden polynomials in the approximations of $Q(x)$, to obtain their *explicit inversion* (Appendix A),
- and a brief survey on the roots of polynomials up to the 4-th degree (Appendix B).

Three interesting photographs of the XIX century works of Gauss and Laplace, about this topic, and a noticeable list of 110 bibliographic references enrich the paper further.

## II. THE FUNCTION *Q*(*X*)

Several functions considered in this paper are based on $\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}$, named $Z(x)$ in the classical Abramowitz & Stegun's book [31] (see Formula 26.2.1) and, more modernly, $\phi(x)$ in many other texts (see, e.g., [40]):

$$Z(x) := \phi(x) := \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}, \tag{1}$$

whose graph is the classical Gaussian bell curve, and is the standard normal pdf.

Basically, through

$$Q(x) := \int_x^{+\infty} \phi(t)\,\mathrm{d}t, \tag{2}$$

the $Q$ function expresses the integral of the right tail of the standard normal pdf (1) and then its meaning is essentially the probability that a standard normal random variable $X$ assumes a value greater than $x$.

### A. THE SEVERAL EQUIVALENT DEFINITIONS OF Q(X)

As a *special function of mathematical analysis*, the function $Q(x)$ is defined on the whole real axis, but for the purposes of telecommunications theory – and then in this paper – it is considered only for $x \geq 0$, or sometimes even $x > 0$, for example when saying (see Section VII-C) that $Q_{\mathrm{Wozencraft}}(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}$ is a bound for $Q(x)$.

Here we recall that by *special function* it is meant a function of large recurrence (in mathematical analysis, physics$\cdots$), usually excluding those which are classified as *elementary functions*.

Here below in Sections II-A1–II-A9 we give several equivalent definitions of $Q(x)$, by real integrals, by a complex integral, by limits, by a differential equation, by the function $\Phi(x)$, by a continuous fraction, by means of power series, by a class of function series, and by random variables, respectively.

#### 1) 5 REAL INTEGRAL DEFINITIONS

$$Q(x) := \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{t^2}{2}}\, dt \tag{3}$$

exactly so defined in Formula 26.2.3 of classical Abramowitz & Stegun [31], but so defined by words (apart from a small oversight)

> “the area of the tail, from $x$ onwards, of the normal curve $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$”

already in the 1926 Mills’ paper [41] in *Biometrika*.

$$Q(x) := 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt \tag{4}$$

immediately derives from (3) and the unit integral of the Gaussian density (1), and

$$Q(x) := \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{x} e^{-\frac{t^2}{2}}\, dt \tag{5}$$

is due to (3), the unit integral of the Gaussian density (1), and its parity.

The following 2 definitions of $Q(x)$, holding only for $x \geq 0$ (which is not a limitation in communications and information theory):

$$Q(x) := \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta \tag{6}$$

and

$$Q(x) := \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\cos^2\theta}\right) d\theta \tag{7}$$

are in [1, Formula 4.2] (which is due to Craig [42]) and [1, Formula 4A.10], respectively.

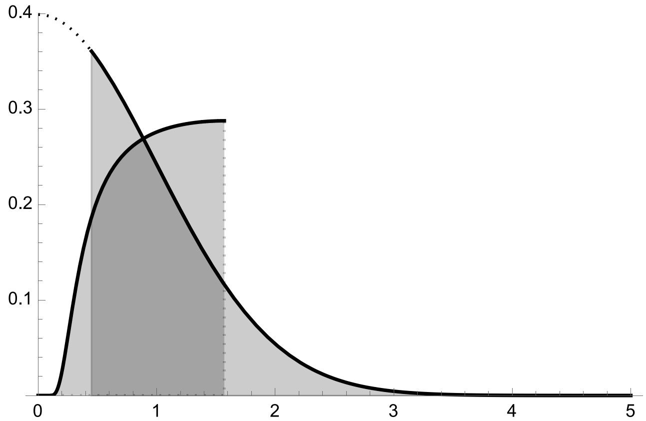



**FIGURE 1.** Fixed in this example $x = 0.45$, the number $Q(0.45) \approx 0.326$ is given by the 2 grey areas: as integral of $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ from 0.45 to $+\infty$, and as integral of $\frac{1}{\pi}\exp(-\frac{0.45^2}{2\sin^2\theta})$ from 0 to $\frac{\pi}{2}$. (For the first case it has to be imagined the graph as extended up to $+\infty$, with a negligible contribution to the total area.).

Remark that (only) the last 3 definitions of $Q(x)$ are based on integrals defined on bounded intervals, which attenuates the practical problem, observed for example in [1], of approximate evaluation by numerical integration.

For any fixed $x \geq 0$, the basic mathematical nature of the number $Q(x)$ is the area under a curve, given by an integral:

- of the fixed function $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ of $t$, on the unbounded interval $[x, +\infty)$ depending from $x$, as in (3);
- of the function $\frac{1}{\pi}\exp(-\frac{x^2}{2\sin^2\theta})$ of $\theta$ but depending from $x$, on the fixed interval $[0, \pi/2]$, as in (6).

In Fig. 1, chosen $x = 0.45$, the number $Q(0.45) \approx 0.326$ is given by the grey areas in both modes. Of course, by changes of variable in the integrals, one may obtain infinite other figures whose area is the number $Q(x)$, and such substitutions sometimes are in fact done, for example in [29].

Notice that, limiting the integral of (6) up to $\pi/4$, one obtains $Q^2(x)$ (see [1, Formula 4.9]), useful in view of, for instance, the quadrature phase shift keying (QPSK) modulation, for which the SEP is given by ([1, Formula 8.19]):

$$P_s(E) = 2\,Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right), \tag{8}$$

where $E_s/N_0$ is the signal-to-noise ratio.

Each of the 5 equations (3)-(7) may be kept as basic definition and the other definitions, reported in the following, may be proved from that adding, when needed, the symmetry formula $Q(-x) = 1 - Q(x)$.

Finally, an integral expression of $Q(x)$ on $[0, +\infty)$ could be obtained (see Table 1) by the integral expression of $\mathrm{erfc}(x)$ in [43, Formula 7.7.1].

#### 2) A COMPLEX INTEGRAL DEFINITION

$$Q(z) := \frac{1}{2}\,\frac{1}{2\pi i}\, e^{-\frac{z^2}{2}} \int_{-\infty}^{(0+)} \frac{e^{\frac{z^2 p}{2}}}{\sqrt{p}\,(1-p)}\, dp$$

with the Hankel-type integration contour. This new [44] definition (Formula (4)) allows to express one of the classical integrals of telecommunications theory, involving the Nakagami-$m$ distribution, by means of the regularized incomplete beta function (see Formula (7) therein).

### 3) DEFINITIONS BY LIMITS

At least from the 3 equivalent definitions (5)-(7) by integrals on bounded intervals, the integration by rectangles immediately gives equivalent definitions by limits and without integrals. In particular, from (6):

$$Q(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{2}\, \mathrm{e}^{-\tilde{a}_i x^2} \qquad \tilde{a}_i := \frac{1}{2\,\sin^2\left(\frac{\pi j}{2N}\right)} \tag{9}$$

with $j = i-1$ or $j = i$ (which correspond, respectively, to left and right Riemann sums of the basic definition of the definite integral) reported in [8, Formula 13a] with the oversight $\pi\,(i-1)/(2N-2)$ instead of $\pi\,(i-1)/(2N)$, corresponding to the left Riemann sum.

Notice that substituting the limit with sufficiently large $N$ one obtains immediately approximations of $Q(x)$ as finite sums of exponential functions of the type *sum* $S(x)$ *of terms* $b_i\,\mathrm{e}^{a_i x^2}$ – called *Type 1* in Section VI, with $a_i$ and $b_i$ negative and positive constants, respectively, and said "Exponential Function Based Approximations" in [45] – which are very appreciated in telecommunications theory, because it is often interesting the averaging of $Q(\alpha\sqrt{x})$ on $[0, +\infty)$ weighted by a fading probability density function (pdf) $p_\gamma(x)$

$$\int_0^{+\infty} Q\big(\alpha\sqrt{\gamma}\big)p_\gamma(\gamma)\mathrm{d}\gamma$$

and $p_\gamma(x)$ is in turn often chosen in a set of classical functions (as, for example, the Raileigh pdf), which allows the expression of the above integral (among other analogous but more complex integrals, see, e.g., [45, Formulas 55a–55c]) in closed form in terms of standard – though not *elementary* – functions computed by software tools, as the hypergeometric function.

Since the integrand function in (6) is increasing, the substitution of the limit (9) with a finite sum up to $N$ gives an approximation of $Q(x)$ which is a lower bound when taking $j = i-1$ (left Riemann sum) and is an upper bound with the choice $j = i$ (right Riemann sum).

The simplest of those approximations of $Q(x)$, obtained with $N = 1$ and $j = i$ in (9), is the classical *improved Chernoff* (upper) bound [46]

$$Q_{\text{Chernoff–impr.}}(x) := \frac{1}{2}\,\mathrm{e}^{-\frac{x^2}{2}}, \tag{10}$$

whereas the more simple and even more classic [47] Chernoff (upper) bound

$$Q_{\text{Chernoff}}(x) := \mathrm{e}^{-\frac{x^2}{2}}, \tag{11}$$

given in [48, Formula 2-1-172], is a majorization of that.

(For the issue of the approximation names see Section IV-D.)

Then with $N = 2$ and $j = i$ one obtains the upper bound

$$Q_{\text{Chiani–1}}(x) := \frac{1}{4}\,\mathrm{e}^{-x^2} + \frac{1}{4}\,\mathrm{e}^{-\frac{x^2}{2}} \tag{12}$$

reported in Table 4, whereas with $j = i-1$ the lower bound

$$Q_{\text{Chang–new}}(x) := \frac{1}{4}\,\mathrm{e}^{-x^2} \tag{13}$$

is obtained, which belongs to the class [46] (For the issue of the name of approximation classes and of the name of new approximations (not already published) derived from the classes, see Section IV-D.)

$$Q_{\text{Chang–class}}(x; \alpha, \beta) := \frac{\alpha}{2}\,\mathrm{e}^{-\frac{\beta}{2}x^2}, \tag{14}$$

(where $\beta > 1$ and $0 < \alpha \le \sqrt{\frac{2\,\mathrm{e}}{\pi}}\,\frac{\sqrt{\beta-1}}{\beta}$) of lower bounds with $\alpha = \frac{1}{2}$ and $\beta = 2$ – the denominators 2 (see Table 1 for the mutual relation between $\operatorname{erfc}(x)$ and $Q(x)$) are due because (14) has been originally published for $\operatorname{erfc}(x)$ – and is overcome by

$$Q_{\text{Wu–1}}(x) := \frac{1}{4}\,\mathrm{e}^{-\frac{2}{\pi}x^2} \tag{15}$$

reported in Table 7.

### 4) DEFINITION BY A DIFFERENTIAL EQUATION

By (1), (2), and (3) it is

$$Q'(x) = -\phi(x) \tag{16}$$

which is a differential equation that, together with an initial value as for example $Q(0) = \frac{1}{2}$, is another definition of $Q(x)$.

### 5) THE RELATION WITH THE FUNCTION $\Phi(X)$

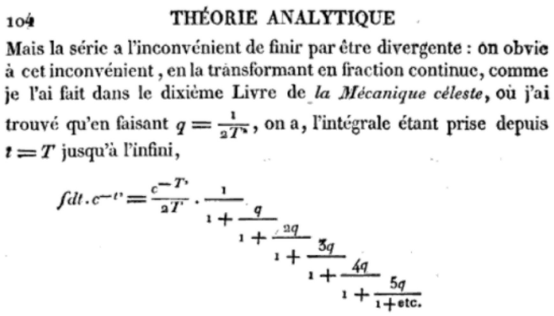
$$Q(x) := 1 - \Phi(x), \tag{17}$$

which is the same of:

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t, \tag{18}$$

where $\Phi(x)$ is the well known *normal cumulative distribution function*, called $P(x)$ in [31, Formula 26.2.2]. For that function one may find online lots of tables of numerical values, with different levels of accuracy, by searching images for "*normal table*".

By (17), any approximate or exact expression for $\Phi(x)$ of the form $\frac{1}{2} + \cdots$ gives for $Q(x)$ the correspondent form $\frac{1}{2} - \cdots$, which is quite frequent in the published approximations of $Q(x)$.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Mais la série a l'inconvénient de finir par être divergente : on obvie à cet inconvénient, en la transformant en fraction continue, comme je l'ai fait dans le dixième Livre de *la Mécanique céleste*, où j'ai trouvé qu'en faisant $q = \frac{1}{2T^2}$, on a, l'intégrale étant prise depuis $t = T$ jusqu'à l'infini,

$$\int dt.c^{-t^2} = \frac{c^{-T^2}}{2T}\cdot\cfrac{1}{1+\cfrac{q}{1+\cfrac{2q}{1+\cfrac{3q}{1+\cfrac{4q}{1+\cfrac{5q}{1+\text{etc.}}}}}}}$$

FIGURE 2. Continuous fraction expansion of the Gaussian integral in Laplace's *Théorie Analytique des Probabilités* (1812), [49, p. 104].

### 6) DEFINITION BY A CONTINUOUS FRACTION

$$Q(x) := \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}\left\{\frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\,\frac{4}{x+}\cdots\right\}$$

(see [31, Formula 26.2.14], originally given (1812) by Laplace ([49, p. 104]) for $\int \mathrm{e}^{-t^2}\mathrm{d}t$, and in Fig. 2 is reported the photograph of this latter formula) whose first convergent (truncation to the first term) is exactly the upper bound [50]

$$Q_{\text{Wozencraft}}(x) := \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x}, \tag{19}$$

treated in Sections VII-C, IX, and X-A, while the second convergent

$$\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}\,\frac{1}{x+\frac{1}{x}}$$

is the lower bound

$$Q^{\diamond}_{\text{Gordon}}(x) := \frac{x}{1+x^2}\cdot\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}, \tag{20}$$

proven by Gordon in 1941 [51] (the small rhombus in the name of the function is explained in Section IV-D) treated in Section X-B, which may be developed, for $x > 1$, as

$$\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}\,\frac{1}{x}\left(1-\frac{1}{x^2}+\cdots\right)$$

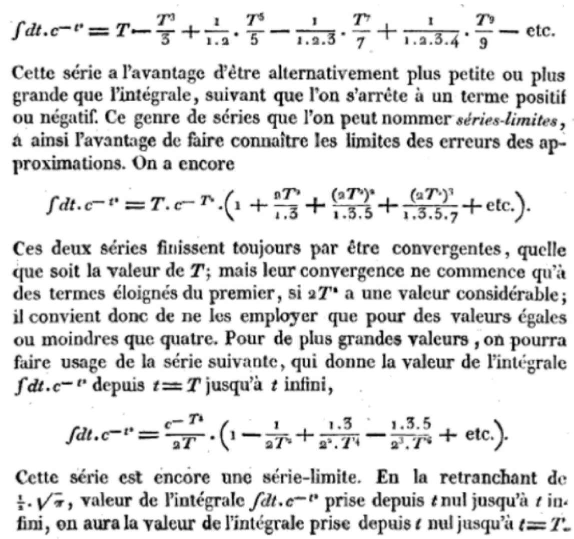
to be compared with the lower bound ([50, Formula 2.121]) for $Q(x)$:

$$Q^{\diamond}_{\text{Wozencraft-lower}}(x) := \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x}\left(1-\frac{1}{x^2}\right), \tag{21}$$

treated in Sections VII-C, IX, and X-B, which is exactly the truncation of (24) to the second term.

### 7) 3 EQUIVALENT DEFINITIONS BY MEANS OF POWER SERIES

$$Q(x) := \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{+\infty}\frac{(-1)^n\,x^{2n+1}}{n!\,2^n\,(2n+1)} \tag{22}$$


$$\int dt.c^{-t^2} = T - \frac{T^3}{3} + \frac{1}{1.2}\cdot\frac{T^5}{5} - \frac{1}{1.2.3}\cdot\frac{T^7}{7} + \frac{1}{1.2.3.4}\cdot\frac{T^9}{9} - \text{etc.}$$

Cette série a l'avantage d'être alternativement plus petite ou plus grande que l'intégrale, suivant que l'on s'arrête à un terme positif ou négatif. Ce genre de séries que l'on peut nommer *séries-limites*, a ainsi l'avantage de faire connaître les limites des erreurs des approximations. On a encore

$$\int dt.c^{-t^2} = T.c^{-T^2}.\left(1+\frac{2T^2}{1.3}+\frac{(2T^2)^2}{1.3.5}+\frac{(2T^2)^3}{1.3.5.7}+\text{etc.}\right).$$

Ces deux séries finissent toujours par être convergentes, quelle que soit la valeur de $T$; mais leur convergence ne commence qu'à des termes éloignés du premier, si $2T^2$ a une valeur considérable; il convient donc de ne les employer que pour des valeurs égales ou moindres que quatre. Pour de plus grandes valeurs, on pourra faire usage de la série suivante, qui donne la valeur de l'intégrale $\int dt.c^{-t^2}$ depuis $t = T$ jusqu'à $t$ infini,

$$\int dt.c^{-t^2} = \frac{c^{-T^2}}{2T}\cdot\left(1-\frac{1}{2T^2}+\frac{1.3}{2^2.T^4}-\frac{1.3.5}{2^3.T^6}+\text{etc.}\right).$$

Cette série est encore une série-limite. En la retranchant de $\frac{1}{2}.\sqrt{\pi}$, valeur de l'intégrale $\int dt.c^{-t^2}$ prise depuis $t$ nul jusqu'à $t$ infini, on aura la valeur de l'intégrale prise depuis $t$ nul jusqu'à $t = T$.

FIGURE 3. Series expansions of the Gaussian integral in Laplace's *Théorie Analytique des Probabilités* (1812), [49, p. 103].

which is a power series (see [31, Formula 26.2.10] for $\Phi(x)$).

$$\begin{aligned} Q(x) &:= \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}\sum_{n=0}^{+\infty}\frac{x^{2n+1}}{(2n+1)!!} = \\ &= \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}\left(x+\frac{x^3}{3}+\frac{x^5}{3\cdot 5}+\cdots\right. \\ &\qquad \left. + \frac{x^{2n+1}}{(2n+1)!!}+\cdots\right) \end{aligned} \tag{23}$$

which is $\frac{1}{2}-$ a power series multiplied by $\phi(x)$ (see [31, Formula 26.2.11] for $\Phi(x)$), originally given by Laplace ([49, 1812, p. 103]) for $\int \mathrm{e}^{-t^2}\mathrm{d}t$, and in Fig. 3 is reported the photograph of this latter formula.

$$\begin{aligned} Q(x) := \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}&\left(\frac{1}{x}-\frac{1}{x^3}+\frac{1\cdot 3}{x^5}-\frac{1\cdot 3\cdot 5}{x^7}+\cdots\right. \\ &\left. + \frac{(-1)^n\,(2n-1)!!}{x^{2n+1}}+\cdots\right) \end{aligned} \tag{24}$$

which is an asymptotic series multiplied by $\phi(x)$ (see [31, Formula 26.2.12] and [52, Formula 3]), originally given as well by Laplace ([49, p. 103]) for $\int \mathrm{e}^{-t^2}\mathrm{d}t$ (see Fig. 3).

Also other series expansions have been found for Gaussian integrals, see for example [31, Formula 26.2.13] and, with Hermite polynomials, but said "unpromising looking", a formula on [52, p. 402]. Finally, a series of functions for $Q(x)$ could be obtained (see Table 1) by the expansion of $\mathrm{erf}(x)$ in series of spherical Bessel functions, see [43, Formula 7.6.8].

### 8) DEFINITION BY A CLASS OF FUNCTION SERIES

Fixing [53] any $T > 0$, being $\omega = \frac{2\pi}{T}$,

$$Q(x) := \frac{1}{2} - \frac{2}{\pi}\sum_{n=1,\, n\ \text{odd}}^{+\infty}\frac{\sin(n\,\omega\,x)}{n}\,\mathrm{e}^{-\frac{n^2\omega^2}{2}}.$$

#### 9) DEFINITION IN TERMS OF RANDOM VARIABLES

$$Q(x) := \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \mathrm{e}^{-(x+X)^2}\, \mathrm{d}X$$

where $X$ is a standard normal random variable (see [8, p. 1274]).

### B. DIFFERENT NAMES OF THE FUNCTION Q(X)

The function $Q(x)$ is also known under several different equivalent names:

1) Gaussian $Q$-function;

one-dimensional

2) Gaussian $Q$-function;
3) Gaussian probability integral $Q(x)$;
4) tail probability of a normal distribution;
5) right normal tail integral;
6) complementary Gaussian cumulative distribution function;
7) survival function of a normal distribution;
8) reliability function $\bar{\Phi}(x)$ [54].

Notice that many Authors speak – essentially – of $Q(x)$ speaking of $\Phi(x) = 1 - Q(x)$ or of the *Mills' ratio*, defined in Section II-C.

Notice also that the function $Q(x)$ should be distinguished from these related functions with similar name:

1) first-order Marcum $Q$-function [55];
2) generalized Marcum $Q$-function [56];
3) Nuttall $Q$-function [57].

### C. THE FUNCTION Q(X) AND THE 4 CLASSICAL SPECIAL FUNCTIONS RELATED TO IT

There are 4 real functions classically related to $Q(x)$:

1) *Error function* ([31, Formula 7.1.1]):

$$\mathrm{erf}(x) := \int_0^x \frac{2}{\sqrt{\pi}} \mathrm{e}^{-t^2} \mathrm{d}t \tag{25}$$

2) *Complementary error function* ([31, Formula 7.1.2]):

$$\mathrm{erfc}(x) := \int_x^{+\infty} \frac{2}{\sqrt{\pi}} \mathrm{e}^{-t^2} \mathrm{d}t \tag{26}$$

3) *(Standard) normal cumulative distribution function* $\Phi(x)$ (18), already present in this form with the same name in [58, Formula (3) of the 1938 paper], translated in English in [59].
Analogously to (2) and (16), since $\Phi(x) = \int_{-\infty}^x \phi(t)\, \mathrm{d}t$ it is $\Phi'(x) = \phi(x)$.
4) *Standard normal Mills' ratio*, considered already in a 1926 paper [41] on *Biometrika* with the name $\mathcal{R}_x$:

$$m(x) := \mathrm{e}^{\frac{x^2}{2}} \int_x^{+\infty} \mathrm{e}^{-\frac{t^2}{2}} \mathrm{d}t = \frac{Q(x)}{\phi(x)}. \tag{27}$$

More generally, the Mills' ratio $m(x) := \frac{\int_x^{+\infty} f(t)\, \mathrm{d}t}{f(x)}$ is defined for any pdf $f(x)$ (see [3, Formula 36]), but in this paper it will be considered only with regard to the standard normal pdf.

Still notice the usual classical preference of statisticians for the function $\Phi(x)$, of physicists for the function $\mathrm{erf}(x)$, and of communications theorists for the functions $Q(x)$ and $\mathrm{erfc}(x)$, while a minority research line keeps on investigating the problem from the point of view of the Mills' ratio. These refer to essentially the same probabilistic problem seen from different perspectives. Furthermore, notice that in telecommunications theory the compound function $Q(\sqrt{x})$ is often considered.

*Remark 1:* The function $Q(x)$ and the 4 classical *special functions* related to it could be referred by the generic term "Gaussian integral", and even others, related, as

$$p(x) := \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \mathrm{e}^{-\frac{t^2}{2}}\, \mathrm{d}t = \int_{-x}^{x} \phi(t)\, \mathrm{d}t, \tag{28}$$

defined (1946) by Williams [60] (which the classical book [31] defines as $A(x)$ in Formula 26.2.4), and this other

$$G(x) := \int_0^x (2\pi)^{-\frac{1}{2}} \mathrm{e}^{-\frac{t^2}{2}}\, \mathrm{d}t = \int_0^x \phi(t)\, \mathrm{d}t, \tag{29}$$

defined (1945-46) by Pólya [61], which is $\frac{1}{2}p(x)$, $\frac{1}{2} - Q(x)$, and $\Phi(x) - \frac{1}{2}$ (which is unfortunately named $\Phi(x)$ in [62]).

### D. THE PROBABILISTIC MEANING OF THE FUNCTIONS Q(X), Φ(X), ERF(X), ERFC(X), AND M(X)

The wide recurrence in Sciences of the equivalent functions $Q(x)$, $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and $m(x)$, is – at a deep level – due to the Central Limit Theorem. The probabilistic nature of the function $Q(x)$ which causes its relevance in communications theory is herein explained, together with the other related *special functions*.

Let us consider a random variable $X$ with standard normal distribution, and then probability density function (pdf) $\phi(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{t^2}{2}}$, then:

$$\begin{aligned}
\Pr\{X \geq x\} &= Q(x) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \\
\Pr\{X \leq x\} &= \Phi(x) = 1 - Q(x) = \\
&= \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) = 1 - \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \\
m(x) &= \frac{1}{h(x)}
\end{aligned}$$

being the reciprocal of the Mills' ratio, $h(x)$, the so-called *failure* or *hazard rate* for a standard normal random variable [54].

Things may be considered in more general terms for a normal random variable with mean $m$ and variance $\sigma^2$ by the Formula 26.2.8 of classical 1964 book [31]:

$$\begin{aligned}
\Pr\{X \leq x\} &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{(t-m)^2}{2\sigma^2}}\, \mathrm{d}t = \cdots \\
&= P\left(\frac{x-m}{\sigma}\right)
\end{aligned}$$

where $P(\cdot)$ is exactly $\Phi(\cdot)$, in nowadays standard. Taking the complement to unity, one has exactly the probabilistic meaning of the function $Q(x)$

$$\Pr\{X \geq x\} = Q\left(\frac{x-m}{\sigma}\right)$$

with respect to a generic normal random variable with mean $m$ and variance $\sigma^2$.

### E. THE GENERAL BEHAVIOUR OF THE FUNCTIONS Q(X), Φ(X), ERF(X), ERFC(X), AND M(X)

All these functions are continuous – and even smooth – and strictly monotonic with these values or limits in $-\infty$, 0, and $+\infty$, respectively [16]:

$$\begin{array}{llll} \mathrm{erf}(x): & -1 & 0 & 1 \\ \Phi(x): & 0 & \frac{1}{2} & 1 \\ Q(x): & 1 & \frac{1}{2} & 0 \\ \mathrm{erfc}(x): & 2 & 1 & 0 \\ m(x): & +\infty & \sqrt{\frac{\pi}{2}} & 0 \end{array}$$

The Reader is kindly addressed to [16] for:

1) The remark that unluckily there are ambiguities in the definitions in the literature (see also the Remark in Section II-C). Here we notice in particular that the classical book [63] (p. 341) defines $\mathrm{erfc}(x)$ without the factor $2/\sqrt{\pi}$, and the classical Hastings' work [64] (p. 185, Sheet 61) defines $\Phi(x)$ the (restriction to $x \geq 0$ of) $\mathrm{erf}(x)$. Nevertheless, the definition of $\mathrm{erfc}(x)$ reported in [16] has to be considered an oversight.
2) The remark that the inverse of an approximation of an invertible function $f(x)$ is an approximation (how good, it has to be seen) of the inverse of $f(x)$.
3) The mutual relations among $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, $\Phi(x)$, and $Q(x)$, in particular:

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) \tag{30}$$
$$Q(x) = 1 - \Phi(x) \tag{31}$$
$$\mathrm{erf}(x) = 2\,\Phi\left(x\sqrt{2}\right) - 1 \tag{32}$$
$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) \tag{33}$$

Here we add also this mutual relation between the Mills' ratio $m(x)$ and $Q(x)$:

$$m(x) = Q(x)\sqrt{2\pi}\,\mathrm{e}^{\frac{x^2}{2}} \tag{34}$$

and these 2 (since of common use in communications theory):

$$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{35}$$
$$\mathrm{erfc}(x) = 2\,Q\left(x\sqrt{2}\right) \tag{36}$$

In Table 1 we report all the 20 mutual relations among the 5 functions $Q(x)$ (3), $\mathrm{erf}(x)$ (25), $\mathrm{erfc}(x)$ (26), $\Phi(x)$ (18), and $m(x)$ (27), and in Table 2 the 12 mutual relations among the inverse functions of the first 4, being the Mills' ratio $m(x)$ (27) not *explicitly invertible* neither by means of elementary functions, nor by the inverses of $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and $Q(x)$ (and even by the Lambert $W$-function (see Section V)).

By means of the 4 simple relations listed in the first column of Table 1, the above listed 4 classical *special functions* $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, $\Phi(x)$, and $m(x)$ in (25), (26), (18), and (27), respectively, can all be approximated as a byproduct of the approximation of $Q(x)$. But the accuracy of such approximations may be severely limited by the considerations exposed in Section VII-A, and of course the complexity of the analytical expressions of the derived approximations may be reduced or increased by the considerations exposed in Section VIII, and the domain may change because of the multiplicative factor $\sqrt{2}$.

### F. THE SYMMETRY FORMULAS

1) When considering exact values, it is sufficient to consider $\mathrm{erf}(x)$ for $x \geq 0$ because this function is odd:

$$\mathrm{erf}(-x) = -\mathrm{erf}(x).$$

Likewise, it is sufficient to consider the other 4 functions $Q(x)$ (3), $\mathrm{erfc}(x)$ (26), $\Phi(x)$ (18), and $m(x)$ (27) for $x \geq 0$ because of the other (following) symmetry formulas.
2) The symmetry formula for the function $\Phi(x)$ is

$$\Phi(-x) = 1 - \Phi(x),$$

from which obviously follows
3) the symmetry formula for the function $Q(x)$:

$$Q(-x) = 1 - Q(x),$$

4) the symmetry formula for the function $\mathrm{erfc}(x)$:

$$\mathrm{erfc}(-x) = 2 - \mathrm{erfc}(x),$$

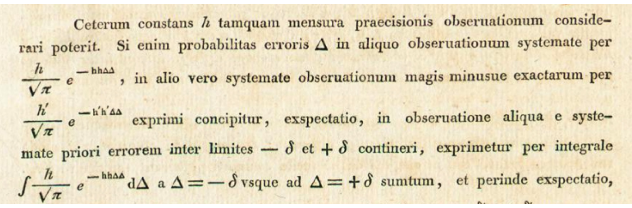
5) and, finally, the symmetry formula for the function $m(x)$:

$$m(-x) = \sqrt{2\pi}\,\mathrm{e}^{\frac{x^2}{2}} - m(x).$$

Thus, as a consequence of the above listed symmetry formulas, for the functions $\mathrm{erf}(x)$, $\Phi(x)$, $Q(x)$, $\mathrm{erfc}(x)$, and $m(x)$ it is sufficient – at least from the point of view of absolute errors (see Section VII-A) – to give approximations and bounds for $x \geq 0$.

### G. A HISTORICAL NOTE

Already in ancient times it was well known the fact that for several physical (natural) quantities the extreme values, low or high, tend to be rare, whereas mean values to be common, which is the general meaning of "bell shaped" densities.



Ceterum constans $h$ tamquam mensura praecisionis obseruationum considerari poterit. Si enim probabilitas erroris $\Delta$ in aliquo obseruationum systemate per $\frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}$, in alio vero systemate obseruationum magis minusue exactarum per $\frac{h'}{\sqrt{\pi}} e^{-h'h'\Delta\Delta}$ exprimi concipitur, exspectatio, in obseruatione aliqua e systemate priori errorem inter limites $-\delta$ et $+\delta$ contineri, exprimetur per integrale $\int \frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta} d\Delta$ a $\Delta = -\delta$ vsque ad $\Delta = +\delta$ sumtum, et perinde exspectatio,

**FIGURE 4.** **"Furthermore the constant $h$ can be considered as the measure of precision of the observation. Then, if the probability of an error $\Delta$ in any system of observations $[\cdots]$ is supposed to be expressed by $\frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}$ $[\cdots]$ the expectation $[\cdots]$ that the error is contained between the limits $-\delta$ and $+\delta$ is expressed by the integral $\int \frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}\, d\Delta$ taken from $\Delta = -\delta$ to $\Delta = +\delta$ $[\cdots]$." [66].**

We read in the classical book *Phaidon* (in [65]) of Plato, V century BC:

> "Of all such things, the extremes of the extremes are rare and few, but those in between are abundant and many."

The meaning of the Gaussian density and the Gaussian integrals, as a measure of the probability of errors, was already clear to Gauss at the beginning of the XIX century. He considered ([66], 1809) the probability density function (see Fig. 4)

$$\frac{h}{\sqrt{\pi}}\, \mathrm{e}^{-h^2 x^2},$$

which is exactly the normal density (with mean 0) expressed in modern terms as $\frac{1}{\sqrt{2\pi}\,\sigma}\, \mathrm{e}^{-\frac{x^2}{2\sigma^2}}$ with the correspondence $h = \frac{1}{\sqrt{2}\sigma}$ and, in particular, is the standard normal density $\phi(x)$ (1) if $\sigma = 1$.

Some years later (1812) the continuous fraction expansion (see Section II-A6) and the series expansion (see Section II-A7) of the Gaussian integral were published by Laplace [49] (see Figs. 2 and 3, respectively).

Historically, the first significant bound [50] for $Q(x)$ is Wozencraft (upper) bound $Q_{\text{Wozencraft}}$ (19) (treated in Section X-A), already present as a bound, limited to $x > 1$, for $Q(x)$ (expressed as $1 - \Phi(x)$) in the 1938 paper [58] translated in English in [59] and stated $\forall x > 0$ for $Q(x)$ (expressed by its integral definition) in 1941 paper [51].

Afterwards (1941) the Gordon lower bound $Q^{\diamond}_{\text{Gordon}}$ (20) [51] was published, and then (1942) also the Birnbaum lower bound [67]

$$Q^{\diamond}_{\text{Birnbaum}}(x) := \frac{\sqrt{4 + x^2} - x}{2} \cdot \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}, \tag{37}$$

both treated in Section X-B.

The successive significant result in chronological order is due to Pólya [61] (1945-46), who proved that for (29) holds

$$G(x) < \frac{1}{2}\left(1 - e^{-2x^2/\pi}\right)^{1/2} \qquad \forall x > 0$$

which, with $Q(x) = 1/2 - G(x)$, is equivalent to

$$Q(x) > \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mathrm{e}^{-2x^2/\pi}} \qquad \forall x > 0. \tag{38}$$

Williams [60] (1946) – independently from Pólya according to Chu [68] – proved similarly that for (28) holds

$$p(x) \le \left[1 - e^{-(2/\pi)\,x^2}\right]^{1/2}$$

which, with $Q(x) = \frac{1}{2} - \frac{1}{2}p(x)$ for $x \ge 0$, is equivalent to (38), with $x = 0$ included.

The successive significant progress is in the 1950's with the so-called Chernoff (upper) bound [47] (see also [48, Formula 2-1-172]) $Q_{\text{Chernoff}}$ (11), more correctly called *Chernoff-Rubin bound* in [11] when equivalently expressed for $\operatorname{erfc}(x)$, since Chernoff himself wrote:

> "the so-called Chernoff bound which was really Rubin's result"

in his short biography *A career in statistics* published in [69].

## III. DOMAIN AND VALUES OF THE FUNCTION $Q(X)$ FOR INFORMATION AND COMMUNICATIONS THEORY

The function $Q(x)$ is defined everywhere on the set of real numbers $\mathbb{R}$.

In this paper the function $Q(x)$ is considered only for $x \ge 0$ or $x > 0$ and notice that, giving up to this latter assumption (often implicit in information and communications theory), several formulas cease to hold, and in particular the so-called Wozencraft bound $Q_{\text{Wozencraft}}$ (19).

### *A. DOMAIN OF PRACTICAL INTEREST OF THE FUNCTION Q(X) IN INFORMATION AND COMMUNICATIONS THEORY*

In information and communications theory, only the domain of the positive numbers $x > 0$ is significant, and, as far as the practical use of the function $Q(x)$ is concerned, we could take as reference the BEP holding for the simple binary digital modulation schemes (see [1, Formula 8.18]):

$$P_b(E) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right), \tag{39}$$

already present in 1965 Wozencraft and Jacobs book [50] in the form (see Formula 2.120c) $P[\varepsilon] = Q(\sqrt{\frac{E_b}{\sigma^2}})$, where $E_b/N_0$ is the signal-to-noise ratio, $E_b$ is the signal energy associated to a bit, and $\sigma^2 = N_0/2$ is the variance of the channel noise.

Calling $\gamma$ the signal-to-noise ratio $E_b/N_0$, (39) may be rewritten as

$$P_b(E) = Q\left(\sqrt{2\gamma}\right). \tag{40}$$

Assuming for $\gamma$, as already done in [22], the following significant range

$$I^{\gamma}_{\text{significant}} := [-10, 10]\text{dB} = \left[\frac{1}{10}, 10\right],$$

the minimum value of the argument $\sqrt{2\gamma}$ of the $Q$-function is

$$\sqrt{2 \cdot \frac{1}{10}} \approx 0.45 \approx 0.5$$

and its maximum value is

$$\sqrt{2 \cdot 10} \approx 4.47 \approx 4.5.$$

In this paper, we mainly focus on the approximation of $Q(x)$ in the above mentioned $x$ range, i.e., on the domain of interest

$$I_{\text{significant}} := [0.45, 4.5].$$

In the literature, other significant intervals have been considered, e.g., [0, 10] dB in [29] and in [45, Table I], and [0, 4] dB in [46], which are strictly contained in $I_{\text{significant}}$.

### B. NOTABLE VALUES IN THE DOMAIN OF INTEREST

We observe that, with 3 significant digits,

$$Q(0.45) \approx 0.326$$

and that

$$Q(4.5) \approx 3.40 \cdot 10^{-6}.$$

Thus, the range of the restriction of the decreasing function $Q(x)$ to the domain of interest $I_{\text{significant}}$ is

$$Q([0.45, 4.5]) \approx [0.00000340, 0.326].$$

Another numerical value of some interest is

$$\Pr(X \geq 1) = 1 - \Phi(1) = Q(1) \approx 0.16,$$

being $X$ a standard normal random variable (see Section II-D), immediately related to the classical probabilistic and statistical relation holding for $X$:

$$\begin{aligned} \Pr(-1 \leq X \leq 1) &= 1 - \Pr(|X| > 1) \\ &= 1 - 2 \cdot \Pr(X > 1) \\ &\approx 1 - 2 \cdot 0.16 = 0.68 \end{aligned} \tag{41}$$

(well known from the classical 68-95-99.7 rule[1]) which is, also by means of the symmetry formula, equal to:

$$\begin{aligned} \Pr(-1 \leq X \leq 1) &= \Phi(1) - \Phi(-1) \\ &= \Phi(1) - (1 - \Phi(1)) \\ &= 2\,\Phi(1) - 1. \end{aligned} \tag{42}$$

From (41) and (42) it follows that

$$\Phi(1) \approx \frac{1 + 0.68}{2} \approx 0.84.$$

From (32), (41), and (42) it follows that:

$$\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) = 2\,\Phi(1) - 1 \approx 0.68$$

and from the last and (33) that:

$$\operatorname{erfc}\left(\frac{1}{\sqrt{2}}\right) \approx 0.32.$$

1. "This empirical rule is the facts that 68.27%, 95.45% and 99.73% of the values in a normal distribution fall within one, two and three standard deviations of the mean, respectively" [70].

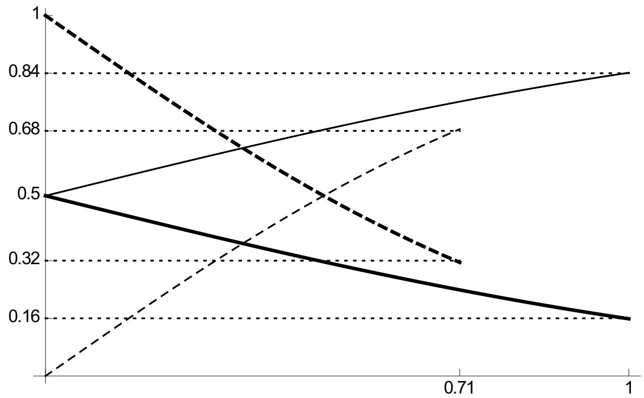



**FIGURE 5.** Some values on graphs of $Q(x)$, $\Phi(x)$, erf$(x)$, and erfc$(x)$. In the order: continuous thick, continuous thin, dashed thin, dashed thick.

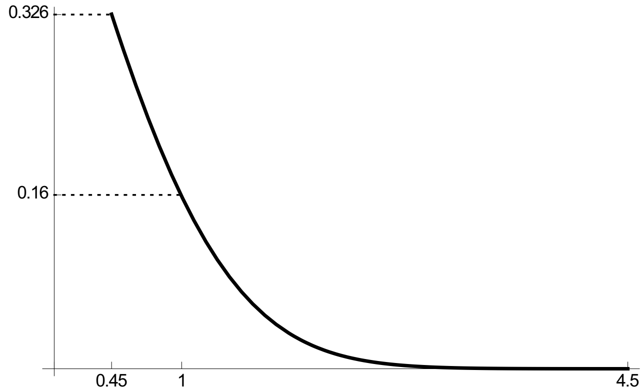



**FIGURE 6.** Graph of $Q(x)$ in the domain of interest [0.45, 4.5].

Remembering, besides these, the obvious and already given (see Section II-E) notable values erf$(0) = 0$, $Q(0) = \Phi(0) = 0.5$ and erfc$(0) = 1$, to draw approximately the graphs of the 4 considered functions, notice that 0, 0.16, 0.32, 0.5, 0.68, 0.84 and 1 are quite near to $\frac{0}{6}$, $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, $\frac{6}{6}$, and that $\frac{1}{\sqrt{2}} \approx 0.71$ is exactly half of the diagonal of the square of vertices $(0, 0)$ and $(1, 1)$. In Fig. 5 are reported parts of the graphs of the functions $Q(x)$, $\Phi(x)$, erf$(x)$, and erfc$(x)$. More accurate values for 0.16, 0.32, 0.5, 0.68, 0.84 are, respectively: 0.159, 0.315, 0.500, 0.685, 0.841.

Finally, from (34) it follows that, for the Mill's ratio, it is:

$$m(1) = Q(1)\sqrt{2\pi}\,e^{\frac{1}{2}} \approx 0.66,$$

still not far from $\frac{4}{6} \approx 0.67$.

The Reader may find some useful graphs in Figs. 6, 7, and 8. In particular, in Fig. 8 is also reported the graph of $\ln Q(x)$, which is the function that has been tentatively approximated in *Type 2* approximations (see Section VI).

Richer lists of selected values of $Q(x)$ are in [71, Tables II.3-1 and II.3-2] and an even richer table may be found in [14, Appendix E].

If one needs a single or few values of $Q(x)$, a good choice is to compute it online for free using WolframAlpha

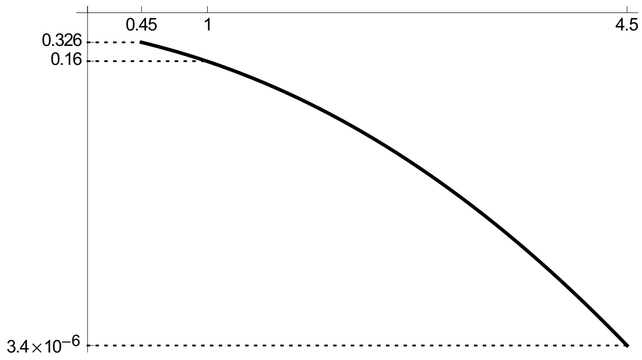



**FIGURE 7.** Log-plot of the graph of $Q(x)$ in the domain of interest [0.45, 4.5].

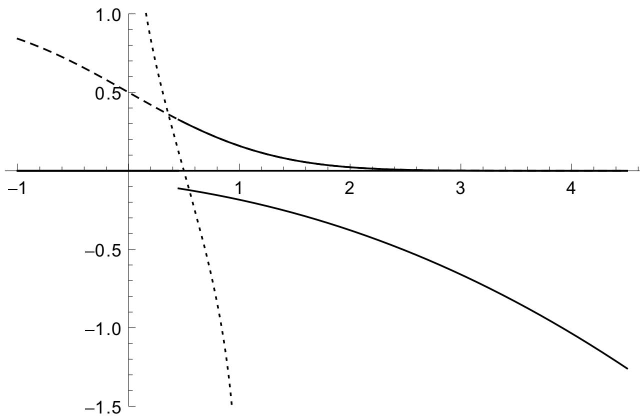



**FIGURE 8.** Graphs of: $Q(x)$ in [0.45, 4.5], positive, continuous; the 10-th part of $\ln Q(x)$ in [0.45, 4.5], negative, continuous; $Q(x)$ before 0.45, dashed; $Q^{-1}(y)$, dotted.

with the instruction `0.5Erfc[x/Sqrt[2]]`, and to compute a value of $Q^{-1}(y)$ one may use the instruction `Sqrt[2]InverseErf[1-2y]`.

## IV. APPROXIMATION OF THE FUNCTION *Q*(*X*)

### A. MOTIVATION

Due to the mathematical complexity associated with the various equivalent definitions of the function $Q(x)$, as presented in Section II-A, and the well-established fact that the $Q$-function cannot be expressed using elementary functions, the academic literature has proposed several approximations and bounds for the Gaussian $Q$-function. These approximations and bounds are introduced in Section II and discussed throughout this survey. Closed-form expressions are crucial in the evaluation of communication systems as they simplify mathematical analysis, hence facilitating performance assessment. In addition, it is important to have closed-form formulations of the system performance in order to facilitate system optimization.

### B. APPLICATION EXAMPLES

It is decisive to have extremely effective closed-form approximations of the Gaussian $Q$-function in the additive white Gaussian noise (AWGN) channel scenario in order to calculate the error probability for a variety of digital modulation methods [1]. Various researchers have suggested different approximations or bounds for the Gaussian $Q$-function to facilitate a basic mathematical analysis of error probabilities. These approximations or bounds are also applicable to evaluating the Symbol Error Probability (SEP) for more complex higher order constellations, as discussed in Section I-A.

Indeed, commencing with elementary binary modulation schemes such as binary amplitude modulation (2-AM) and binary phase shift keying (BPSK), the bit error probability (BEP) is associated with the Gaussian $Q$-function (see to (39)):

$$P_b(E) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

In the context of complex higher order constellations, there is a significant interest in developing accurate closed-form approximations also for integer powers of the Gaussian $Q$-function. This is particularly important because the symbol error probability (SEP) in quadrature phase shift keying (QPSK) modulation relies on the Gaussian $Q$-function and its square, as shown in equation (8):

$$P_s(E) = 2\,Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right),$$

or on the integer powers of the Gaussian $Q$-function up to $Q^4(x)$ as far as the differentially encoded QPSK modulation is concerned (see, e.g., [1, Formula 8.38]):

$$P_s(E) = 4\,Q\left(\sqrt{\frac{E_s}{N_0}}\right) - 8\,Q^2\left(\sqrt{\frac{E_s}{N_0}}\right) + 8\,Q^3\left(\sqrt{\frac{E_s}{N_0}}\right) - 4\,Q^4\left(\sqrt{\frac{E_s}{N_0}}\right). \quad (43)$$

### C. FUNDAMENTAL ATTRIBUTES OF AN APPROXIMATION OF Q(X)

In this survey, we will focus on various approximations or bounds for the Gaussian $Q$-function that have been proposed in the literature. We will exclude bounds or approximations specifically targeting integer powers of the Gaussian $Q$-function. This decision is also influenced by the Remark on the sum of powers of $Q(x)$ discussed in Section VII-C.

In this section, we present a set of four essential attributes that, in our perspective, an approximation or bound – published for $Q(x)$ or the other 4 related functions $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$ and Mills' ratio $m(x)$, from which $Q(x)$ may be immediately obtained (see Table 1) – should possess. Indeed, these attributes aim to facilitate the mathematical analysis and enhance the ease of computing the aforementioned exemplifying formulas. Precisely, an approximation of $Q(x)$ should:

1) hold at least for any $x > 0$;
2) be defined by a single expression, that is to say not piecewise defined;

3) be defined in closed form by means of elementary functions with standard names used in mathematics (for the issues of elementary functions and standard names used in mathematics see Remarks 1 and 2 in Section V, respectively);
4) be *explicitly invertible* (see Definition in Section V).

For comparison, in this paper we also review several other approximations of $Q(x)$ not having all the above stated properties.

*Remark 1:* The above Properties 3 and 4 together imply that the approximations are continuous functions.

*Remark 2:* The above Property 2 excludes for example [62][2]

$$Q_{\text{Cadwell}}(x) := \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mathrm{e}^{-\frac{2}{\pi}x^2 + \frac{2(\pi-3)}{3\pi^2}x^4}}$$

not defined for $x > 8.158\cdots$ (and similarly its refinement in the same [62], involving an 8-th degree polynomial). Another published [21] approximation not fulfilling Property 2 is

$$Q_{\text{Lin-3}}(x) := \frac{1}{1 + \mathrm{e}^{\frac{4.2\pi x}{9-x}}}$$

which is defined for $0 \le x < 9$ (and, in theory, also for $x > 9$, becoming an extremely bad approximation). (By the same Author, $Q_{\text{Lin-1}}(x)$ and $Q_{\text{Lin-2}}(x)$ are listed in Tables 6 and 8, respectively; for the issue of the names of the approximations, see Section IV-D.)

*Remark 3:* An example of approximation [72] not fulfilling the above Property 3 is this classical scholastic

$$Q^{\diamond}_{\text{Shah}}(x) := \begin{cases} \frac{1}{2} - \frac{x(4.4-x)}{10} & 0 \le x \le 2.2 \\ 0.01 & 2.2 < x < 2.6 \\ 0 & x \ge 2.6 \end{cases} \tag{44}$$

originally published for $\Phi(x) - \frac{1}{2}$, and for that function even not so bad in the sense of the relative error. (The small rhombus in the name of the function is explained in Section IV-D.)

Another example [73] of piecewise – therein said "composite" – approximation, much more recent and precise, and published directly for $Q(x)$ in the environment of telecommunications theory, is

$$Q^{\diamond}_{\text{Peric}}(x) := \begin{cases} \mathrm{e}^{-0.35054x^2 - 0.78995x - 0.69354} & 0 \le x \le 0.7 \\ \frac{1}{2\pi}\frac{\mathrm{e}^{-\frac{x^2}{2}}}{0.6797x + 0.3202\sqrt{x^2+5.9735}} & x > 0.7 \end{cases} \tag{45}$$

## D. ON THE NAMES OF THE APPROXIMATIONS

The many published approximations of $Q(x)$ found in this research have been (re)named in the form $Q_{<\cdot>}$ and precisely:

- $Q_{\text{<Author>}}$: as a general rule for the names of the approximations, they have been named with the letter $Q$ with a subscript reporting the name of the first Author of the paper in which the approximation was introduced – eventually for any equivalent function $\Phi(x)$, $\mathrm{erfc}(x)\cdots$ – as for instance $Q_{\text{Powari}}$ reported in Table 4, eventually adding a cardinal number if there are more approximations with the same first Author's name: e.g., $Q_{\text{Benitez–1}}$ and $Q_{\text{Benitez–2}}$, reported in Table 6.

For clarity, we have made 2 exceptions:

- the very classical so-called *Chernoff bound* $Q_{\text{Chernoff}}$ (11) follows the above said rule, but for the so-called *improved Chernoff bound* (10) the name $Q_{\text{Chernoff–impr.}}$ has been used (both shown in Table 7, see also Section X-A);
- analogously, the very classical upper bound $Q_{\text{Wozencraft}}$ (19) follows the above said rule, but for the lower bound (21) due to Wozencraft the name $Q^{\diamond}_{\text{Wozencraft–lower}}$ has been used (see also Sections X-A and X-B, respectively).

Moreover:

- $Q_{\text{<Author–class>}}$: such a name has been used for classes (or, families) of approximations, as for instance $Q_{\text{Chang–class}}$ (14), eventually adding a cardinal number if there are more classes with the same first Author's name: e.g., $Q_{\text{Wu–class–1}}$ (70) and $Q_{\text{Wu–class–2}}$ (71);
- $Q_{\text{<Author–new>}}$: such a name has been used for a new approximation (not already published), chosen by us in a class of approximations, as for instance $Q_{\text{Chang–new}}$ (13), eventually adding a cardinal number if the here chosen new approximations are more than one: e.g., $Q_{\text{Chiani–new–1}}$ (65) and $Q_{\text{Chiani–new–2}}$ (66);
- $Q_{\text{<Author–[number–]equivalent>}}$: such a name has been used for an alternative analytic expression of an approximation $Q_{\text{<Author–[number]>}}$: e.g., $Q_{\text{Chiani–1}}$ (12) and $Q_{\text{Chiani–1–equivalent}}$ (68);
- $Q_{\text{<Author–inverted>}}$: such a name has been used for the only case in which the new approximation has been obtained (in this research) inverting an approximation of $Q^{-1}(y)$: $Q_{\text{Hamaker–inverted}}$ (47) (see Section IV-G);
- $Q^{\diamond}_{\text{<Author>}}$: such a name has been used for any not *explicitly invertible* (see Section V) published approximation, for instance $Q^{\diamond}_{\text{Gordon}}$ (20).

## E. VALUABLE MERITS OF AN APPROXIMATION OF Q(X)

Already [9] has considered some desirable characteristics that the approximations of $Q(x)$ and $\mathrm{erfc}(x)$ should have: namely, being highly accurate, *exponential-type* (see Remark 1 in Section VI), closed-form approximations for the function $\mathrm{erfc}(x)$ (strictly speaking $\mathrm{erfc}\sqrt{x}$ is considered in [9]) that are easily (*explicitly*, see Remark 3 in Section V) invertible, differentiable, and facilitating the statistical averaging over fading distributions. Moreover, considering also [74] and [75], where we have reported some other merits an approximation should have, referred to another function (used in information theory), here we present, extending the 4 characteristics listed in Section IV-C, a tentatively all-encompassing list of valuable merits – from

2. Originally published for $\frac{1}{2} - Q(x)$, unfortunately named $\Phi(x)$ as explained in the Remark of Section II-C.

the point of view of telecommunications theory – which should be possessed by an approximation of $Q(x)$, or of the other 4 related functions $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and Mills' ratio $m(x)$, from which an approximation of $Q(x)$ may be immediately (see Table 1) obtained:

1) to be appreciable - with low error(s) - on a wide domain: the best would be $\mathbb{R}$, subordinately $x \geq 0$, then $x > 0$, then $(0, b]$ with (large) $b > 0$, and, finally, $[a, b]$ with (small) $a > 0$ as $I_{\text{significant}}$;
2) to be defined by a single expression, i.e., not piecewise;[3]
3) to be expressed in closed form, without integrals, series, continuous fractions, and limits, by means of elementary functions with standard names used in mathematics (for the issues of elementary functions and standard names used in mathematics see Remarks 1 and 2 in Section V, respectively);
4) to allow the expression in closed form[4] of integrals involved in the error probabilities [45] computations as
$$\int_0^{+\infty} \tilde{Q}\big(\alpha\sqrt{\gamma}\big)\, p_\gamma(\gamma)\, d\gamma \tag{46}$$
where $\tilde{Q}(x)$ is the chosen approximation of $Q(x)$ and $p_\gamma(\gamma)$ is a fading probability density distribution (see Section II-A3);
5) to be as accurate as possible, i.e., present a low relative error in absolute value and/or a low absolute error, on some domain, particularly $I_{\text{significant}}$: this topic is treated in Section VII;
6) to be as simple as possible: this topic is treated in Section VIII;
7) to be *explicitly invertible* (see Definition in Section V) and, possibly, with an inversion as simple as possible, in particular, even to have only one entry of $x$: this topic is treated in Section IX;
8) to be an upper bound of $Q(x)$ (more widely used in telecommunications theory) and, alternatively, a lower bound: this topic is treated in Section X.

Notice that almost all the listed valuable merits may also be considered – mutatis mutandis – for the approximations of the integer powers of $Q(x)$, which play a role in telecommunications theory, too, but will not be treated in this paper.

### F. UTILITY OF EXPLICITLY INVERTIBLE APPROXIMATIONS OF Q(X)

An important need of a communications system designer is to derive, given the target symbol error or bit error probability of a communications system, the operating signal-to-noise ratio needed by the considered application. For this purpose, the simple and *explicit invertibility* of the approximation of the Gaussian $Q$-function considered assumes a very important role. Already many papers and books as [6], [9], [11], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39] explicitly address the inversions of the approximations of $Q(x)$ or $\Phi(x)$ or $\mathrm{erf}(x)$ or $\mathrm{erfc}(x)$ (and $\mathrm{erfc}\sqrt{x}\cdots$) – which are all equivalent by formulas of Table 2 – and in particular [6], [9], [11], [22], and [29] considered the importance of the explicit inversion in the environment of communications theory.

The first two (namely, [6] and [9]) of them address the problem of developing highly accurate, *exponential-type* (see *Type 1* in Section VI with the Remark 1) sums of $a_k\, \mathrm{e}^{-k\, b\, x^2}$, with positive integer $k$ and $b > 0$ (corresponding to our $b_i\, \mathrm{e}^{a_i\, x^2}$), closed-form approximations for the function $\mathrm{erfc}(x)$ (strictly speaking $\mathrm{erfc}\sqrt{x}$ is considered) that are easily (explicitly, see Remark 3 in Section V) invertible, differentiable and facilitating the statistical averaging over fading distributions (in the sense of the explicit integration in (46)). (In [9], in the Formula (2) for $\mathrm{erfc}\sqrt{x}$ the term $\sqrt{2\,x}$ is written as $\sqrt{2}\,x$ for an oversight.) As well explained in [9], the explicit invertibility of the $Q$-function can also

> "facilitate discrete-rate adaptive modulation designs and establish a closed-form relation between the outage probability measure and symbol error rate (SER) of common digital modulation schemes such as M-ary phase shift keying (MPSK) and M-ary quadrature amplitude modulation (MQAM)".

The third paper (i.e., [11]) addresses the problem of finding bounds of the type called *Type 1* in this paper (see Section VI), for $\mathrm{erfc}(x)$ and its inverse over additive white Gaussian noise channels: in this sense these bounds, or better the bounds for $Q(x)$ immediately obtained by

$$\begin{gathered} u(x) < \mathrm{erfc}(x) < v(x) \\ \Rightarrow \frac{1}{2}\, u\!\left(\frac{x}{\sqrt{2}}\right) < Q(x) < \frac{1}{2}\, v\!\left(\frac{x}{\sqrt{2}}\right) \end{gathered}$$

have the desirable attributes, stated in [9], that a $Q$-function approximation should have: more precisely, the bounds are in the form of the sum of exponential functions that are highly accurate, *exponential-type*, closed-form and easily invertible (of course, implicitly in the sense of *explicitly invertible*).

Paper [22] in the environment of communications theory determines $I_{\text{significant}}$ (see Section III-A) and optimizes the coefficients of the nested exponential (*Type 6* of Section VI) $Q_{\text{Soranzo–3}}$ of [16] to minimize the relative error for $Q(x)$ in that interval, preserving the same invertibility level (see Section IX).

Finally, paper [29] deals with a lower bound on the $Q$-function, and presents a method to optimize its accuracy and

3. If the approximation $a(x)$ is defined on its domain by a single expression, in any point $x$ it has always an analytical expression allowing its simple algebraic manipulation, in contrast to piecewise defined functions [76], having different analytical expressions depending on the value taken by the argument $x$. Unfortunately, the distinction between piecewise and not piecewise defined functions, which is intrinsic using elementary functions, is not intrinsic for non elementary functions because of the existence of the Heaviside step function.

4. Obviously it is meant in closed form by means of functions having standard names in mathematics, even if not elementary, in particular the hypergeometric function.

an iterative procedure to produce its inverse. Furthermore, therein it is noticed that

> "There are also applications in which one has to invert the [error probability] $P(e)$ expressions in order to determine the change in signal-to-noise ratio (SNR) required to maintain a certain $P(e)$ value due to changes in channel parameters. Although some commercial software has a built-in inverse Gaussian $Q$-function, online applications in receivers usually lack the software, high processing power or storage space. Hence, a tight invertible bound on the Gaussian $Q$-function or a simple inversion algorithm is desirable."

### G. APPROXIMATING Q(X) INVERTING AN APPROXIMATION OF $Q^{-1}(Y)$

In theory it would be possible to search for approximations of the inverse $Q^{-1}(y)$ or equivalently (see Table 2) of $\Phi^{-1}(y)$, $\mathrm{erf}^{-1}(y)$, and $\mathrm{erfc}^{-1}(y)$. If these approximations were *explicitly invertible*, we could obtain, with the inversion of the inverse, approximations of $Q(x)$ or equivalently (see Table 1) of $\Phi(x)$, $\mathrm{erf}(x)$, and $\mathrm{erfc}(x)$. As far as we know, not very much has been done in this sense.

We start citing these 2 simple approximations of $x = Q^{-1}(y)$, reported in [35] for $\Phi^{-1}(y)$, which are both not *explicitly invertible*:

$$x(y) := \frac{(1-y)^{0.135} - y^{0.135}}{0.1975}$$
$$x(y) := 0.2 + \frac{(1-y)^{0.14} - y^{0.09}}{0.1596}$$

and we proceed with one of other 4 approximations, reported in [36] for $\Phi^{-1}(y)$, which is instead *explicitly invertible*:

$$x(y) := -5.531\left(\left(\frac{y}{1-y}\right)^{0.1193} - 1\right)$$

even if the inverse was not given in [36].

Moreover, we quote these 2 rational approximations of $x = Q^{-1}(y)$, both *explicitly invertible* with *InvLev 3* and *InvLev 1*, respectively (see Section IX) given by Zelen and Severo in the classical book [31] (credited to C. Hastings, Jr. [64]), valid for $0 < y \le \frac{1}{2}$,

$$t := \sqrt{\ln\frac{1}{y^2}}$$
$$x(y) := t - \frac{a_0 + a_1 t}{1 + b_1 t + b_2 t^2} \quad \text{Formula 26.2.22}$$
$$x(y) := t - \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} \quad \text{Formula 26.2.23}$$
$$\begin{aligned} a_0 &= 2.30753 \quad a_1 = 0.27061 \\ b_1 &= 0.99229 \quad b_2 = 0.04481 \\ c_0 &= 2.515517 \quad c_1 = 0.802853 \quad c_2 = 0.010328 \\ d_1 &= 1.432788 \quad d_2 = 0.189269 \quad d_3 = 0.001308, \end{aligned}$$

and another quite old (1973) approximation is given – for $0 < y < \frac{1}{2}$ but in fact holding for $0 < y \le \frac{1}{2}$ – in [37], analogous to the above mentioned Formula 26.2.23, giving several possibilities for the coefficients. A further refinement has been obtained in [38] substituting the 2-nd degree numerator polynomial of the same Formula 26.2.23 with a 4-th degree polynomial, and the 3-rd degree denominator polynomial with another 4-th degree polynomial, with appropriate coefficients, so reducing – with respect to the original $4.5 \cdot 10^{-4}$ – the absolute error of 4 magnitude orders: $1.5 \cdot 10^{-8}$, but loosing the *explicit invertibility*, since the inversion would require the solution of a 5-th degree equation (see Section IX and Remark in Appendix B).

Moreover, in [17] is reported (Formula (2)) an approximation for $\Phi^{-1}(y)$, *explicitly invertible* with *InvLev 6* (see Section IX) – obtained simplifying the above mentioned [31, Formula 26.2.22] (reported, for $\Phi^{-1}(y)$, in [17, Formula 1]) with a small loss in accuracy – that here we report for $x = Q^{-1}(y)$, valid again for $0 < y \le \frac{1}{2}$,

$$x(y) := t - \frac{1}{0.5 + 0.3\,t} \qquad t = \sqrt{\ln\frac{1}{y^2}}.$$

Its inverse

$$Q_{\text{Hamaker–inverted}} := \mathrm{e}^{-\frac{1}{72}\left(\sqrt{9x^2+30x+145}+3x-5\right)^2} \tag{47}$$

computed in the present research, has on $I_{\text{significant}}$, as approximation of $Q(x)$ (see Remark (on *Type 2*) in Section VI), errors $\varepsilon < 3.2 \cdot 10^{-3}$ and $\varepsilon_r < 9.7 \cdot 10^{-2}$ (for the issue of the errors, see Section VII-A).

The second approximation for $\Phi^{-1}(y)$ reported in [17] (Formula (3)) to overcome the above limitation on $y$, and reported here for $x = Q^{-1}(y)$, is

$$\begin{aligned} x(y) &:= \mathrm{sign}\left(\frac{1}{2} - y\right) 1.238\,t\,(1 + 0.0262\,t) \\ t &:= \sqrt{-\ln 4\,y\,(1-y)} \end{aligned} \tag{48}$$

whose inversion, obtained solving the second equation for $y$ and the first equation for $t$, gives the *explicitly invertible* approximation of $Q(x)$ called in this paper $Q_{\text{Hamaker}}$ and reported in Table 8, and of course, since in this paper we consider $x \ge 0$, the above mentioned $\mathrm{sign}(\frac{1}{2} - y)$ is always $+1$ and then may be omitted. Both $Q_{\text{Hamaker}}$ and its inverse (48) (when restricted to $x \ge 0$), reported in Table 16, have *InvLev 6* (see Section IX).

In [20] an alternative to the approximation (48) is proposed

$$x(y) := \frac{1.237\,t}{1 - 0.0249\,t} \qquad t := \sqrt{-\ln 4\,y\,(1-y)} \tag{49}$$

whose inversion, obtained solving the second equation for $y$ and the first equation for $t$, gives the *explicitly invertible* approximation of $Q(x)$ called in this paper $Q_{\text{Lin–2}}$ reported in Table 8. As shown in [20], the approximation $Q_{\text{Lin–2}}$ and its inverse (49), reported in Table 16, are almost as accurate as $Q_{\text{Hamaker}}$ and its inverse (48), respectively.

Finally, the classical book [39] in 3.8 lists approximations directly given for the inverse of $\Phi(x)$ (from which one could immediately obtain approximations for $Q^{-1}(y)$) some of which are *explicitly invertible*.

## V. EXPLICIT INVERTIBILITY OF AN APPROXIMATION

*The Lambert W-function:* Here we recall the Lambert $W$-function. In $\mathbb{C}$ [77]

> "[t]he Lambert W function is defined to be the multivalued inverse of the function $w \mapsto w\,e^w$ $[\cdots]$ If $x$ is real, then for $-1/e \le x < 0$ there are two possible real values of $W(x)$ (see figure 1 [therein]). We denote the branch satisfying $-1 \le W(x)$ by $W_0(x)$ or just $W(x)$ [as we will do in this paper] when there is no possibility for confusion, and the branch satisfying $W(x) \le -1$ by $W_{-1}(x)$. $W_0(x)$ is referred to as the *principal branch* of the $W$ function."

In this research, in the environment of telecommunications theory, we will be concerned with $W(x)$ only for $x \ge 0$. This function is very classic, with a standard name used in mathematics, has well established algorithms for computation (incorporated in the common software tools), is *explicitly invertible* by means of elementary functions with standard names used in mathematics (its inverse is $x\,e^x$), *but* is not [78] an elementary function. It is

$$W\big(x\,e^x\big) \equiv x \qquad \forall x \ge 0 \quad \big(\text{in fact } \forall x \ge -1\big) \tag{50}$$

$$W^{-1}(x) \equiv x\,e^x \qquad \forall x \ge 0 \quad \Big(\text{in fact } \forall x \ge -\frac{1}{e}\Big). \tag{51}$$

It has already been shown the relation between the Lambert $W$-function and the *explicit inversion* of a function related with the approximations of $Q(x)$: precisely that function in [76] is named $\psi(x)$ and in [79] is named $\hat{\phi}(x)$, and in fact it is $\pi\, Q_{\text{Wozencraft}}(\sqrt{\frac{x}{2}})$, see (19).

*Definition (of Explicitly Invertible):* Throughout this paper we call *explicitly invertibile* a function which is, in fact, explicitly invertibile by elementary functions having standard names in mathematics, and/or the Lambert $W$-function. For example $x^3$, $\log x$, $\log^3 x$, $x^3 + \log x$, but not $x + \log^3 x$ and $x^3 + \log^3 x$.

*Remark 4:* To be an elementary function or the Lambert $W$-function excludes, in the above Definition, for example the logarithmic integral (function with standard name in mathematics but not elementary and different from Lambert $W$-function).

*Remark 5:* To have a standard name in mathematics excludes, in the above Definition, for example the inverse of $y(z) = 0.208\,z^{971} + 0.147\,z^{525}$ for $z \ge 0$ (obtained operating on $Q^{\diamond}_{\text{Loskot}}$, see Remark 1 in Section VI), which is elementary (it is even algebraic), but does not have a standard name in mathematics.

*Remark 6:* Sometimes, in engineering literature, the word "invertible" is used meaning a similar thing, namely *explicitly invertible* by means of elementary functions having standard names in mathematics. So, for example, $e^x + \arcsin x$ is excluded, although being invertible, since increasing.

*Remark 7:* The *explicit invertibility* of an approximation $\tilde{f}(x)$ grants that there is no need [74] to resort to any interpolated function approximating it by points $(x_k, \tilde{f}(x_k))$ and its inverse by points $(\tilde{f}(x_k), x_k)$.

## VI. MAIN TYPES OF APPROXIMATIONS OF THE FUNCTION $Q(X)$

Here we produce a classification defining 7 fundamental types (from *Type 0* to *Type 6*) of functions in which almost all the already published approximations of $Q(x)$ fall, leaving in *miscellanea* (*Type 7*) the few remaining. A kind of 9-th class is the *Chernoff type* of approximations (strictly belonging to both *Type 1* and *Type 2*, see Remark 2 in this section). As far as possible, in treating (in this section) the fundamental types of approximations of $Q(x)$ we have given approximations which are *explicitly invertible* (see Definition in Section V) and approximations which are not *explicitly invertible* (denoted by $Q^{\diamond}_{\text{<Author>}}$). Only in *Type 3* and *Type 6* not *explicitly invertible* examples are lacking.

*Definition (of Irrational Function):* Though there is not a completely standard definition of *irrational function*, in this paper it is meant a function defined by the 4 operations and roots, which is not rational (then roots must be present). So not $e^x$, neither rational nor irrational, which has to be considered a *transcendent function*. Notice that with this definition $x^{3.14}$ is the irrational function $\sqrt[100]{x^{314}}$, while $x^{\pi}$ is the transcendent function $e^{\pi\,\ln x}$.

Hereafter, the above announced description of the fundamental types of published approximations of $Q(x)$.

- *Type 0*: *rational and irrational* functions **reported in Table 3**. In this type we find, e.g., the ancient (1955) Chu's (upper, for $x \ge 0$) bound $Q_{\text{Chu}}$ [68], $Q_{\text{Hastings-1}}$ (see also *InvLev 1* in Section IX), and 2 fifth degree polynomials reported in [82] (here we report only the first, $Q^{\diamond}_{\text{Zogheib–3}}$). As prototypes of this type, here we report:

$$Q_{\text{Chu}}(x) := \frac{1}{2} - \frac{x}{\sqrt{2\big(\pi + 2x^2\big)}}$$

$$Q^{\diamond}_{\text{Zogheib–3}}(x) := 0.5 + 0.398942\,x + 0.066490\,x^3 - 0.09974\,x^5$$

(see $\Phi_1(x)$ in [82])

  where $Q_{\text{Chu}}$ was originally published for $\frac{1}{2} - Q(x)$, and obviously is *explicitly invertible* (see *InvLev 6* in Section IX), whereas $Q^{\diamond}_{\text{Zogheib–3}}$, not *explicitly invertible*, was originally published for $\Phi(x)$ (and looses quickly precision for large $x$, since it tends to $-\infty$).
- *Type 1*: *sum* $S(x)$ *of terms* $b_i\,e^{a_i x^2}$, with $a_i$ and $b_i$ negative and positive constants, respectively, **reported in Tables 4 and 5**. The approximations of this type are said "Exponential Function Based Approximations"

**TABLE 3.** Already published *Type 0 explicitly invertible* approximations of $Q(x)$.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [80] $Q_{\text{Burr}}$ (Origin. for $\Phi(x)$) | $(1+(a+b\,x)^c)^{-d}$ $a=0.644693$ $b=0.161984$ $c=4.874$ $d=6.158$ | $2.3\cdot10^{-3}$ | $5.6\cdot10^{0}$ | 35, 2, 4 | 7.49 | 7 (18) | N |
| [68] Formula 7 $Q_{\text{Chu}}$ (Origin. for $\frac{1}{2}-Q(x)$) | $\frac{1}{2}-\frac{x}{\sqrt{2\,(\pi+2\,x^2)}}$ | $5.4\cdot10^{-2}$ | $5.4\cdot10^{3}$ | 15, 1, 0 | 3.11 | 6 (16) | upper ($\forall x\geq0$) |
| [81] $Q_{\text{Boiroju-1}}$ | $\frac{1}{2}\,(1+\frac{5}{238}\,x+\frac{x^2}{115}+\frac{x^3}{991})^{-37.9435}$ | $2.2\cdot10^{-5}$ | $1.8\cdot10^{-1}$ | 35, 1, 1 | 5.19 | 3 | N |
| [31] Formula 26.2.18 $Q_{\text{Hastings-1}}$ (Origin. for $\Phi(x)$) | $\frac{1}{2}\,(1+a\,x+b\,x^2+c\,x^3+d\,x^4)^{-4}$ $a=0.196854$ $b=0.115194$ $c=0.000344$ $d=0.019527$ | $2.4\cdot10^{-4}$ | $5.6\cdot10^{0}$ | 51, 0, 4 | 6.34 | 1 | N |

in [45] (there, our $a_i$ is $-\frac{1}{2}\,\beta_i$ and our $b_i$ is $\frac{1}{2}\,\alpha_i$). As prototypes of this type, here we report:

$$Q_{\text{Chiani-1}}(x) := \frac{1}{4}\,\mathrm{e}^{-x^2} + \frac{1}{4}\,\mathrm{e}^{-\frac{x^2}{2}}$$
(see (12) and Table 4)
$$Q^{\diamond}_{\text{Loskot}}(x) := 0.208\,\mathrm{e}^{-0.971\,x^2} + 0.147\,\mathrm{e}^{-0.525\,x^2}$$
(see [8, Formula 13c]).

*Remark 8:* Particularly interesting is the case – called "exponential-type" in [6], [9], and [11] – when all $a_i$ are integer multiple of some real $c$, that is to say $a_i = n_i \cdot c$ with positive integer $n_i$, possibly small (namely $\leq 4$), because the substitution $z := \mathrm{e}^{c\,x^2}$ converts the sum $S(x)$ into a polynomial in $z$. For example by the substitution $z := \mathrm{e}^{-\frac{1}{2}\,x^2}$, the above written $Q_{\text{Chiani-1}}$ becomes $\frac{1}{4}\,z^2 + \frac{1}{4}\,z$, allowing easy *explicit invertibility* (see *InvLev 6* in Section IX). Operating in the same way for $Q^{\diamond}_{\text{Loskot}}$, with $c = -0.001$ the polynomial $0.208\,z^{971} + 0.147\,z^{525}$ can be obtained, but with degree 971 (and better cannot be done, since 971 and 525 are relatively prime numbers). Another case in which the conversion to a polynomial is possible but the resulting degree is very high is $Q^{\diamond}_{\text{Sadhwani}}(x) := \frac{1}{16}\mathrm{e}^{-\frac{x^2}{2}} + \frac{1}{8}\mathrm{e}^{-x^2} + \frac{1}{8}\mathrm{e}^{-\frac{10}{3}\,x^2} + \frac{1}{8}\mathrm{e}^{-\frac{10}{17}\,x^2}$ [12] (originally given for $\mathrm{erfc}(x)$) for which $c = -\frac{1}{102}$ (of course 102 is the least common multiple of the denominators 2, 3, and 17) gives a polynomial of degree 102 (and better cannot be done). And in other cases the conversion of such a sum $S(x)$ to a polynomial is even impossible, as for example in $Q^{\diamond}_{\text{Wu-3}}(x) := \frac{1}{12}\,\mathrm{e}^{-6\,(\sqrt{3}-1)\,x^2/\pi} + \frac{1}{12}\,\mathrm{e}^{-2\,(3-\sqrt{3})\,x^2/\pi} + \frac{1}{6}\,\mathrm{e}^{-\sqrt{3}\,x^2/\pi}$ [83].

- *Type 2: exponentials of polynomials* $A\cdot\mathrm{e}^{P(x)}$ (equivalently written $\mathrm{e}^{P(x)+\ln A}$, and then, $\mathrm{e}^{T(x)}$, with $P(x)$ and $T(x)$ polynomials), or even (in a single case [88])[5] *exponentials of rational functions* $A\cdot\mathrm{e}^{R(x)}$ (similarly as above, the exponential $A\cdot\mathrm{e}^{R(x)}$ of a rational function $R(x)$ may also be equivalently expressed by $\mathrm{e}^{S(x)}$ if $S(x) = R(x) + \ln A$), **reported in Table 6**. The approximations of this type are said "Exponent Polynomial Based Approximations" in [45].
As prototypes of this type, here we report:

$$Q_{\text{Benitez-2}}(x) := \mathrm{e}^{-0.4774\,x^2-0.4484\,x-0.9049}$$
$$Q^{\diamond}_{\text{Phong}}(x) := \mathrm{e}^{0.0000018643\,x^6+\cdots-0.698740}$$
(see [89], Formula 8 and Table IV)

5. The unique example is $Q_{\text{Derenzo}}$ in Table 6 (in which the rational function is not expressed in the canonical form of quotient of polynomials).

**TABLE 4. Already published *Type 1 explicitly invertible* approximations of *Q*(*x*) - Part I.**

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [11] Formula 10<br>$Q_{\text{Chiani-1}}$<br>(Origin. for $\text{erfc}(x)$) | $\frac{1}{4}\,\mathrm{e}^{-x^2}+\frac{1}{4}\,\mathrm{e}^{-\frac{x^2}{2}}$ | $1.1\cdot 10^{-1}$ | $2.0\cdot 10^{0}$ | 17, 2, 0 | 3.71 | 6<br>with<br>$z:=\mathrm{e}^{-\frac{x^2}{2}}$<br>becomes<br>$\frac{z^2}{4}+\frac{z}{4}$<br>(16) | upper<br>$(\forall x\geq 0)$ |
| [83] Formula 9<br>$Q_{\text{Wu-2}}$ | $\frac{1}{6}\,\mathrm{e}^{-\frac{2\sqrt{3}}{\pi}\,x^2}+\frac{1}{6}\,\mathrm{e}^{-\frac{\sqrt{3}}{\pi}\,x^2}$ | $4.4\cdot 10^{-2}$ | $3.1\cdot 10^{-1}$ | 24, 2, 0 | 4.16 | 6<br>with<br>$z:=\mathrm{e}^{-\frac{\sqrt{3}}{\pi}\,x^2}$<br>becomes<br>$\frac{z^2}{6}+\frac{z}{6}$<br>(16) | lower |
| [84] Table 1<br>$Q_{\text{Powari}}$<br>(Origin. for $\text{erfc}(x)$) | $\frac{1}{3}\,\mathrm{e}^{-x^2}+\frac{1}{12}\mathrm{e}^{-\frac{x^2}{2}}$ | $2.9\cdot 10^{-2}$ | $3.1\cdot 10^{-1}$ | 18, 2, 0 | 3.78 | 6<br>with<br>$z:=\mathrm{e}^{-\frac{x^2}{2}}$<br>becomes<br>$\frac{z^2}{3}+\frac{z}{12}$<br>(16) | N |
| [9] Formula 2<br>with $N=2$<br>$Q_{\text{Olabiyi-2}}$<br>(Origin. for $\text{erfc}\sqrt{x}$) | $a\,\mathrm{e}^{-b\,x^2}+c\,\mathrm{e}^{-2\,b\,x^2}$<br>$a=0.15085$<br>$b=0.5255$<br>$c=0.21945$ | $1.4\cdot 10^{-2}$ | $6.6\cdot 10^{-2}$ | 36, 2, 3 | 7.56 | 6<br>with<br>$z:=\mathrm{e}^{-b\,x^2}$<br>becomes<br>$a\,z+c\,z^2$<br>(16) | N |

and, of course, the exponential of a polynomial may be expressed in bases different from e, in particular 2:

$$\mathrm{e}^{a_n x^n+\cdots+a_0}=2^{\frac{a_n}{\ln 2}x^n+\cdots+\frac{a_0}{\ln 2}} \tag{52}$$

and this may increase simplicity.

*Remark (on Type 2).* Allowing also square roots in the exponent, overcoming the strict definitions of polynomials and rational functions, the originally derived $Q_{\text{Hamaker–inverted}}$ (47) could be included in *Type 2*.

*Remark 2.* Since functions $b\,\mathrm{e}^{a\,x^2}$, said "Chernoff type" in [46], belong to both the 2 classes *Type 1* and *Type 2*, they are **listed** separately **in Table 7**. Among them there are the classical *Chernoff bound* $Q_{\text{Chernoff}}(x) := \mathrm{e}^{-x^2/2}$ (see (11) and Section X-A) and the *improved Chernoff bound* $Q_{\text{Chernoff–impr.}}(x) := \frac{1}{2}\,\mathrm{e}^{-x^2/2}$ (see (10) and Section X-A). Obviously all *Chernoff type* approximations are *(very simply) explicitly invertible* (*InvLev 7*, see Section IX).

- *Type 3: Pólya type approximations* of the form $\frac{1}{2}-\frac{1}{2}\sqrt{1-\mathrm{e}^{R(x)}}$, being $R(x)$ a rational function, or even (in a single case [91][6]) $a-b\sqrt{1-\mathrm{e}^{R(x)}}$, **reported in Tables 8 and 9**. As prototype of this type, here we report this very classical *explicitly invertible* approximation:

$$Q_{\text{Polya}}(x) := \frac{1}{2}-\frac{1}{2}\sqrt{1-\mathrm{e}^{-\frac{2}{\pi}x^2}} \text{ (see Table 8)}$$

originally published (1945, [61]), *mutatis mutandis*, for $\frac{1}{2}-Q(x)$, being the oldest generally recognized approximation of the "Gaussian integral" class of functions (see Remark in Section II-C).

6. The unique example is $Q_{\text{Abderrahmane–1}}$ in Table 8.

**TABLE 5.** Already published *Type 1 explicitly invertible* approximations of *Q*(*x*) - Part II.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [9] Formula 2 with $N=3$ $Q_{\text{Olabiyi}-3}$ (Origin. for $\operatorname{erfc}\sqrt{x}$) | $a\,e^{-b\,x^2}+c\,e^{-2\,b\,x^2}+d\,e^{-3\,b\,x^2}$ <br> $a=0.16785$ <br> $b=0.53245$ <br> $c=0.16805$ <br> $d=0.01525$ | $3.0\cdot10^{-2}$ | $9.0\cdot10^{-2}$ | 58, 3, 4 | 10.51 | 3 with $z:=e^{-b\,x^2}$ becomes a cubic equation | N |
| [11] Formula 11 $Q_{\text{Chiani}-3}$ (Origin. for $\operatorname{erfc}(x)$) | $\frac{1}{6}\,e^{-2\,x^2}+\frac{1}{12}\,e^{-x^2}+\frac{1}{4}\,e^{-\frac{x^2}{2}}$ | $7.9\cdot10^{-2}$ | $2.0\cdot10^{0}$ | 27, 3, 0 | 4.76 | 2 with $z:=e^{-\frac{x^2}{2}}$ becomes $\frac{z^4}{6}+\frac{z^2}{12}+\frac{z}{4}$ | upper ($\forall x\geq0$) |
| [11] Formula 14 $Q_{\text{Chiani}-2}$ (Origin. for $\operatorname{erfc}(x)$) | $\frac{1}{12}\,e^{-\frac{x^2}{2}}+\frac{1}{4}\,e^{-2\,\frac{x^2}{3}}$ | $3.3\cdot10^{-2}$ | $2.7\cdot10^{-1}$ | 21, 2, 0 | 3.98 | 1 with $z:=e^{-\frac{x^2}{6}}$ becomes $\frac{z^4}{4}+\frac{z^3}{12}$ | N |
| There is also the class of approximations $Q_{\text{Chiani}-\text{class}}$ (64) to which $Q_{\text{Chiani}-2}$, $Q_{\text{Chiani}-\text{new}-1}$ (65), and $Q_{\text{Chiani}-\text{new}-2}$ (66) belong. | | | | | | | |

- *Type 4: functions* $e^{-\frac{x^2}{2}}A(x)$ being $A(x)$ a (non constant) rational or irrational function (see the Definition above) **reported in Table 10**.
  Among them we find asymptotic approximations of $Q(x)$ (considered in Section VII-C), starting from $Q_{\text{Wozencraft}}$ (19). Many of them are modifications of this approximation, obtained substituting one or both the recurrences of $x$ in $Q_{\text{Wozencraft}}(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}$ with functions with *similar* behaviour.
  As prototypes of this type, here we report:
  $$Q_{\text{Borjesson-1}}(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}\,\sqrt{x^2+1}}$$
  $$Q^{\diamond}_{\text{Borjesson-2}}(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}\left((1-0.339)\,x+0.339\,\sqrt{x^2+5.51}\right)}$$
  (see [94], Formula 13 and Table I).
- *Type 5:* the *(complementary) logistic-type* approximations of $Q(x)$
  $$1-\frac{1}{1+e^{P(x)}}=\frac{e^{P(x)}}{1+e^{P(x)}}$$
  $$=\frac{1}{1+e^{T(x)}}=\frac{1}{2}-\frac{1}{2}\tanh S(x)$$
  **reported in Table 11** (where we have listed all the approximations of this type in the third form), or even (in a single case [91][7])
  $$1-\frac{a}{b+e^{P(x)}}$$
  (reported in Table 11 in this form), being $P(x)$ a polynomial tending to $-\infty$ for $x\to+\infty$, $T(x)=-P(x)$, and $S(x)=\frac{1}{2}\,T(x)$.
  As prototypes of this type, here we report:
  $$Q_{\text{Tocher}}(x) := \frac{1}{1+e^{\sqrt{8/\pi}\,x}}$$
  $$Q^{\diamond}_{\text{Zogheib-4}}(x) := \frac{1}{1+e^{-0.000345\,x+0.039547\,x^3+1.604326\,x^5}}$$
  (called $\Phi_3(x)$ in [82]).

Notice that the above considered expression $\frac{1}{1+e^{T(x)}}$ approximating $Q(x)$ gives immediately the corresponding approximation for $\Phi(x)$ substituting $T(x)$ with $-T(x)$.

*Remark 3.* This type $\frac{1}{1+e^{-T(x)}}$ of approximation is well appreciated for $\Phi(x)$ because it may be quite tight also

7. The unique example is $Q_{\text{Abderrahmane-3}}$ in Table 11.

**TABLE 6.** Already published *Type 2 explicitly invertible* approximations of $Q(x)$.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [85] Formula 1 $Q_{\text{Ordaz}}$ (Origin.$^{(*)}$ for $\Phi(x)$) | $0.6931\,\mathrm{e}^{-\left(\frac{9\,x+8}{14}\right)^2}$ | $4.1 \cdot 10^{-3}$ | $2.6 \cdot 10^{-1}$ | 18, 1, 1 | 4.16 | 7 (18) | N |
| [33] Table 2 $Q_{\text{Hanandeh}-2}$ (Origin. for $\Phi(x)$) | $0.688182\,\mathrm{e}^{-\left(\frac{9\,x+8}{14}\right)^2}$ | $1.9 \cdot 10^{-3}$ | $2.5 \cdot 10^{-1}$ | 20, 1, 1 | 4.31 | 7 (18) | N |
| [86] Formula 6 $Q_{\text{Mastin}-2}$ | $\frac{1}{2}\,\mathrm{e}^{-\frac{1}{\pi}\,x^2-\sqrt{\frac{2}{\pi}}\,x}$ | $7.1 \cdot 10^{-3}$ | $5.5 \cdot 10^{0}$ | 16, 1, 0 | 3.17 | 6 (16) | upper ($\forall x \geq 0$) |
| [86] Formula 7 $Q_{\text{Mastin}-3}$ | $\frac{1}{2}\,\mathrm{e}^{-a\,x^2-b\,x}$ $a = 0.374$ $b = 0.777$ | $9.4 \cdot 10^{-4}$ | $1.3 \cdot 10^{0}$ | 19, 1, 2 | 4.85 | 6 (16) | N |
| [13] Formula 8 with min-SSE criterion $Q_{\text{Benitez}-1}$ | $\mathrm{e}^{-a\,x^2-b\,x-c}$ $a = 0.3845$ $b = 0.7635$ $c = 0.6966$ | $5.8 \cdot 10^{-4}$ | $9.7 \cdot 10^{-1}$ | 25, 1, 3 | 5.85 | 6 | N |
| [86] Formula 5 $Q_{\text{Mastin}-1}$ | $\frac{1}{2}\,\mathrm{e}^{-\frac{1}{2}\,x^2-\sqrt{\frac{2}{\pi}}\,x}$ | $2.3 \cdot 10^{-2}$ | $8.4 \cdot 10^{-1}$ | 16, 1, 0 | 3.17 | 6 (16) | lower ($\forall x \geq 0$) |
| [87] Formula 6 $Q_{\text{Sofotasios}}$ | $0.49\,\mathrm{e}^{-\frac{x^2}{2}-\frac{8}{13}\,x}$ | $9.4 \cdot 10^{-3}$ | $6.4 \cdot 10^{-1}$ | 16, 1, 1 | 4.00 | 6 | N |
| [18] Formula 1 $Q_{\text{Lin}-1}$ | $\frac{1}{2}\,\mathrm{e}^{-a\,x^2-b\,x}$ $a = 0.416$ $b = 0.717$ | $6.6 \cdot 10^{-3}$ | $2.9 \cdot 10^{-1}$ | 19, 1, 2 | 4.85 | 6 (16) | N |
| [13] Formula 8 with min-MARE criterion $Q_{\text{Benitez}-2}$ | $\mathrm{e}^{-a\,x^2-b\,x-c}$ $a = 0.4774$ $b = 0.4484$ $c = 0.9049$ | $2.7 \cdot 10^{-2}$ | $8.1 \cdot 10^{-2}$ | 25, 1, 3 | 5.85 | 6 | N |
| [88] Formula 4 $Q_{\text{Derenzo}}$ (Origin. for $2\,Q(x)$) | $\frac{1}{2}\,\mathrm{e}^{-\frac{(83\,x+351)\,x+562}{703/x+165}}$ | $7.2 \cdot 10^{-5}$ | $4.2 \cdot 10^{-4}$ | 29, 1, 0 | 3.87 | 3 | N |

$(*)$ Published with the oversights $9x-$ instead of $9x+$ and $0.6931$ instead of $1-0.6931$.

for negative $x$, which is of no interest for $Q(x)$ in telecommunications theory but is of practical utility in statistics for $\Phi(x)$.

- *Type 6: nested exponentials* approximations of $Q(x)$

$$a \cdot b_1^{\wedge}\left(b_2^{\wedge}\left(\cdots^{\wedge}\left(b_n^{F(x)}\right)\cdots\right)\right)$$

**TABLE 7.** Already published *Chernoff type explicitly invertible* approximations of $Q(x)$.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [48] Formula 2-1-172 $Q_{\text{Chernoff}}$ | $\mathrm{e}^{-\frac{x^2}{2}}$ | $5.9 \cdot 10^{-1}$ | $1.1 \cdot 10^{1}$ | 6, 1, 0 | 2.29 | 7 (18) | upper $(\forall x \geq 0)$ |
| [71] Formula 2.3-15 $Q_{\text{Chernoff-impr.}}$ | $\frac{1}{2}\,\mathrm{e}^{-\frac{x^2}{2}}$ | $1.6 \cdot 10^{-1}$ | $4.9 \cdot 10^{0}$ | 9, 1, 0 | 2.62 | 7 (18) | upper $(\forall x \geq 0)$ |
| [90] Formula 10 $Q_{\text{Ermolova}-2}$ (Origin. for $\operatorname{erfc}(x)$) | $a\,\mathrm{e}^{-b\,x^2}$ $a = 0.3$ $b = 0.505$ | $5.6 \cdot 10^{-2}$ | $2.2 \cdot 10^{0}$ | 12, 1, 2 | 4.16 | 7 (18) | N |
| [40] p. 1850 $Q_{\text{Gasull}}$ | $\frac{1}{\sqrt{2\,\pi}}\mathrm{e}^{-\frac{11}{10}\,x^2}$ | $3.4 \cdot 10^{-1}$ | $1.0 \cdot 10^{0}$ | 14, 1, 0 | 3.04 | 7 (18) | lower $(\forall x \geq 0)$ |
| [9] Formula 2 with $N = 1$ $Q_{\text{Olabiyi}-1}$ (Origin. for $\operatorname{erfc}\sqrt{x}$) | $a\,\mathrm{e}^{-b\,x^2}$ $a = 0.24015$ $b = 0.5616$ | $2.8 \cdot 10^{-1}$ | $8.4 \cdot 10^{-1}$ | 17, 1, 2 | 4.67 | 7 (18) | N |
| [83] Formula 8 $Q_{\text{Wu}-1}$ | $\frac{1}{4}\mathrm{e}^{-\frac{2}{\pi}\,x^2}$ | $1.1 \cdot 10^{-1}$ | $8.2 \cdot 10^{-1}$ | 10, 1, 0 | 2.71 | 7 (18) | lower $(\forall x \geq 0)$ |
| [90] Formula 9 $Q_{\text{Ermolova}-1}$ (Origin. for $\operatorname{erfc}(x)$) | $a\,\mathrm{e}^{-b\,x^2}$ $a = 0.28$ $b = 0.6375$ | $8.1 \cdot 10^{-2}$ | $8.0 \cdot 10^{-1}$ | 14, 1, 2 | 4.38 | 7 (18) | lower |
| [46] Formula 31 with $\beta_{\text{opt.}}$ of Table I $Q_{\text{Chang}}$ (Origin. for $\operatorname{erfc}(x)$) | $\sqrt{\frac{\mathrm{e}}{2\,\pi}}\,\frac{\sqrt{1.080-1}}{1.080}\,\mathrm{e}^{-\frac{1.080}{2}\,x^2}$ | $1.8 \cdot 10^{-1}$ | $5.4 \cdot 10^{-1}$ | 27*, 1, 1** * see Section VIII-A ** see Section VIII-C | 4.76 | 7 (18) | lower $(\forall x \geq 0)$ |

There are also 4 classes of lower bounds:
1. $Q_{\text{Chang-class}}$ (14), to which the above reported $Q_{\text{Chang}}$ and $Q_{\text{Chang-new}}$ (13) belong;
2. $Q_{\text{Cote-class}}$ (69);
3. $Q_{\text{Wu-class}-1}$ (70);
4. and $Q_{\text{Wu-class}-2}$ (71), to which the above reported $Q_{\text{Wu}-1}$ belongs.

with $a, b_1, \ldots, b_n \in \mathbb{R}^+$, $n = 2, 3, \ldots,$, being $F(x)$ a rational or irrational function and being obviously ^ power elevation, **reported in Table 12**. Of course the level of easiness of *explicit invertibility* (called *InvLev* in Section IX) is the same of that of $F(x)$.

So far, approximations of this type have been published with 2 and 3 nested exponentials, all *explicitly invertible*, as this [82] with 2 exponentials (originally published for $\Phi(x)$) which we report here as prototype:

$$Q_{\text{Zogheib-2}}(x) := \frac{1}{2}\,\mathrm{e}^{-1.2\,x^{1.3}} = \frac{1}{2}\left(\mathrm{e}^{-1.2}\right)^{x^{1.3}}$$

**TABLE 8.** Already published *Type 3 explicitly invertible* approximations of $Q(x)$ - Part I.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [91] Formula 2.3<br>$Q_{\text{Abderrahmane}-1}$<br>(Origin. for $\Phi(x)$) | $a - b\sqrt{1-\mathrm{e}^{-c\,x^2}}$<br>$a = 0.49897$<br>$b = 0.49794$<br>$c = 0.62632$ | $1.1 \cdot 10^{-3}$ | $3.1 \cdot 10^{2}$ | 29, 2, 3 | 7.03 | 7<br>(18) | N |
| [61] Formula 1.5<br>$Q_{\text{Polya}}$<br>(Origin. for $\frac{1}{2} - Q(x)$) | $\frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-\frac{2}{\pi}x^2}}$ | $3.2 \cdot 10^{-3}$ | $8.2 \cdot 10^{-1}$ | 17, 2, 0 | 3.71 | 7<br>(18) | lower<br>$(\forall x \geq 0)$ |
| [81]<br>$Q_{\text{Boiroju}-2}$ | $\frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-0.635\,x^2}}$ | $3.0 \cdot 10^{-3}$ | $8.1 \cdot 10^{-1}$ | 19, 2, 1 | 4.85 | 7<br>(18) | lower |
| [19] called $\Phi_5(x)$<br>$Q_{\text{Aludaat}}$<br>(Origin. for $\Phi(x)$) | $\frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-\sqrt{\frac{\pi}{8}}\,x^2}}$ | $2.0 \cdot 10^{-3}$ | $7.8 \cdot 10^{-1}$ | 18, 2, 0 | 3.78 | 7<br>(18) | N |
| [28] called $\Phi_5(x)$<br>$Q_{\text{Eidous}}$<br>(Origin. for $\Phi(x)$) | $\frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-\frac{5}{8}x^2}}$ | $1.9 \cdot 10^{-3}$ | $7.7 \cdot 10^{-1}$ | 17, 2, 0 | 3.71 | 7<br>(18) | N |
| [92]<br>$Q_{\text{Abderrahmane}-2}$<br>(Origin. for $\Phi(x)$) | $\frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-0.62306179\,x^2}}$ | $1.7 \cdot 10^{-3}$ | $7.6 \cdot 10^{-1}$ | 24, 2, 1 | 5.24 | 7<br>(18) | N |
| [34] called $G_9(x)$<br>$Q_{\text{Hanandeh}-4}$<br>(Origin. for $\Phi(x)$) | $\frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-\frac{81}{130}x^2}}$ | $1.7 \cdot 10^{-3}$ | $7.6 \cdot 10^{-1}$ | 20, 2, 0 | 3.91 | 7<br>(18) | N |
| [17] Formula 4<br>$Q_{\text{Hamaker}}$<br>(Origin. for $\Phi(x)$ on $\mathbb{R}$) | $\frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-(a\,x\,(1-b\,x))^2}}$<br>$a = 0.806$<br>$b = 0.018$ | $5.7 \cdot 10^{-4}$ | $1.1 \cdot 10^{-1}$ | 31, 2, 2 | 6.53 | 6<br>(16) | N |
| [20] Formula 4<br>$Q_{\text{Lin}-2}$ | $\frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-\left(\frac{x}{a+b\,x}\right)^2}}$<br>$a = 1.237$<br>$b = 0.0249$ | $9.2 \cdot 10^{-4}$ | $8.3 \cdot 10^{-2}$ | 30, 2, 2 | 6.46 | 6<br>(16) | N |

(where $x^{1.3}$, or $\sqrt[10]{x^{13}}$, is an irrational function, see Definition above in this section) and this [16] with 3 exponentials

$$\Phi(x) \approx 2^{-22^{1-41^{x/10}}} = 1 \cdot \left(2^{-22}\right)^{\left(22^{-1}\right)^{\left(41^{1/10}\right)^{x}}} \tag{53}$$

reported for $Q(x)$ as $Q_{\text{Soranzo–3}}$ in Table 12, which has been modified and optimized for $Q(x)$ on $I_{\text{significant}}$ in [22] obtaining $Q_{\text{Soranzo–4}}$ reported in the same Table 12.

*Remark (on InvLev):* Because of the conversion formulas of Table 1, any 4 mutually corresponding approximations of $Q(x)$, $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$ (but not the corresponding approximation of $m(x)$) have all the same level of easiness of *explicit invertibility* (called *InvLev* in Section IX). For example the above mentioned $Q_{\text{Soranzo–3}}$ and (53).

- *Type 7: miscellanea* of other types of various nature **reported in Table 13**. As prototypes of this type, here we report

$$Q_{\text{Kundu}}(x) := 1 - \left(1 - \mathrm{e}^{-\mathrm{e}^{0.3820198\,x+1.0792510}}\right)^{12.8},$$

and some interesting not *explicitly invertible* approximations (thus not appearing in Table 13):

**TABLE 9.** Already published *Type 3 explicitly invertible* approximations of $Q(x)$ - Part II.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [24] Formula 1 <br> $Q_{\text{Soranzo}-1}$ <br> (Origin. for $\text{erf}(x)$) | $\frac{1}{2}-\frac{1}{2}\sqrt{1-\mathrm{e}^{-x^2\,\frac{a+b\,x^2}{2+c\,x^2+d\,x^4}}}$ <br> $a=1.2735457$ <br> $b=0.0743968$ <br> $c=0.1480931$ <br> $d=0.0002580$ | $1.2\cdot10^{-5}$ | $2.0\cdot10^{-1}$ | 60, 2, 4 | 9.65 | 5 | N |
| [23] Formula 3 <br> $Q_{\text{Winitzki}}$ <br> (Origin. for $\text{erf}(x)$) | $\frac{1}{2}-\frac{1}{2}\sqrt{1-\mathrm{e}^{-x^2\,\frac{4/\pi+0.147\,x^2/2}{2+0.147\,x^2}}}$ | $6.3\cdot10^{-5}$ | $3.1\cdot10^{-2}$ | 37, 2, 1 | 6.05 | 5 (15) | N |
| [25] Formula 5 <br> $Q_{\text{Soranzo}-2}$ <br> (Origin. for $\Phi(x)$) | $\frac{1}{2}-\frac{1}{2}\sqrt{1-\mathrm{e}^{-x^2\,\frac{17+x^2}{26.694+2\,x^2}}}$ | $4.0\cdot10^{-5}$ | $2.2\cdot10^{-2}$ | 30, 2, 1 | 5.65 | 5 (15) | N |
| [93] called Cadwell (modified) <br> $Q_{\text{Brophy}}$ <br> (Origin.$^{(*)}$ for $\Phi(x)$) | $\frac{1}{2}-\frac{1}{2}\sqrt{1-\mathrm{e}^{-x^2\,(a-x^2\,(b-c\,x^2))}}$ <br> $a=0.6366197724$ <br> $b=0.009564223505$ <br> $c=0.0004$ | $3.3\cdot10^{-5}$ | $6.7\cdot10^{-1}$ | 56, 2, 3 | 8.76 | 3 with $z:=x^2$ gives a cubic equation | N |

$(*)$ Published with the oversight $.5-$ instead of $.5+$.

**TABLE 10.** Already published *Type 4 explicitly invertible* approximations of $Q(x)$.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [50] Formula 2.121 <br> $Q_{\text{Wozencraft}}$ | $\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}\,x}$ | $4.8\cdot10^{-1}$ | $1.5\cdot10^{0}$ | 11, 1, 0 | 2.80 | 6.5 (17) | upper ($\forall x>0$) |
| [94] Formula 9 <br> $Q_{\text{Borjesson}-1}$ | $\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}\,\sqrt{x^2+1}}$ | $1.5\cdot10^{-2}$ | $8.1\cdot10^{-2}$ | 15, 2, 0 | 3.56 | 6.5 (17) | upper |

$$Q^{\diamond}_{\text{Bagby}}(x) := \frac{1}{2}-\frac{1}{2}\cdot\sqrt{1-\frac{1}{30}\left(7\,\mathrm{e}^{-\frac{x^2}{2}}+16\,\mathrm{e}^{-x^2\left(2-\sqrt{2}\right)}+\left(7+\frac{\pi\,x^2}{4}\right)\mathrm{e}^{-x^2}\right)},$$

appearing on [100, p. 46], resembling *Type 3* (*Pólya type approximations*), and

$$Q^{\diamond}_{\text{Moran}}(x) := 0.5 - \frac{1}{\pi}\sum_{n=0}^{12}\left\{\frac{\mathrm{e}^{-\frac{(n+0.5)^2}{9}}}{n+0.5}\sin\left(\frac{\sqrt{2}}{3}\,(n+0.5)\,x\right)\right\}$$

originally published in [101] for $\Phi(x)$ and only for $x\in[-7,7]$ (which in any case includes $I_{\text{significant}}$)

**TABLE 11.** Already published *Type 5 explicitly invertible* approximations of $Q(x)$.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [91] Formula 3.2 $Q_{\text{Abderrahmane}-3}$ (Origin. for $\Phi(x)$) | $1-\frac{a}{b+e^{-c\,x}}$ $a=0.97186$ $b=0.96628$ $c=1.69075$ | $5.8\cdot10^{-3}$ | $1.6\cdot10^{3}$ | 28, 1, 3 | 6.07 | 7 (18) | N |
| [32] p. 32 $Q_{\text{Tocher}}$ (Origin.$^{(*)}$ for $\frac{1}{2}-Q(x)$) | $\frac{1}{1+e^{\sqrt{8/\pi}x}}$ | $1.8\cdot10^{-2}$ | $2.3\cdot10^{2}$ | 10, 1, 0 | 2.71 | 7 (18) | upper |
| [33] Table 2 $Q_{\text{Hanandeh}-1}$ (Origin. for $\Phi(x)$) | $\frac{1}{1+e^{1.7017\,x}}$ | $9.5\cdot10^{-3}$ | $1.4\cdot10^{2}$ | 12, 1, 1 | 3.63 | 7 (18) | N |
| [95] Equation 8 $Q_{\text{Bowling}-1}$ (Origin. for $\Phi(x)$) | $\frac{1}{1+e^{1.702\,x}}$ | $9.5\cdot10^{-3}$ | $1.4\cdot10^{2}$ | 11, 1, 1 | 3.53 | 7 (18) | N |
| [96] p. 119 $Q_{\text{Johnson}-1}$ (Origin. for $\Phi(x)$) | $\frac{1}{1+e^{\pi\,x/\sqrt{3}}}$ | $2.3\cdot10^{-2}$ | $8.3\cdot10^{1}$ | 10, 1, 0 | 2.71 | 7 (18) | N |
| [97] called $\Phi_{19}(x)$ in [98] $Q_{\text{Divgi}}$ (Origin. for $\Phi(x)$) | $\frac{1}{1+e^{a\,x(1+b\,x)}}$ $a=1.526$ $b=0.1034$ | $2.2\cdot10^{-3}$ | $1.2\cdot10^{1}$ | 22, 1, 2 | 5.09 | 6 | N |
| [27] Formula 5 $Q_{\text{Vedder}}$ (Origin. for $\Phi(x)$) | $\frac{1}{1+e^{\sqrt{8/\pi}\,x+\sqrt{2/\pi}(4-\pi)\,x^3/(3\,\pi)}}$ | $3.2\cdot10^{-4}$ | $7.1\cdot10^{-1}$ | 27, 1, 0 | 3.78 | 4 (14) | upper |
| [26] p. 75 $Q_{\text{Page}}$ (Origin. for $\Phi(x)$) | $\frac{1}{1+e^{\sqrt{8/\pi}\left(x+0.044715\,x^3\right)}}$ | $1.8\cdot10^{-4}$ | $6.7\cdot10^{-1}$ | 23, 1, 1 | 4.51 | 4 (14) | N |
| [95] Equation 10 $Q_{\text{Bowling}-2}$ (Origin. for $\Phi(x)$) | $\frac{1}{1+e^{a\,x+b\,x^3}}$ $a=1.5976$ $b=0.07056$ | $1.5\cdot10^{-4}$ | $6.5\cdot10^{-1}$ | 22, 1, 2 | 5.09 | 4 (14) | N |
| [82] called $\Phi_4(x)$ $Q_{\text{Zogheib}-1}$ (Origin. for $\Phi(x)$) | $\frac{1}{1+e^{-a+b\,x+c\,x^3}}$ $a=0.0054$ $b=1.6101$ $c=0.0674$ | $5.8\cdot10^{-4}$ | $5.5\cdot10^{-1}$ | 29, 1, 3 | 6.14 | 4 | N |

$(*)$ Published with the lack of the factor $\frac{1}{\sqrt{2\,\pi}}$ (see (5)).

but clearly may be intended as holding for all $x \geq 0$. Another interesting example, with the quality of being a good approximation on the whole real axis, is [102]

$$Q^{\diamond}_{\text{Leal}}(x) := \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{39\,x}{2\sqrt{2\,\pi}} - \frac{111}{2}\arctan\left(\frac{35\,x}{111\sqrt{2\,\pi}}\right)\right)$$

**TABLE 12.** Already published *Type 6 explicitly invertible* approximations of $Q(x)$.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [33] Table 2 $Q_{\text{Hanandeh}-3}$ (Origin. for $\Phi(x)$) | $\frac{1}{2}\,\mathrm{e}^{-a\,x^b}$ $a = 1.2$ $b = 1.275247$ | $9.2 \cdot 10^{-3}$ | $4.1 \cdot 10^{1}$ | 17, 2, 2 | 5.35 | 7 (18) | N |
| [82] called $\Phi_5(x)$ $Q_{\text{Zogheib}-2}$ (Origin. for $\Phi(x)$) | $\frac{1}{2}\,\mathrm{e}^{-a\,x^b}$ $a = 1.2$ $b = 1.3$ | $8.5 \cdot 10^{-3}$ | $3.0 \cdot 10^{1}$ | 12, 2, 2 | 4.76 | 7 (18) | N |
| [16] Formula A $Q_{\text{Soranzo}-3}$ (Origin. for $\Phi(x)$) | $1 - 2^{-22^{1-41^{x/10}}}$ | $1.3 \cdot 10^{-4}$ | $6.8 \cdot 10^{-1}$ | 14, 3, 0 | 3.83 | 7 (18) | N |
| [22] Table I $Q_{\text{Soranzo}-4}$ | $1 - 2^{-54^{1-1.3671^{x}}}$ | $1.3 \cdot 10^{-2}$ | $7.7 \cdot 10^{-2}$ | 15, 3, 1 | 4.93 | 7 (18) | N |

**TABLE 13.** Already published *Type 7 explicitly invertible* approximations of $Q(x)$.

| Bibliographic reference and name of the approximation of $Q(x)$ | Analytic expression of the approximation of $Q(x)$ | Reasonable majoriz. of abs. err. in $I_{\text{significant}}$ $[0.45, 4.5]$ | Reasonable majoriz. of rel. err. in abs. val. in $[0.45, 4.5]$ | Compl-exities: typogr., comput., decim. | Total compl-exity | Inverti-bility level (inverse in Table) | Is it a bound in $[0.45, 4.5]$? |
|---|---|---|---|---|---|---|---|
| [30] Formula 1.1 $Q_{\text{Lipoth}}$ (Origin. for $\Phi(x)$) The Authors give also other 5-uples of parameters in Table 3 | $1 - \left(1 + a\left(\ln\left(1 + \mathrm{e}^{-\frac{x}{h}+c}\right)\right)^b\right)^{-d}$ $a = 0.00161826615$ $b = 3.38692114553$ $c = 3.26862849061$ $d = 7.80500878654$ $h = 0.82116764005$ | $2.4 \cdot 10^{-5}$ | $7.4 \cdot 10^{-1}$ | 85, 4, 5 | 13.66 | 7 (18) | N |
| [99] Formula 17 $Q_{\text{Kundu}}$ (Origin. for $\Phi(x)$) | $1 - \left(1 - \mathrm{e}^{-\mathrm{e}^{a\,x+b}}\right)^c$ $a = 0.3820198$ $b = 1.0792510$ $c = 12.8$ | $3.2 \cdot 10^{-4}$ | $7.3 \cdot 10^{-1}$ | 32, 3, 3 | 8.00 | 7 (18) | N |

resembling *Type 5* approximations in the fourth form (but not included because $S(x)$ is not a polynomial nor even a rational function).

## VII. THE ACCURACY OF AN APPROXIMATION OF A FUNCTION

The struggle of the scientific researches considered in this survey is exactly to achieve high precision with low complexity, both defined in any reasonable way. In this sense, as a measure of precision of an approximation of the $Q$-function we have chosen (a majorization of) the relative error in absolute value $\varepsilon_r$ and, secondarily, (a majorization of) the absolute error $\varepsilon$, both defined in the following Section VII-A, where we also explain why the (majorization of) $\varepsilon_r$ has been chosen as relevant measure of precision of an approximation of the $Q$-function.

Other Authors have chosen measures of precision different from maximal relative and absolute error. A rich list of measures of precision, used just for the $Q$-function, is in [45, Table II].

### A. NOTES ON ABSOLUTE AND RELATIVE ERRORS

Renewing the observations in [22], let us denote, for a function $f(x)$ and an approximation $\tilde{f}(x)$, by

$$\varepsilon^{\left(f\tilde{f}\right)}(x) := |\tilde{f}(x) - f(x)|, \text{ or simply } \varepsilon^{\left(\tilde{f}\right)}(x), \text{ or even } \varepsilon(x)$$

the absolute error, intended as a function with the same domain $D$ of $f$ and $\tilde{f}$, and by

$$\varepsilon^{\left(f\tilde{f},D\right)} := \sup_D \varepsilon^{\left(f\tilde{f}\right)}(x), \text{ or simply } \varepsilon^{\left(f\tilde{f}\right)}, \quad \text{or simply } \varepsilon^{\left(\tilde{f}\right)}, \text{ or even } \varepsilon \tag{54}$$

the absolute error, intended as a number, summarizing the distance of the approximation from the function.

Analogously, for relative errors, if $f(x) \neq 0$ on the domain $D$, let us denote with

$$\varepsilon_r^{\left(f\tilde{f}\right)}(x) := \left|\frac{\tilde{f}(x) - f(x)}{f(x)}\right| = \frac{\varepsilon^{\left(f\tilde{f}\right)}(x)}{|f(x)|}, \quad \text{or simply } \varepsilon_r^{\left(\tilde{f}\right)}(x), \text{ or even } \varepsilon_r(x) \tag{55}$$

the relative error in absolute value, intended as a function with the same domain $D$ of $f$ and $\tilde{f}$, and by

$$\varepsilon_r^{\left(f\tilde{f},D\right)} := \sup_D \varepsilon_r^{\left(f\tilde{f}\right)}(x), \text{ or simply } \varepsilon_r^{\left(f\tilde{f}\right)}, \quad \text{or simply } \varepsilon_r^{\left(\tilde{f}\right)}, \text{ or even } \varepsilon_r \tag{56}$$

the relative error in absolute value, intended as a number, summarizing the distance of the approximation from the function, normalized with respect to the function $f$ itself, if $\neq 0$ in $D$.

From the point of view of the absolute errors, due to the mutual relations listed in Table 1, it is essentially the same thing to approximate any of the functions $\text{erf}(x)$, $\text{erfc}(x)$, $\Phi(x)$, $Q(x)$, eventually – if $D = [0, +\infty)$, which is the most impacting case – doubling or halving the absolute error. Instead, observe that this is not valid for $m(x)$ because of the exponential factors, see Table 1.

Quite different is the matter for relative errors. Although the functions $\Phi(x)$ and $Q(x)$ are mutually related by the linear relation $Q(x) = 1 - \Phi(x)$, their approximations for $x \geq 0$ are quite different matters, because $\frac{1}{2} \leq \Phi(x) < 1$ whereas $\frac{1}{2} \geq Q(x) > 0$ and, when $x \to +\infty$, $\Phi(x) \to 1$ whereas $Q(x) \to 0$, and then a small absolute error $\varepsilon^{(\tilde{\Phi})}(\bar{x})$ affecting an approximation $\tilde{\Phi}(x)$ of $\Phi(x)$ in $\bar{x}$ sufficiently large results in a relative error (in absolute value) $\varepsilon_r^{(\tilde{\Phi})}(\bar{x}) = \frac{\varepsilon^{(\tilde{\Phi})}(\bar{x})}{\Phi(\bar{x})} \approx \varepsilon^{(\tilde{\Phi})}(\bar{x})$ for $\Phi(\bar{x})$, being $\Phi(\bar{x}) \approx 1$, but the same absolute error $\varepsilon^{(\tilde{Q})}(\bar{x}) = \varepsilon^{(\tilde{\Phi})}(\bar{x})$ gives for $Q(\bar{x})$ a relative error $\frac{\varepsilon^{(\tilde{Q})}(\bar{x})}{Q(\bar{x})}$ which is very large, being $Q(\bar{x}) = 1 - \Phi(\bar{x}) \approx 0$ and positive. And in communications and information theory the consideration of the $Q$-function for large values of $x$ is quite common. Similar things may be said about approximating $\text{erfc}(x)$, having limit 0 in $+\infty$.

In general, it may be said that, when comparing the accuracy of 2 approximations of a function, the consideration of the relative error is more appropriate if that function has a zero limit. In fact, for instance, often it is of little interest to say that an approximated value of about $10^{-5}$ has an absolute error less than $10^{-4}$.

All the 5 considered *special functions* $Q(x)$, $\Phi(x)$, $\text{erf}(x)$, $\text{erfc}(x)$, and $m(x)$ have a zero limit in their domain (included $\text{erf}(x)$ which has value and limit 0 in 0), but $\frac{1}{2} \leq \Phi(x) < 1$ for $x \geq 0$, and then for this restricted function the consideration of the absolute error is appropriate and generally used in literature.[8] Instead, for the above mentioned reasons, for $\text{erfc}(x)$ and $Q(x)$ the consideration of relative errors is more appropriate and generally used in literature, at least from a mathematical point of view. On the other hand, as far as $\text{erf}(x)$ is concerned, classical books such as [31] (more precisely in Chapter 7 authored by W. Gautschi) consider absolute errors for evaluating its approximations, since, although the consideration of the relative error appears to be somehow more appropriate than the absolute error, because $\text{erf}(x) = 0$ for $x = 0$, also the absolute error appears to be meaningful, for these 2 reasons:

- $\text{erf}(x)$ takes (in absolute value) very small values – where a low absolute error of an approximation may correspond to a high relative error – only on a very small interval, near 0, very differently from what happens to $\text{erfc}(x)$ and to $Q(x)$;
- these small values, for $x$ near 0, may be of scarce interest in many applications.

Notice that for 3 of the 5 considered *special functions*, i.e., for $\text{erf}(x)$, $\Phi(x)$, and $Q(x)$, it is, considering (56),

$$\varepsilon(x) < \varepsilon_r(x) \tag{57}$$

because these 3 functions take values $< 1$, in absolute value, wherever $\varepsilon_r(x)$ exists, which is the entire domain of the real numbers $\mathbb{R}$, except 0 for $\text{erf}(x)$. For $\text{erfc}(x)$, the inequality (57) holds for $x > 0$ (where this function is $< 1$, see Section II-E), and for $m(x)$ it holds for $x > 0.3026\cdots$ (where this function is $< 1$).

Furtherly, for $\Phi(x)$ and its approximation $\tilde{\Phi}(x)$, it is

$$\varepsilon_r^{\left(\tilde{\Phi}\right)}(x) = \left|\frac{\tilde{\Phi}(x) - \Phi(x)}{\Phi(x)}\right| = \frac{\varepsilon^{\left(\tilde{\Phi}\right)}(x)}{\Phi(x)} \qquad \forall x \geq 0$$

and then by this, (57), and being $\Phi(x) \geq \frac{1}{2}$ for $x \geq 0$,

$$\varepsilon^{\left(\tilde{\Phi}\right)}(x) < \varepsilon_r^{\left(\tilde{\Phi}\right)}(x) \leq 2\,\varepsilon^{\left(\tilde{\Phi}\right)}(x) \qquad \forall x \geq 0,$$

8. The approximation of $\Phi(x)$ for $x \leq 0$ is essentially the problem of approximating $Q(x)$ for $x \geq 0$ because $\Phi(-x) = Q(x)$.

and, taking sup, for the approximations of $\Phi(x)$ it is

$$\varepsilon^{\left(\tilde{\Phi}\right)} < \varepsilon_r^{\left(\tilde{\Phi}\right)} \leq 2\,\varepsilon^{\left(\tilde{\Phi}\right)} \qquad \text{in } [0, +\infty)$$

or, which is equivalent,

$$\frac{\varepsilon_r^{\left(\tilde{\Phi}\right)}}{2} \leq \varepsilon^{\left(\tilde{\Phi}\right)} < \varepsilon_r^{\left(\tilde{\Phi}\right)} \qquad \text{in } [0, +\infty),$$

so the optimization of an approximation of $\Phi(x)$ in $[0, +\infty)$ with respect to the relative error (in absolute value) is essentially equivalent to the optimization with respect to the absolute error.

Instead, the 2 errors in $[0, +\infty)$ are completely different for $Q(x)$, $\mathrm{erf}(x)$ and $\mathrm{erfc}(x)$ since these 3 functions have zero limits: a small absolute error $\varepsilon(x)$ for any of these functions may correspond to a huge relative error (in absolute value) $\varepsilon_r(x)$ if located in a point $x$ where the function takes a very small value, or, viceversa, to a small relative error $\varepsilon_r(x)$ if located in a point $x$ where the function takes a value very far from 0.

This paper is essentially devoted to the *explicitly invertible* approximations of $Q(x)$, and the stress is pointed on the single values of this function and, correspondingly, its approximations, for which, as explained above, the relative error is more significant. The absolute error plays again an important role when considering the approximations globally on the whole $I_{\text{significant}}$, which is made when considering the weighted integrals as (46) even limiting the integration domain to $I_{\text{significant}}$.

For the approximations of $Q(x)$ the case of the domain $I_{\text{significant}}$ is very different from the case of the domain $[0, +\infty)$, as we are going to show in 2 different perspectives. In fact, when limiting the domain to $I_{\text{significant}} = [0.45, 4.5]$, the absolute error $\varepsilon$ is less than the third part of the relative error in absolute value $\varepsilon_r$:

$$\begin{aligned}
\varepsilon_r^{\left(\tilde{Q}\right)} &= \sup_{I_{\text{significant}}} \left| \frac{\tilde{Q}(x) - Q(x)}{Q(x)} \right| \\
&= \sup_{I_{\text{significant}}} \frac{\left|\tilde{Q}(x) - Q(x)\right|}{Q(x)} \\
&\geq \sup_{I_{\text{significant}}} \frac{\left|\tilde{Q}(x) - Q(x)\right|}{\max_{I_{\text{significant}}} Q(x)} \\
&= \sup_{I_{\text{significant}}} \frac{\left|\tilde{Q}(x) - Q(x)\right|}{Q(0.45)} \\
&= \frac{1}{Q(0.45)} \sup_{I_{\text{significant}}} \left|\tilde{Q}(x) - Q(x)\right| \\
&= \frac{1}{Q(0.45)}\, \varepsilon^{\left(\tilde{Q}\right)} = \frac{\varepsilon^{\left(\tilde{Q}\right)}}{0.326\cdots} > 3\,\varepsilon^{\left(\tilde{Q}\right)}
\end{aligned}$$

and then

$$\varepsilon^{\left(\tilde{Q}\right)} < \frac{\varepsilon_r^{\left(\tilde{Q}\right)}}{3} \qquad \text{in } I_{\text{significant}}.$$

So, for $Q(x)$ in $I_{\text{significant}}$, optimizing an approximation of $Q(x)$ to minimize the relative error (in absolute value), which is the perspective mainly taken in this paper, leads to a good approximation in the sense of the absolute error. Instead, an analogous computation, replacing, in the above expression, $\geq$, max, and 0.45 with $\leq$, min, and 4.5, respectively, gives, for $Q(x)$ in $I_{\text{significant}}$, the very poor

$$\varepsilon_r^{\left(\tilde{Q}\right)} \leq \frac{1}{Q(4.5)}\, \varepsilon^{\left(\tilde{Q}\right)} < 3 \cdot 10^5\, \varepsilon^{\left(\tilde{Q}\right)} \qquad \text{in } I_{\text{significant}} \quad (58)$$

which means that optimizing an approximation of $Q(x)$ to minimize the absolute error in $I_{\text{significant}}$ does not necessarily lead to a good approximation in the sense of the relative error (in absolute value). A striking example is $Q_{\text{Abderrahmane–1}}$, reported in Table 8, presenting $\varepsilon < 1.1 \cdot 10^{-3}$ but $\varepsilon_r < 3.1 \cdot 10^2$, where the ratio $\frac{\varepsilon_r}{\varepsilon}$ attains almost the huge value $3 \cdot 10^5$, before said in (58).

For each approximation, the absolute error $\varepsilon$, defined in (54), and the relative error in absolute value $\varepsilon_r$, defined in (56), have been reported in Tables 3–13, evaluated with reference to the domain $I_{\text{significant}} = [0.45, 4.5]$. Notice that they have been expressed as majorizations, so for example 2.42, though more similar to 2.4, gives 2.5. This is what we mean by “reasonable majorization”, using 2 significant digits.

### B. TIGHTNESS OF THE APPROXIMATIONS OF Q(X) IN TELECOMMUNICATIONS SYSTEMS

As far as telecommunications systems are concerned, we are interested in good approximations of the $Q(x)$ function especially for quite large values of the signal-to-noise ratio, since this region is characterized by very low SEP and BEP values, which are difficult to be obtained by simulating the system behaviour (with appropriate software or hardware tools). The difficulty is due to the length of the simulation needed to find these very low SEP and BEP values: namely, if a BEP value of the order of $10^{-6}$ has to be obtained, it is necessary to simulate a transmission with bit length of the order of $10^6$. In this sense, the analytic performance of the system, usually expressed in terms of the $Q$-function, is very useful to get an insight in its behaviour especially at quite high signal-to-noise ratios, giving the so-called *asymptotic performance*.

Although this is not generally remarked, in this particular field the difference between absolute and relative errors is remarkable: a small absolute error of $10^{-3}$ where the function $Q(x)$ is about $10^{-5}$ gives a relative error of the order of $10^2$ on the approximation or majorization of SEP or BEP. Notice that this small absolute error is negligible at the beginning of the significant interval $I_{\text{significant}}$ (see Section III-A) where $Q(x)$ is about 0.326 (see Section III-B).

### C. ASYMPTOTICITY AND ASYMPTOTIC EQUIVALENCE OF AN APPROXIMATIONS OF THE FUNCTION Q(X)

In [50] one finds this Formula 2.121 (here rewritten with the variable $x$)

$$\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x}\left(1-\frac{1}{x^2}\right) < Q(x) < \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x} \qquad \forall x>0. \quad (59)$$

In this paper we name the lower bound in (59) $Q^{\diamond}_{\text{Wozencraft–lower}}$, reported in (21), and the upper bound in (59)

$$Q_{\text{Wozencraft}}(x) := \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x} \sim Q(x) \tag{60}$$

where by $\sim$ we mean the asymptotic equivalence, i.e., the fact that $Q_{\text{Wozencraft}}$ is characterized by this limit

$$\lim_{x\to+\infty} \frac{Q(x)}{\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x}} = 1 \tag{61}$$

which may be easily obtained dividing the above inequality chain (59) by $Q_{\text{Wozencraft}}$.

Already we have seen the lower bound $Q^{\diamond}_{\text{Wozencraft–lower}}$, reported in (21), and here we show that it is asymptotically equivalent to $Q(x)$ for $x \to +\infty$:

$$\begin{aligned}&\frac{Q^{\diamond}_{\text{Wozencraft–lower}}(x)}{Q(x)}\\ &= \frac{Q^{\diamond}_{\text{Wozencraft–lower}}(x)}{Q_{\text{Wozencraft}}(x)} \cdot \frac{Q_{\text{Wozencraft}}(x)}{Q(x)}\\ &= \left(1-\frac{1}{x^2}\right)\cdot\frac{Q_{\text{Wozencraft}}(x)}{Q(x)} \to 1\end{aligned}$$

because of (59), (60), and (61).

Also the lower bound $Q^{\diamond}_{\text{Gordon}}(x) := \frac{x}{1+x^2}\cdot\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}$ (20) is asymptotically equivalent to $Q(x)$ for $x \to +\infty$, and here this fact may be proved in this way, taking in account (61),

$$\begin{aligned}&\frac{Q(x)}{Q^{\diamond}_{\text{Gordon}}(x)}\\ &= \frac{Q(x)}{Q_{\text{Wozencraft}}(x)} \cdot \frac{Q_{\text{Wozencraft}}(x)}{Q^{\diamond}_{\text{Gordon}}(x)}\\ &= \frac{Q(x)}{\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x}} \cdot \frac{1+x^2}{x^2} \to 1.\end{aligned}$$

In [71, Fig. 2.3-2] there are, superimposed, graphs of $Q(x)$, $Q_{\text{Chernoff–impr.}}$, $Q_{\text{Wozencraft}}$ and $Q^{\diamond}_{\text{Gordon}}$.

In 1942 paper [67] the improvement $Q^{\diamond}_{\text{Birnbaum}}$ (37) is presented larger than $Q^{\diamond}_{\text{Gordon}}$ but, nevertheless, lower than $Q(x)$. Also this approximation is asymptotically equivalent to $Q(x)$.

Furthermore, the lower bound $Q^{\diamond}_{\text{Wozencraft–lower}}$ (21) enlightens the tightness of the upper bound $Q_{\text{Wozencraft}}$ (19) for large $x$, up to $+\infty$. For example it is:

$$Q^{\diamond}_{\text{Wozencraft–lower}}(x) = 0.99\, Q_{\text{Wozencraft}}(x) \quad \text{for } x = 10,$$

and in the thin strip between (the graphs of) $Q^{\diamond}_{\text{Wozencraft–lower}}(x)$ and $Q_{\text{Wozencraft}}(x)$ lies (the graph of) $Q(x)$. In other words, for $x > 10$ both these functions have (at least) the $\frac{1}{99} \approx 1\%$ precision. All that has been already highlighted in [50, Fig. 2.36] which also shows that the very classical and widely used $Q_{\text{Chernoff–impr.}}$ Eq. (10) is really very much less precise, for large $x$.

Remember that the asymptotic equivalence means limit 1 of the quotient of the function $Q(x)$ and its approximation, *not* limit 0 of the difference between them, which is the asymptoticity. For $Q(x)$ and any positive infinitesimal function for $x \to +\infty$, asymptotical equivalence implies asymptoticity, but the viceversa is not true. For instance, the upper bound $Q_{\text{Chernoff–impr.}}(x) = \frac{1}{2}\mathrm{e}^{-\frac{x^2}{2}}$ (10) and its double $Q_{\text{Chernoff}}$ (11) both have difference 0 from $Q(x)$ in $+\infty$ and then they are asymptotic (limit of the difference 0) to $Q(x)$ but they are not asymptotically equivalent (limit of the quotient 1) to $Q(x)$ and precisely, taking in account (61),

$$\begin{aligned}&\frac{Q_{\text{Chernoff–impr.}}(x)}{Q(x)}\\ &= \frac{Q_{\text{Chernoff–impr.}}(x)}{Q_{\text{Wozencraft}}(x)} \cdot \frac{Q_{\text{Wozencraft}}(x)}{Q(x)}\\ &= \frac{1}{2}\sqrt{2\pi}\,x \cdot \frac{1}{\frac{Q(x)}{\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x}}} \to +\infty.\end{aligned}$$

Furthermore notice that, for $Q(x)$ (and similarly for any positive infinitesimal function for $x \to +\infty$) and its approximation $\tilde{Q}(x)$, it is:

$$\begin{aligned}\text{asymptoticity} &\Leftrightarrow \varepsilon^{(\tilde{Q})}(+\infty) = 0\\ \text{asymptoticity} &\Leftarrow \varepsilon_r^{(\tilde{Q})}(+\infty) = 0 (\text{not} \Rightarrow)\\ \text{asymptoticequivalence} &\Rightarrow \varepsilon^{(\tilde{Q})}(+\infty) = 0 (\text{not} \Leftarrow)\\ \text{asymptoticequivalence} &\Leftrightarrow \varepsilon_r^{(\tilde{Q})}(+\infty) = 0.\end{aligned}$$

Any function $F(x)$ asymptotically equivalent to $Q_{\text{Wozencraft}}$ is asymptotically equivalent to $Q(x)$ because

$$\begin{aligned}&\lim_{x\to+\infty}\frac{F(x)}{Q(x)}\\ &= \lim_{x\to+\infty}\left(\frac{F(x)}{Q(x)}\cdot\frac{Q(x)}{Q_{\text{Wozencraft}}(x)}\right)\\ &= \lim_{x\to+\infty}\frac{F(x)}{Q_{\text{Wozencraft}}(x)} = 1,\end{aligned} \tag{62}$$

where the first equivalence is due to (61) and the last to the hypothesis.

Several approximations of $Q(x)$ considered in this paper are refinements of the simple asymptotic approximation $Q_{\text{Wozencraft}}$ (60). This latter will be also considered in Section X-A as a bound: it is actually an upper bound for $Q(x)$.

The above considerations imply that all the (widely used, see *Type 1* in Section VI) approximations of $Q(x)$ as *sum* $S(x)$ *of terms* $b_i\,\mathrm{e}^{a_i x^2}$, where $a_i$ and $b_i$ are negative and positive constants, respectively, have, as proved below:

1) unbounded relative errors (in absolute value) on $[0, +\infty)$, and precisely $+\infty$ in $+\infty$, if at least one of the $a_i$'s is $\geq -\frac{1}{2}$ (as $Q_{\text{Chernoff}}$ and $Q_{\text{Chernoff–impr.}}$ in Table 7, $Q_{\text{Chiani–1}}$ in Table 4, $Q_{\text{Chiani–2}}$ and $Q_{\text{Chiani–3}}$ in Table 5);
2) relative error 1 or 100% in the other case, i.e., if all the $a_i$'s are $< -\frac{1}{2}$ (as $Q_{\text{Olabiyi–1}}$, $Q_{\text{Wu–1}}$, and $Q_{\text{Chang}}$ in Table 7, $Q_{\text{Olabiyi–2}}$ in Table 4, and $Q_{\text{Olabiyi–3}}$ in Table 5).

In fact

$$\begin{aligned}
&\lim_{x\to+\infty} \varepsilon_r^{(\tilde{Q})}(x) \\
&\quad = \lim_{x\to+\infty} \left| \frac{\sum_{i=1}^m b_i\, e^{a_i x^2} - Q(x)}{Q(x)} \right| \\
&\quad = \lim_{x\to+\infty} \left| \frac{\sum_{i=1}^m b_i\, e^{a_i x^2}}{Q(x)} - 1 \right| \\
&\quad = \lim_{x\to+\infty} \left| \frac{\sum_{i=1}^m b_i\, e^{a_i x^2}}{\frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}} \cdot \frac{\frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}}{Q(x)} - 1 \right| \\
&\quad = \lim_{x\to+\infty} \left| \frac{\frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}}{Q(x)} \sum_{i=1}^m b_i \sqrt{2\pi}\, x\, e^{(a_i+1/2)\,x^2} - 1 \right|
\end{aligned}$$

and, using (61), one finds the limits $+\infty$ and 1 in the two cases (namely, if at least one of the $a_i$'s is $\geq -\frac{1}{2}$ and if all the $a_i$'s are $< -\frac{1}{2}$), respectively. Then, those approximations may be precise, in the sense of the relative error, only on bounded domains.

Since $Q_{\text{Wozencraft}}$ (60) is asymptotically equivalent to $Q(x)$, its relative error in $+\infty$, when considered as approximation of $Q(x)$, is 0, but, essentially due to its divergence in 0, its relative error at the beginning of $I_{\text{significant}}$ is large, about 145% (and for $x \to 0^+$ even diverges).

Slight modifications of $Q_{\text{Wozencraft}}$ (60) as $Q_{\text{Borjesson–1}}(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}\,\sqrt{x^2+1}}$ in Table 10, greatly reduce the relative error (in absolute value). In fact, $Q_{\text{Borjesson–1}}$, besides being analytically simple and asymptotically equivalent to $Q(x)$, since, through (60) and (61),

$$\begin{aligned}
&\lim_{x\to+\infty} \frac{Q_{\text{Borjesson–1}}(x)}{Q(x)} \\
&\quad = \lim_{x\to+\infty} \frac{Q_{\text{Borjesson–1}}(x)}{Q_{\text{Wozencraft}}(x)} \cdot \frac{Q_{\text{Wozencraft}}(x)}{Q(x)} \\
&\quad = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} \cdot 1 = 1,
\end{aligned}$$

presents $\varepsilon_r < 0.15$ or 15% (defined in (56)) on the interval $[0, +\infty)$, and so $Q_{\text{Borjesson–1}}$ really reveals – at a glance, it could be said – the global behavior of $Q(x)$, and this is absolutely not achieved by any truncation of the Taylor series (see Section II-A7) or even of a class of function series (see Section II-A8), nor by the (quite complicated) integral definitions (see Sections II-A1 and II-A2).

The path of successive refinements of the basic asymptotically equivalent approximation $Q_{\text{Wozencraft}}$ (60) passes through $Q^{\diamond}_{\text{Wozencraft–lower}}$ (21)[9] and ([14, Formulas 2–25])

$$Q^{\diamond}_{\text{Cooper}}(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}\left(1 - \frac{1}{2\,x^2}\right), \tag{63}$$

which is the arithmetic mean of the lower bound $Q^{\diamond}_{\text{Wozencraft–lower}}$ (21) (function already mentioned in Section II-A6) and the upper bound $Q_{\text{Wozencraft}}$ (19) (there is no one who can fail to see that the first ratio in (63) is $Q_{\text{Wozencraft}}$), till this [103] approximation

$$\begin{aligned}
&Q^{\diamond}_{\text{Byrc}}(x) := e^{-x^2/2} \\
&\cdot \frac{x^2 + 5.575192695\,x + 12.77436324}{x^3\sqrt{2\pi} + 14.38718147\,x^2 + 31.53531977\,x + 25.548726}
\end{aligned}$$

which is, as $Q^{\diamond}_{\text{Cooper}}$ (63), clearly asymptotically equivalent to $Q_{\text{Wozencraft}}$ and then, through (62), to $Q(x)$.

As an example of a widely used approximation (of *Type 4*, see Section VI), which is not asymptotically equivalent to $Q(x)$, we show (see [31, Formula 26.2.17])

$$\begin{aligned}
&Q^{\diamond}_{\text{Hastings-2}}(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot \\
&\quad \cdot \left(b_1\,t + b_2\,t^2 + b_3\,t^3 + b_4\,t^4 + b_5\,t^5\right) \\
&t := \frac{1}{1 + p\,x} \qquad p = 0.231\,641\,9 \\
&b_1 = 0.319\,381\,53 \qquad b_2 = -0.356\,563\,782 \\
&b_3 = 1.781\,477\,937 \qquad b_4 = -1.821\,255\,978 \\
&b_5 = 1.330\,274\,429
\end{aligned}$$

reported for $\Phi(x)$ (with absolute error less than $7.5 \cdot 10^{-8}$, which is the same for $Q(x)$) in the classical book of Abramowitz and Stegun [31] in a chapter authored by M. Zelen and N.C. Severo, and credited to Hastings, Jr. [64].

It is not asymptotically equivalent to $Q(x)$ because, through (60) and (61),

$$\begin{aligned}
&\lim_{x\to+\infty} \frac{Q^{\diamond}_{\text{Hastings-2}}(x)}{Q(x)} \\
&\quad = \lim_{x\to+\infty} \left( \frac{Q^{\diamond}_{\text{Hastings-2}}(x)}{Q_{\text{Wozencraft}}(x)} \cdot \frac{Q_{\text{Wozencraft}}(x)}{Q(x)} \right) \\
&\quad = \frac{b_1}{p} \cdot 1 \neq 1
\end{aligned}$$

This approximation is so widely used in practice, that the advanced search tool of Google declares (May 2023) about 7400 results searching (contemporarily) the 3 strings 0.2316419 0.31938153 1.781477937 (and at a glance the results refers just to that formula).

A kind of optimization of the parameters of $Q^{\diamond}_{\text{Hastings-2}}$ for the relative error in absolute value on $I_{\text{significant}}$ is in [104].

9. Replacing in (63) the term $2\,x^2$ with $x^2$ one obtains the lower bound $Q^{\diamond}_{\text{Wozencraft–lower}}$ (21) for $Q(x)$, function already mentioned in Section II-A6.

*Remark on the sum of powers of* $\boldsymbol{Q(x)}$*:* Notice that $2\,Q_{\mathrm{Wozencraft}}$ is asymptotically equivalent to $2\,Q(x)-Q^2(x)$, considered in (8) for the QPSK modulation:

$$P_s(E)=2\,Q\left(\sqrt{\frac{E_s}{N_0}}\right)-Q^2\left(\sqrt{\frac{E_s}{N_0}}\right).$$

In fact, through (61),

$$\begin{aligned}&\lim_{x\to+\infty}\left|\frac{2\,Q(x)-Q^2(x)}{2\,Q_{\mathrm{Wozencraft}}}\right|\\&=\lim_{x\to+\infty}\left|\frac{Q(x)}{Q_{\mathrm{Wozencraft}}}-\frac{1}{2}\cdot\frac{Q(x)}{Q_{\mathrm{Wozencraft}}}\cdot Q(x)\right|\\&=\left|1-\frac{1}{2}\cdot 1\cdot 0\right|.\end{aligned}$$

Analogously $a_1\,Q_{\mathrm{Wozencraft}}\sim a_1\,Q(x)+a_2\,Q^{m_2}(x)+\cdots+a_n\,Q^{m_n}(x)$, useful for evaluating the SEP of the differentially encoded QPSK modulation, considered in (43):

$$\begin{aligned}P_s(E)=4\,Q\left(\sqrt{\frac{E_s}{N_0}}\right)-8\,Q^2\left(\sqrt{\frac{E_s}{N_0}}\right)\\+8\,Q^3\left(\sqrt{\frac{E_s}{N_0}}\right)-4\,Q^4\left(\sqrt{\frac{E_s}{N_0}}\right).\end{aligned}$$

where $E_s/N_0$ is the signal-to-noise ratio. So the approximation on $I_{\mathrm{significant}}$ of the function $2\,Q(x)-Q^2(x)$ considered in (8) is a problem really not far from the approximation of $Q(x)$ (and this is obviously still true on intervals extending on the right $I_{\mathrm{significant}}$, even up to $+\infty$). For example, in the family of functions $f(x;a,b):=\frac{a\,\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,\sqrt{x^2+b}}$ (a family including $Q_{\mathrm{Wozencraft}}$ for $a=1$ and $b=0$ and $Q_{\mathrm{Borjesson-1}}$ for $a=1$ and $b=1$) an approximation for $Q(x)$ with relative error 6% is obtained with $a=1$ and $b=1.15$ and an approximation analogously tight (with relative error 5%) for $2\,Q(x)-Q^2(x)$ is simply obtained with $a=2$ and $b=1.63$.

*Remark on the twofold nature of* $\boldsymbol{Q(x)}$*:* Notice that $Q(x)$, for $x\geq 0$, has a somehow twofold nature. For large $x$, say $x>3$ to fix ideas, $Q(x)$ is essentially $\approx Q_{\mathrm{Wozencraft}}(x):=\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,x}$ with both low absolute and relative errors (see Section VII-A). For $0\leq x\leq 3$, it is simply a decreasing positive function *far from zero*, and so it may be well approximated, quite easily, even by a polynomial, with both low absolute and relative errors.

*Remark on the bad behaviour in* $+\infty$ *of almost all approximations:* Simple developments of the above treated arguments show that all approximations not belonging to *Type 4* (Wozencraft-like) and *Type 7* (miscellanea) have relative error (in absolute value) $+\infty$ in $+\infty$.

### D. TIGHTNESS OF THE INVERSE OF AN APPROXIMATION OF Q(X) IN $I_{\mathrm{SIGNIFICANT}}$

Generally speaking, the inverse of an invertible approximation of $Q(x)$ is an approximation of the inverse of $Q(x)$ with the concerns treated – specifically for bounds – in Section X-C.

More precisely, the inverse of a sufficiently tight invertible approximation of $Q(x)$ on $I_{\mathrm{significant}}$ is an approximation, generally quite tight, of the inverse of $Q(x)$ at least on an interval approximately equal to $Q(I_{\mathrm{significant}})$. If, as it is reasonable, the lower extreme of $Q(I_{\mathrm{significant}})$, which is to say $Q(4.5)$ (about $3\cdot 10^{-6}$), is considered practically indistinguishable from 0, and if we consider a sufficiently tight approximation $\tilde{Q}(x)$ of $Q(x)$, so that also $\tilde{Q}(4.5)$ is practically indistinguishable from 0, the situation is as follows. Among approximations, we may distinguish between 2 extreme cases, i.e., the approximation could be an upper or a lower bound for $Q(x)$.

1) The most favorable case is when the approximation is an invertible tight upper bound for $Q(x)$ on $I_{\mathrm{significant}}$: in this case, its inverse (existing, and findable in the case of explicit invertibility) is a tight upper bound for $Q^{-1}(y)$ on an interval which contains $Q(I_{\mathrm{significant}})$ and, when restricted to $Q(I_{\mathrm{significant}})$, remains a tight upper bound for $Q^{-1}(y)$.
2) The other extreme case, less favorable, is when the approximation is an invertible tight lower bound for $Q(x)$ on $I_{\mathrm{significant}}$: its inverse (existing, and findable in the case of explicit invertibility) is a tight lower bound for $Q^{-1}(y)$ on an interval which is – unfortunately – strictly contained in $Q(I_{\mathrm{significant}})$, starting from (approximately) 0 and ending slightly before $Q(0.45)$.

In the first and more favorable case above said, one may even investigate on the relation between the (majorization of the) relative error of the upper bound $Q^+(x)$ (with respect to the exact $Q(x)$) and the (majorization of the) relative error of its inverse (with respect to the exact $Q^{-1}(y)$): our experience shows that in the interval of relative errors [0.1%, 10%], the relative error of the inverse may be at most very slightly more than double of that of $Q^+(x)$ and, generally speaking, it is always about double.

## VIII. COMPLEXITY OF THE APPROXIMATIONS OF $Q(X)$

Leaving out, for the moment, *explicit invertibility*, the struggle of the scientific researches considered in this survey is exactly to achieve high precision with low complexity, since it is always desirable to search, in the words of Boiroju [81], for

> "an adequate balance between the accuracy and analytical tractability."

There is no universally recognized standard for evaluating the complexity of an analytic expression and, in particular, of an approximation, and several methods have been used over time by various Authors, for instance by measuring [105] the average time to compute approximations of the $Q$-function, or by measuring the computation complexity of integrals involved in the SEP expression, summarized in [45, Table IX] for various approximations of the $Q$-function over the Nakagami-$m$ fading channel. Here, considering the experience of the past and also innovating, we will consider 3 completely different indexes of complexity and a 4-th index that summarizes them.

*Remark:* Of course one may have an approximation of $Q(x)$ on a bounded interval (as $I_{\text{significant}}$) with relative error as small as desired, using sufficiently many terms of the continued fraction (see Section II-A6), or of the Taylor series (see Section II-A7), or even of a class of function series (see Section II-A8). Nevertheless nobody gives approximations of the $Q(x)$ function in closed form which are truncations of Taylor or other series because reaching any reasonable precision, on $I_{\text{significant}}$ or any other interval of practical interest, would require a huge number of terms. As remarked also in [45] about complex approximations,

> "These approximations are less significant when evaluating the error probabilities in communication systems due to mathematical complexity."

### A. TYPOGRAPHIC COMPLEXITY

This is a formalization of what generally Authors implicitly – basically – mean when distinguishing between "simple" and "complicated" expressions, and in particular approximations of $Q(x)$. The typographic complexity is here meant as the number of typographic characters of an analytic expression, *reasonably written*. Nevertheless, finally the form of the analytic expression of an approximation remains, in part, a personal choice of the Authors. For example, the approximation $Q_a$ (by us named $Q^{\diamond}_{\text{Borjesson–2}}$ in Section VI) reported[10] in [94, Formula 13]:

$$Q_a(x) = \frac{1}{(1-a)\,x + a\,\sqrt{x^2+b}}\,\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-x^2/2}$$

and reported in [45, Table I] as $E_{\text{Borj}}$:

$$E_{\text{Borj}}(x) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}}\left(\frac{1}{(1-0.339)\,x + 0.339\,\sqrt{x^2+5.510}}\right)$$

presents a typographic complexity of 39 characters (or 38, if not counting the final 0 as explained here below). The same, reported as $Q_{\text{a-Borjesson-1}}$ in [105]:

$$Q_{\text{a-Borjesson-1}}(x) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}\left(0.661\,x + 0.339\,\sqrt{x^2+5.51}\right)}$$

presents a typographic complexity of 32 characters (but with 1 more decimal constant, acquiring a higher decimal complexity when evaluated from the point of view of the number of decimal constants used in the formula, as explained in the following Section VIII-C).

Some tricks have been avoided, such as:

- $\mathrm{e}^{-\frac{x^2}{2}}$ or $\mathrm{e}^{-x^2/2}$ (which have both 6 characters) have *not* been written as $\sqrt{\mathrm{e}^{-x^2}}$ to spare 1 character;
- $\frac{a}{b\,(c+d)}$ has *not* been written as $\frac{\frac{a}{b}}{c+d}$ to spare 1 character.

Such subtleties cause a character complexity of, say, 60, to be regarded, in general, as much greater than 30, but more or less comparable with, say, 58.

Furtherly, notice that the typographic complexity is influenced by the number of digits of decimal constants. For example, classical [31] uses many digits in constants, but essentially the same precision may be achieved with fewer digits. In this paper we counted the digits as Authors put them, except excluding final zeroes. For instance,

$$Q_{\text{Chang}}(x) := \sqrt{\frac{\mathrm{e}}{2\,\pi}}\,\frac{\sqrt{1.080-1}}{1.080}\,\mathrm{e}^{-1.080x^2/2},$$

reported in Table 7, presents a typographic complexity of 27 characters (and not 30) since the last zero in the decimal constant (1.080, appearing 3 times) has not been counted.

Still, note that a low typographic complexity is useful essentially to *catch the soul* of an approximation and even of the function $Q(x)$ itself, to understand, so to say, its deep nature. This concept is well explained by, for example, this approximation, *asymptotically equivalent to* $Q(x)$ (see Section VII-C), defined in [94] as

> "a simple analytical approximation of $Q(x)$ for all $x \geq 0$",

and therein called $P(x)$:

$$P(x) = \frac{1}{\sqrt{2\,\pi}}\,\frac{1}{\sqrt{1+x^2}}\,\mathrm{e}^{-x^2/2},$$

having a typographic complexity of 18 characters. The same approximation is called $E_2(x)$ in [45]:

$$E_2(x) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}}\left(\frac{1}{\sqrt{x^2+1}}\right)$$

with a typographic complexity of 19 characters, and $Q_{\text{a-Borjesson-2}}$ in [105]:

$$Q_{\text{a-Borjesson-2}}(x) = \frac{1}{\sqrt{2\,\pi}}\,\frac{1}{\sqrt{x^2+1}}\,\mathrm{e}^{-x^2/2}$$

with a typographic complexity of 18 characters. In Table 10 we have called it $Q_{\text{Borjesson–1}}$:

$$Q_{\text{Borjesson–1}}(x) := \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}\,\sqrt{x^2+1}}$$

with a typographic complexity of 15 characters. We could say shortly:

*Have a look at* $Q_{\text{Borjesson–1}}$ *and you will understand the deep nature of the Q-function – apart from a 15% error.*

Notice that many Authors have not felt as a necessity to reduce the number of decimal digits in the presented constants: obviously, in view of a reduction of the typographic complexity, their constants could be rounded, reducing the number of decimal digits, without affecting substantially the precision of the approximations.

The typographic complexity, as considered in this paper, has been, e.g., previously considered also in [18], where 2 approximations of $Q(x)$ were compared, one requiring about 20 keystrokes on a pocket calculator, and the other 32 keystrokes. A kind of typographic complexity, measured in bytes, was also considered in [106], essentially devoted to the comparison of several previously published approximations

10. From [94, Table I], $a = 0.339$ and $b = 5.510$ are the values giving the best approximation for $x \geq 0$, i.e., the approximation minimizing $\varepsilon_r$, defined in (56).

of $Q^{-1}(y)$, some of which – *explicitly invertible* – have been considered in Section IV-G.

Finally, a low typographic complexity of an approximation allows also to *remember* it, for the work on the field, and even for teaching.

### B. COMPUTATIONAL COMPLEXITY

The computational complexity is a widely considered concept, but it could be measured in several ways, no one of which nowadays used as a fixed standard in Numerical Analysis.

Here, an index of computational complexity is used, that is integer and well related to the real computing time. This choice is due to the observation that the *exact* computational time of an expression is not so easy to evaluate: it depends on the machine, and, given the machine, is not perfectly stable, it depends on the value of the argument $x$ in which to evaluate the function, it depends on the use of single or double precision, and on the way the function is written. For instance, the function $\mathrm{e}^{-\frac{x^2}{2}}$ may be written as:

$$\mathrm{e}^{-\frac{x^2}{2}} = \mathrm{e}^{-\frac{1}{2}x^2} = \mathrm{e}^{-0.5\,x^2} = \frac{1}{\mathrm{e}^{\frac{x^2}{2}}} = \cdots$$

On the basis of these observations, in this paper the index of computational complexity of an approximation has been evaluated by counting the number of non-rational functions applied to the argument $x$. So, for example, all rational expressions, including constants given by transcendental numbers like $\pi$ and e, such as $x+\sqrt{\pi}$, $2\,x$, $-\frac{x^2}{2}$, $3\,x^2+2\,x+\mathrm{e}$, $\frac{x^4+\pi}{x^3+x^2}$, count 0, and any non-rational expression, such as $\exp(\cdot)$, $\ln(\cdot)$, $\sqrt{\cdot}$, of any rational expression, all count 1, even if repeated.

As example consider $Q_{\text{Borjesson–1}}$ reported in Table 10:

$$Q_{\text{Borjesson–1}}(x) := \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}\,\sqrt{x^2+1}}$$

which counts 2 because there are 2 non-rational expressions (the exponential function and the square root) of a rational expression and the constant $\frac{1}{\sqrt{2\,\pi}}$ counts 0 because in general it has to be computed only one time while the complete expression will be computed many times. (If the approximation has to be computed only once, the consideration of the machine time is nowadays completely useless, irrelevant, due to its extreme shortness.)

Tricks have been avoided as, for example, rewriting $Q_{\text{Chiani–2}}$ reported in Table 5:

$$Q_{\text{Chiani–2}}(x) := \frac{1}{12}\,\mathrm{e}^{-\frac{x^2}{2}} + \frac{1}{4}\,\mathrm{e}^{-2\,\frac{x^2}{3}}$$

which counts 2 because there are 2 non-rational expressions (the 2 exponential functions) of a rational expression and the constants count 0, as

$$Q_{\text{Chiani-2-equivalent}}(x) := \frac{1}{12}\left(\mathrm{e}^{-\frac{x^2}{6}}\right)^3 + \frac{1}{4}\left(\mathrm{e}^{-\frac{x^2}{6}}\right)^4,$$

in order to count a computational complexity 1 (since the non-rational expression $\mathrm{e}^{-\frac{x^2}{6}}$ is repeated and thus counts only 1) instead of 2. (Of course, such tricks could in fact be used in practice, if on a particular machine they grant a computation time sparing.)

### C. DECIMAL COMPLEXITY

Classically it is also considered how many decimal constants are present in an approximation, as a simple and immediate measure of complexity. For example, in [17] we read:

> "(⋯) which, though containing only two numerical constants, turns out to be almost as accurate as (⋯)."

Of course, nor integer numbers, nor $\pi$ and e are computed as decimal constants.

Analogously to the other kinds of complexity, the decimal complexity is not intrinsic and in different equivalent forms of an expression it may vary. In particular

$$Q_{\text{Chang}}(x) := \sqrt{\frac{\mathrm{e}}{2\,\pi}}\,\frac{\sqrt{1.080-1}}{1.080}\,\mathrm{e}^{-1.080\,x^2/2},$$

reported in Table 7 as having a decimal complexity 1 (since the same decimal constant 1.080 appears 3 times) may be expressed in this quite different way:

$$Q_{\text{Chang-equivalent}}(x) := \frac{5\,\mathrm{e}^{\frac{1}{2}-\frac{27}{50}\,x^2}}{27\,\sqrt{\pi}}$$

as having a decimal complexity 0, since no decimal constants are present.

### D. TOTAL COMPLEXITY

Since we have introduced 3 different complexity indexes, we propose a further index which takes into account all them together. Said $t$ the typographic complexity (see Section VIII-A), $c$ the computational complexity (see Section VIII-B), and $d$ the decimal complexity (see Section VIII-C), we compute the geometrical mean of $t$, $c+1$, and $d+1$ as:

$$\text{Total complexity} := \sqrt[3]{t\cdot(c+1)\cdot(d+1)}.$$

The $+1$ is intended to avoid the collapsing in 0 of the result when $c$ or $d$ is 0.

The value of the total complexity has always been approximated with 2 decimals. (It is not always a majorization, differently from absolute and relative errors.)

An example is reported in Section XIII-C, going from the minimum total complexity of $Q_{\text{Chernoff}}$ (2.29, reported in Table 7), through the intermediate one of $Q_{\text{Olabiyi–2}}$ (7.56, reported in Table 4) up to the highest complexity of $Q_{\text{Lipoth}}$ (13.66, reported in Table 13) approximately sextuple (with respect to the minimum total complexity of $Q_{\text{Chernoff}}$).

## IX. LEVELS OF EASINESS OF EXPLICIT INVERTIBILITY OF INCREASING FUNCTIONS

There exist plenty of approximations for $\Phi(x)$, or $Q(x)$, or $\mathrm{erf}(x)$, or $\mathrm{erfc}(x)$ (equivalently): many of them are *explicitly invertible* (see below in this section) at different levels, and in this research we have found about 60 of them, listed in Tables 3–13.

In [16] one already finds this classification of the approximations from the point of view of the *explicit invertibility*:

1) not *explicitly invertible*;
2) *explicitly invertible* solving a quartic equation;
3) *explicitly invertible* solving a generic cubic equation;
4) *explicitly invertible* solving a depressed cubic equation $x^3 + ax + b = 0$;
5) simply *explicitly invertible* solving a quadratic (or biquadratic) equation;
6) *very simply explicitly invertible*, with only 1 entry of $x$. (Concept recently used in [30].)

The *explicit invertibility* in the above classification has to be meant in the sense of *explicit invertibility* by means of elementary functions having standard names used in mathematics (see Definition in Section V) and, in the present paper, this is here plainly stated. Notice also that the above mentioned classification refers to the formal writings of the approximations chosen by their Authors (see Section IX-A).

Improving the above reported classification [16] for the levels of easiness of *explicit invertibility* of functions by means of elementary functions, here we give a new classification, which in this paper we consider only for increasing (in the considered domains) functions, so eliminating the $\pm$ ambiguity in the solution of the 2-nd degree equations and the even more complex ambiguities for 3-rd and 4-th degree equations, illustrated in Appendix B. With respect to [16], the case of the quartic equation has been splitted into *generic* and *depressed* quartic equation (see Appendix B), the quadratic and biquadratic equations have been separated, and *InvLev 6.5* has been defined, requiring the Lambert $W$-function in the *explicit inversion*, finally obtaining the following 9 levels, going from no *explicit invertibility* (*InvLev 0*) to the easiest one (*InvLev 7*):

- *InvLev 0:* no *explicit invertibility* by elementary functions and the Lambert $W$-function;
- *InvLev 1:* invertibility by elementary functions solving a *generic* quartic equation;
- *InvLev 2:* invertibility by elementary functions solving a *depressed* quartic equation;
- *InvLev 3:* invertibility by elementary functions solving a *generic* cubic equation;
- *InvLev 4:* invertibility by elementary functions solving a *depressed* cubic equation;
- *InvLev 5:* invertibility by elementary functions solving a biquadratic equation;
- *InvLev 6:* invertibility by elementary functions solving a quadratic equation;
- *InvLev 6.5:* invertibility by elementary functions and the Lambert $W$-function, with only 1 entry of $x$ when expressed using $W^{-1}(x)$;
- *InvLev 7:* invertibility by elementary functions, with only 1 entry of $x$.

In all the above cases, the invertibility by elementary functions has to be meant in the sense of elementary functions having standard names used in mathematics, and for this issue see Remarks 1 and 2 in Section V. Moreover, the solution of the polynomial equations could require also obvious substitutions, as explained in Appendix A.

*Definition:* We call *explicitly invertible, not simply*, the approximations with *InvLev* from 1 to 4, whereas we call *simply explicitly invertible* the approximations with *InvLev* from 5 to 7: in the first group, in fact, the inversion requires the solution of cubic or quartic equations, generally considered quite complicated. (This definition will be especially applied in the summarizing Table 19.)

Now we are going to illustrate the above mentioned 9 cases, in general, and with reference to the approximations of $Q(x)$.

- *InvLev 0*: not *explicitly invertible* by means of elementary functions and the Lambert $W$-function, for example
$$y = x^3 + \mathrm{e}^x$$
which, though lacking of explicit inverse, is indeed invertible being increasing because sum of increasing functions.
With reference to the approximations of $Q(x)$, let us consider, as an example, $Q^{\diamond}_{\mathrm{Cooper}}$ (63) which (at best of our knowledge) is not *explicitly invertible* by means of elementary functions and even the Lambert $W$-function. Notice that, as already mentioned in Section VII-C, the approximation (63) is the arithmetic mean of the lower bound $Q^{\diamond}_{\mathrm{Wozencraft-lower}}$ (21) with the same *InvLev 0* (function already mentioned in Section II-A6 and in Section VII-C), and the upper bound $Q_{\mathrm{Wozencraft}}$ (19), which is instead *explicitly invertible* by means of elementary functions and the Lambert $W$-function (see *InvLev 6.5* below).
- *InvLev 1*: *explicitly invertible* by means of elementary functions solving a *generic* quartic equation (or 4-th degree equation). With reference to the approximation of the $Q$-function, an *InvLev 1* approximation is $Q_{\mathrm{Chiani-2}}$, reported in Table 5,
$$y = \frac{1}{12}\,\mathrm{e}^{-\frac{x^2}{2}} + \frac{1}{4}\,\mathrm{e}^{-2\,\frac{x^2}{3}}$$
which, with the obvious substitution $z := \mathrm{e}^{-\frac{x^2}{6}}$, gives the quartic equation (in $z$ with parameter $y$)
$$y = \frac{z^3}{12} + \frac{z^4}{4},$$
and then
$$x(y) = \sqrt{-6\,\ln z(y)}.$$

A different example is $Q_{\text{Hastings-1}}$, reported in Table 3 with its values $a$, $b$, $c$, and $d$,

$$y = \frac{1}{2}\left(1 + a\,x + b\,x^2 + c\,x^3 + d\,x^4\right)^{-4}$$

which gives, without any substitution, the quartic equation (in $x$ with parameter $y$)

$$(2\,y)^{-\frac{1}{4}} = 1 + a\,x + b\,x^2 + c\,x^3 + d\,x^4.$$

- *InvLev 2*: *explicitly invertible* by means of elementary functions solving a *depressed* quartic equation, lacking of the cubic (or 3-rd degree) term; of course the quartic equation of *InvLev 2* is intended not to be $y = x^4 + a$, which would lead to *InvLev7*, nor $y = x^4{+}ax^2{+}b$, which would lead to *InvLev5* (see above in this section). As far as the approximation of the $Q$-function is concerned, an *InvLev 2* approximation is $Q_{\text{Chiani–3}}$, reported in Table 5,

$$y = \frac{1}{6}\,\mathrm{e}^{-2\,x^2} + \frac{1}{12}\,\mathrm{e}^{-x^2} + \frac{1}{4}\,\mathrm{e}^{-\frac{x^2}{2}}$$

which, with the obvious substitution $z := \mathrm{e}^{-\frac{x^2}{2}}$, gives the *depressed* quartic equation (in $z$ with parameter $y$)

$$y = \frac{z^4}{6} + \frac{z^2}{12} + \frac{z}{4}$$

and then

$$x(y) = \sqrt{-2\,\ln z(y)}.$$

- *InvLev 3*: *explicitly invertible* by means of elementary functions solving a *generic* cubic equation (or 3-rd degree equation). Some straightforward examples of approximation of the $Q$-function of this invertibility level are $Q_{\text{Boiroju-1}}$, reported in Table 3 and $Q_{\text{Derenzo}}$, reported in Table 6, needing no substitutions.
Another example, needing instead a substitution, is $Q_{\text{Olabiyi–3}}$, reported in Table 5,

$$y = 0.16785\,\mathrm{e}^{-0.53245\,x^2} + 0.16805\,\mathrm{e}^{-1.0649\,x^2} + 0.01525\,\mathrm{e}^{-1.59735\,x^2}$$

which, with the obvious substitution $z := \mathrm{e}^{-0.53245\,x^2}$, gives the cubic equation (in $z$ with parameter $y$)

$$y = 0.16785\,z + 0.16805\,z^2 + 0.01525\,z^3$$

and then

$$x(y) = \sqrt{-\frac{1}{0.53245}\,\ln z}.$$

Finally, a quite different example is $Q_{\text{Brophy}}$, reported in Table 9 with its values $a$, $b$, and $c$ (and a correction note),

$$y = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mathrm{e}^{-x^2\,(a - x^2\,(b - c\,x^2))}}$$

which gives, with the obvious substitution $z := x^2$, the cubic equation (in $z$ with parameter $y$)

$$\ln\left(1 - \left(2\left(\frac{1}{2} - y\right)\right)^2\right) = -z\,(a - z\,(b - c\,z))$$

and then

$$x(y) = \sqrt{z(y)}.$$

- *InvLev 4*: *explicitly invertible* by means of elementary functions solving a *depressed* cubic equation, lacking of the quadratic (or 2-nd degree) term. An example of approximation of the $Q$-function of this invertibility level is $Q_{\text{Page}}$, reported in Table 11,

$$y = \frac{1}{1 + \mathrm{e}^{\sqrt{8/\pi}\,(x + 0.044715\,x^3)}} = \frac{1}{2} - \frac{1}{2}\,\tanh\left(\sqrt{\frac{2}{\pi}}\left(x + 0.044715\,x^3\right)\right)$$

which gives the *depressed* cubic equation (in $x$ with parameter $y$)

$$\ln\left(\frac{1}{y} - 1\right) = \sqrt{\frac{8}{\pi}}\left(x + 0.044715\,x^3\right).$$

Other similar examples are $Q_{\text{Vedder}}$, $Q_{\text{Bowling-2}}$, and $Q_{\text{Zogheib-1}}$, reported in the same table, whose inversion proceeds in the same way. The *explicit inverses* of $Q_{\text{Page}}$, $Q_{\text{Vedder}}$ [27], and $Q_{\text{Bowling-2}}$ are **reported in Table 14**.
Starting, to consider another example, from the class of approximations

$$Q_{\text{Chiani-class}}(x;\,\theta) := \left(\frac{1}{4} - \frac{\theta}{2\pi}\right)\mathrm{e}^{-x^2/2} + \frac{1}{4}\mathrm{e}^{-\frac{x^2}{2\sin^2\theta}}, \tag{64}$$

originally given in [11] for $\operatorname{erfc}(x)$, here another approximation of this invertibility level is obtained with $\theta := \arcsin\frac{1}{\sqrt{3}}$, which we name $Q_{\text{Chiani–new–1}}$ in (65), cited at the end of Table 5,

$$y = \left(\frac{1}{4} - \frac{1}{2\pi}\,\arcsin\frac{1}{\sqrt{3}}\right)\mathrm{e}^{-\frac{x^2}{2}} + \frac{1}{4}\,\mathrm{e}^{-\frac{3}{2}x^2} \tag{65}$$

which, with the obvious substitution $z := \mathrm{e}^{-x^2/2}$, gives the *depressed* cubic equation (in $z$ with parameter $y$)

$$y = \left(\frac{1}{4} - \frac{1}{2\pi}\,\arcsin\frac{1}{\sqrt{3}}\right)z + \frac{1}{4}\,z^3$$

and then

$$x(y) = \sqrt{-2\,\ln z(y)}.$$

- *InvLev 5*: *explicitly invertible* by means of elementary functions solving a biquadratic equation (4-th degree equation lacking of the 3-rd degree and 1-st degree terms). An example of approximation of the $Q$-function of this invertibility level is $Q_{\text{Winitzki}}$, reported in Table 9,

$$y = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mathrm{e}^{-x^2\,\frac{4/\pi + 0.147\,x^2/2}{2 + 0.147\,x^2}}},$$

TABLE 14. A published inverse of already published *InvLev 4* approximations of $Q(x)$.

| Types of the approximations of $Q(x)$ | Names of the approximations of $Q(x)$ | Formula approximating $Q(x)$ | Inverse |
|---|---|---|---|
| *Type 5* | $Q_{\mathrm{Vedder}}$ [27] <br> $Q_{\mathrm{Page}}$ <br> $Q_{\mathrm{Bowling-2}}$ | $\frac{1}{1+e^{a\,x\,+b\,x^3}}$ | $-2\,\sqrt{A}\sinh\left(\frac{\sinh^{-1}\left(B/\sqrt{A^3}\right)}{3}\right)$ <br> $A=\frac{a}{3\,b}$ <br> $B=\frac{\ln(y/(1-y))}{2\,b}$ |

TABLE 15. A published inverse of already published *InvLev 5* approximations of $Q(x)$.

| Types of the approximations of $Q(x)$ | Names of the approximations of $Q(x)$ | Formula approximating $Q(x)$ | Inverse |
|---|---|---|---|
| *Type 3* | $Q_{\mathrm{Winitzki}}$ [23] <br> $Q_{\mathrm{Soranzo-2}}$ | $\frac{1}{2}-\frac{1}{2}\sqrt{1-e^{-x^2\,\frac{a+b\,x^2}{c+2\,b\,x^2}}}$ | $\sqrt{-\left(\frac{a}{2\,b}+t\right)+\sqrt{\left(\frac{a}{2\,b}+t\right)^2-\frac{t\,c}{b}}}$ <br> $t=\ln\left(1-(1-2\,y)^2\right)$ |

whose inversion proceeds in this way:

$$2\left(\frac{1}{2}-y\right)=\sqrt{1-e^{-x^2\,\frac{4/\pi+0.147\,x^2/2}{2+0.147\,x^2}}}$$

$$1-4\,y+4\,y^2=1-e^{-x^2\,\frac{4/\pi+0.147\,x^2/2}{2+0.147\,x^2}}$$

$$e^{-x^2\,\frac{4/\pi+0.147\,x^2/2}{2+0.147\,x^2}}=4\,y-4\,y^2$$

$$\frac{-(4/\pi)\,x^2-0.147\,x^4/2}{2+0.147\,x^2}=\ln\left(4\,y-4\,y^2\right)$$

$$-\frac{4}{\pi}\,x^2-0.147\,\frac{x^4}{2}$$
$$-\left(2+0.147\,x^2\right)\,\ln\left(4\,y-4\,y^2\right)=0$$

which is a biquadratic equation (in $x$ with parameter $y$). Other similar examples are $Q_{\text{Soranzo-1}}$ and $Q_{\text{Soranzo-2}}$, reported in the same table, whose inversion proceeds in the same way. The *explicit inverses* of $Q_{\text{Winitzki}}$ [23] and $Q_{\text{Soranzo-2}}$ are **reported in Table 15**.

- *InvLev 6*: *explicitly invertible* by means of elementary functions solving a quadratic equation (or 2-nd degree equation), having the linear term. An immediate example of approximation of the $Q$-function of this invertibility level is $Q_{\text{Chu}}$, reported in Table 3,

$$y=\frac{1}{2}-\frac{x}{\sqrt{2\left(\pi+2\,x^2\right)}}$$

which, for $x\geq 0$, gives the second degree equation (in $x$ with parameter $y$)

$$2\left(\pi+2\,x^2\right)\left(\frac{1}{2}-y\right)^2=x^2$$

from which, successively,

$$2\,\pi\left(\frac{1}{2}-y\right)^2+4\,x^2\left(\frac{1}{2}-y\right)^2-x^2=0$$

$$2\,\pi\left(\frac{1}{2}-y\right)^2-4\,x^2\left(-\left(\frac{1}{2}-y\right)^2+\frac{1}{4}\right)=0$$

$$2\,\pi\left(\frac{1}{2}-y\right)^2=4\,x^2\left(y-y^2\right)$$

$$x^2=\frac{\pi\,\left(\frac{1}{2}-y\right)^2}{2\,\left(y-y^2\right)}$$

and, without ambiguity of sign, the *explicit inverse*

$$x(y)=\frac{\frac{1}{2}-y}{\sqrt{\frac{2}{\pi}\,\left(y-y^2\right)}}$$

**reported in Table 16**.

A quite different example, requiring logarithms, is $Q_{\text{Sofotasios}}$, reported in Table 6,

$$y=0.49\,e^{-\frac{x^2}{2}-\frac{8}{13}\,x}$$

which gives the second degree equation (in $x$ with parameter $y$)

$$\ln\left(\frac{y}{0.49}\right)=-\frac{x^2}{2}-\frac{8}{13}\,x.$$

Analogous examples are $Q_{\text{Mastin–1}}$, $Q_{\text{Mastin-2}}$, $Q_{\text{Mastin-3}}$, $Q_{\text{Lin-1}}$, $Q_{\text{Benitez–1}}$, and $Q_{\text{Benitez–2}}$ of Table 6, which clearly have *InvLev 6* since their inversion obviously produces second degree equations after taking logarithms. The *explicit inverses* of $Q_{\text{Mastin–1}}$, $Q_{\text{Mastin-2}}$, $Q_{\text{Mastin-3}}$, and $Q_{\text{Lin-1}}$ [18] are **reported in Table 16**.

As further different examples here we may consider

$$Q_{\text{Chiani–new–2}}(x):=\frac{3}{8}\,e^{-\frac{x^2}{2}}+\frac{1}{4}\,e^{-x^2}=y \tag{66}$$

cited at the end of Table 5, which is immediately obtained from the class (64) with $\theta=\pi/4$, whose

**TABLE 16.** **Some new and published inverses of already published *InvLev 6* approximations of $Q(x)$.**

| Types of the approximations of $Q(x)$ | Names of the approximations of $Q(x)$ | Formula approximating $Q(x)$ | Inverse |
|---|---|---|---|
| *Type 0* | $Q_{\text{Chu}}$ | $\frac{1}{2}-\frac{x}{\sqrt{2(\pi+2x^2)}}$ | $\frac{\frac{1}{2}-y}{\sqrt{\frac{2}{\pi}\,(y-y^2)}}$ |
| *Type 1* | $Q_{\text{Chiani}-1}$ [11] $Q_{\text{Wu}-2}$ $Q_{\text{Powari}}$ $Q_{\text{Olabiyi}-2}$ [9] | $a\,\mathrm{e}^{-b\,x^2}+c\,\mathrm{e}^{-2\,b\,x^2}$ | $\sqrt{-\frac{1}{b}\,\ln\frac{-a+\sqrt{a^2+4\,c\,y}}{2\,c}}=$ $=\sqrt{\frac{1}{b}\,\ln\frac{a+\sqrt{a^2+4\,c\,y}}{2\,y}}$ |
| *Type 2* | $Q_{\text{Mastin}-2}$ $Q_{\text{Mastin}-3}$ $Q_{\text{Mastin}-1}$ $Q_{\text{Lin}-1}$ [18] | $\frac{1}{2}\,\mathrm{e}^{-a\,x^2-b\,x}$ | $\frac{-b+\sqrt{b^2-4\,a\,\ln(2\,y)}}{2\,a}$ |
| *Type 3* | $Q_{\text{Hamaker}}$ [17] | $\frac{1}{2}-\frac{1}{2}\sqrt{1-\mathrm{e}^{-(a\,x\,(1-b\,x))^2}}$ $a=0.806$ $b=0.018$ | $a\sqrt{-\ln(4\,y\,(1-y))}\,(1+b\sqrt{-\ln(4\,y\,(1-y))}$ $a=1.238$ $b=0.0262$ |
| *Type 3* | $Q_{\text{Lin}-2}$ [20] | $\frac{1}{2}-\frac{1}{2}\sqrt{1-\mathrm{e}^{-\left(\frac{x}{a+b\,x}\right)^2}}$ | $\frac{\sqrt{-a^2\,\ln\left(1-(1-2\,y)^2\right)}}{1-\sqrt{-b^2\,\ln\left(1-(1-2\,y)^2\right)}}=\frac{a\sqrt{-\ln(4\,y\,(1-y))}}{1-b\sqrt{-\ln(4\,y\,(1-y))}}$ |

inversion is made substituting $z := e^{-\frac{x^2}{2}}$, giving the second degree equation (in $z$ with parameter $y$)

$$y=\frac{3}{8}\,z+\frac{z^2}{4}$$

and then

$$x(y)=\sqrt{-2\,\ln z(y)},$$

and $Q_{\text{Chiani-1}}$ (which is better) of Table 4, whose inversion proceeds similarly with the same substitution (reported in the table). Other similar examples shown in the same table are $Q_{\text{Wu-2}}$, $Q_{\text{Powari}}$, and $Q_{\text{Olabiyi-2}}$, whose inversion proceeds similarly but with different substitutions (reported in the table). The *explicit inverses* of $Q_{\text{Chiani-1}}$ [11], $Q_{\text{Wu-2}}$, $Q_{\text{Powari}}$, and $Q_{\text{Olabiyi-2}}$ [9] are **reported in Table 16**.

Another approximation of the same invertibility level is $Q_{\text{Divgi}}$ reported in Table 11, whose inversion proceeds at the beginning similarly to the inversion of the *InvLev 4* approximation $Q_{\text{Page}}$ shown before, but in the end solving a 2-nd degree equation instead of a *depressed* cubic one.

Finally, for $Q_{\text{Hamaker}}$ [17] and $Q_{\text{Lin-2}}$ [20], reported in Table 8, the *explicit inverses* are **shown in Table 16**.

- *InvLev 6.5: explicitly invertible* by means of elementary functions and the Lambert $W$-function, with only 1 entry of $x$ when expressed using $W^{-1}(x)$, inverse of the Lambert $W$-function $W(y)$ – more precisely of its restriction for $x \geq 0$, considered in this paper, as illustrated in Section V – as we are just going to show in the following example. All the functions of *InvLev 6.5* would belong to *InvLev 7* if the principal branch of the real Lambert $W$-function was considered an elementary function, which historically has not been done.

The basic example of this invertibility level is $Q_{\text{Wozencraft}}$ reported in Table 10 (see also Section VII-C):

$$\begin{aligned}y&=\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}\,x}\\&=\left(\left(\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\,\pi}\,x}\right)^2\right)^{\frac{1}{2}}\\&=\left(\frac{\mathrm{e}^{-x^2}}{2\,\pi\,x^2}\right)^{\frac{1}{2}}\\&=\left(\frac{1}{2\,\pi\,x^2\mathrm{e}^{x^2}}\right)^{\frac{1}{2}}\end{aligned}$$

and, by means of the function $W^{-1}(x)=x\,e^x$ (see (51)), inverse of the Lambert $W$-function $W(y)$,

$$y=\left(\frac{1}{2\,\pi\,W^{-1}\left(x^2\right)}\right)^{\frac{1}{2}}$$

which shows clearly the single recurrency of $x$, when admitting $W^{-1}(x)$ in the set of allowable functions. It follows that

$$W^{-1}\left(x^2\right)=\frac{1}{2\,\pi\,y^2}$$

from which, inverting,

$$x^2=W\left(\frac{1}{2\,\pi\,y^2}\right)$$

**TABLE 17.** Some new inverses of already published *InvLev 6.5* approximations of $Q(x)$.

| Types of the approximations of $Q(x)$ | Names of the approximations of $Q(x)$ | Formula approximating $Q(x)$ | Inverse |
|---|---|---|---|
| *Type 4* | $Q_{\text{Wozencraft}}$ | $\frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}$ | $\sqrt{W\left(\frac{1}{2\pi y^2}\right)}$ |
| *Type 4* | $Q_{\text{Borjesson}-1}$ | $\frac{e^{-x^2/2}}{\sqrt{2\pi}\sqrt{x^2+1}}$ | $\sqrt{W\left(\frac{e}{2\pi y^2}\right)-1}$ |

and then, with no ambiguity of sign,

$$x(y) = Q^{-1}_{\text{Wozencraft}}(y) = \sqrt{W\left(\frac{1}{2\pi y^2}\right)}$$

is **reported in Table 17**, determining *InvLev 6.5* (instead of *InvLev 7*) because of the use of the (not elementary) Lambert $W$-function in the *explicit inversion*.

As a further more complex example here we consider $Q_{\text{Borjesson–1}}$, reported in Table 10, essentially an improvement of $Q_{\text{Wozencraft}}$ obtained substituting $x$ with $\sqrt{x^2+1}$, granting a better behaviour near 0:

$$y = \frac{e^{-x^2/2}}{\sqrt{2\pi}\sqrt{x^2+1}} \tag{67}$$

whose inversion proceeds similarly, only a bit more complex:

$$\begin{aligned} y &= \left(\left(\frac{e^{-x^2/2}}{\sqrt{2\pi}\sqrt{x^2+1}}\right)^2\right)^{\frac{1}{2}} \\ &= \left(\frac{e^{-x^2}}{2\pi\left(x^2+1\right)}\right)^{\frac{1}{2}} \\ &= \left(\frac{e\, e^{-x^2-1}}{2\pi\left(x^2+1\right)}\right)^{\frac{1}{2}} \\ &= \left(\frac{e}{2\pi\, e^{x^2+1}\left(x^2+1\right)}\right)^{\frac{1}{2}} \\ &= \left(\frac{e}{2\pi\, W^{-1}\left(x^2+1\right)}\right)^{\frac{1}{2}}, \end{aligned}$$

determining *InvLev 6.5* (instead of *InvLev 7*) because of the use of the (not elementary) Lambert $W$-function in the *explicit inversion*:

$$x(y) = Q^{-1}_{\text{Borjesson–1}}(y) := \sqrt{W\left(\frac{e}{2\pi y^2}\right)-1}.$$

This *explicit inverse* is **reported in Table 17**, too.

- *InvLev 7*: *explicitly invertible* by means of elementary functions with only 1 entry of $x$, for example $Q_{\text{Soranzo–3}}$, reported in Table 12,

$$y = 1 - 2^{-22^{1-41^{x/10}}}$$

whose *explicit inverse* is

$$x(y) = 10\,\log_{41}\left(1-\log_{22}(-\log_2(1-y))\right),$$

**reported** in terms of natural logarithms in [22] and here **in Table 18**.

**In Table 18** are **reported** the *explicit inverses* of all the other *InvLev 7* approximations (including $Q_{\text{Soranzo–3}}$ [22]):

- $Q_{\text{Burr}}$ (reported in Table 3);
- $Q_{\text{Ordaz}}$, $Q_{\text{Hanandeh–2}}$ [33] (reported in Table 6);
- $Q_{\text{Chernoff}}$, $Q_{\text{Chernoff–impr.}}$, $Q_{\text{Ermolova–2}}$, $Q_{\text{Gasull}}$ $Q_{\text{Olabiyi–1}}$, $Q_{\text{Wu–1}}$, $Q_{\text{Ermolova–1}}$, $Q_{\text{Chang}}$ (reported in Table 7);
- $Q_{\text{Abderrahmane–1}}$, $Q_{\text{Polya}}$, $Q_{\text{Boiroju–2}}$, $Q_{\text{Aludaat}}$ [19], $Q_{\text{Eidous}}$ [28], $Q_{\text{Abderrahmane-2}}$, $Q_{\text{Hanandeh-4}}$ [34] (reported in Table 8);
- $Q_{\text{Abderrahmane–3}}$, $Q_{\text{Tocher}}$, $Q_{\text{Hanandeh–1}}$ [33], $Q_{\text{Bowling–1}}$, $Q_{\text{Johnson–1}}$ (reported in Table 11);
- $Q_{\text{Hanandeh-3}}$ [33], $Q_{\text{Zogheib–2}}$, $Q_{\text{Soranzo–4}}$ [22] (reported in Table 12);
- $Q_{\text{Lipoth}}$, and $Q_{\text{Kundu}}$ (reported in Table 13).

## A. SOME NOTES ON TYPES AND INVERTIBILITY LEVELS

The types (from *Type 0* to *Type 7*), considered in Section VI, and the above considered invertibility levels (from *InvLev 0* to *InvLev 7*) are essentially independent. The tables described and commented in Section XI have been organized according to types and, secondarily, to decreasing *InvLev*, reported for any approximation.

The distinction among the invertibility levels is not always *intrinsic*: sometimes it is essentially a personal choice of the Authors. In particular, of course all the *InvLev 6* approximations may be written in an *InvLev 7* form with only 1 entry of $x$. For example,

$$Q_{\text{Sofotasios}}(x) := 0.49\, e^{-\frac{x^2}{2}-\frac{8}{13}x}$$

reported in Table 6 may be written as

$$Q_{\text{Sofotasios-equivalent}}(x) := 0.49\, e^{-\frac{1}{2}\left(x+\frac{8}{13}\right)^2+\frac{32}{169}}.$$

A different example of the same conversion from *InvLev 6* to *InvLev 7* may be obtained with

$$Q_{\text{Chiani–1}}(x) := \frac{1}{4}e^{-x^2} + \frac{1}{4}e^{-\frac{x^2}{2}}$$

reported in Table 4 which may be written as

$$Q_{\text{Chiani–1–equivalent}}(x) := \frac{1}{4}\left(e^{-\frac{x^2}{2}}+\frac{1}{2}\right)^2 - \frac{1}{16}. \tag{68}$$

Conversely, some *InvLev 7* approximations as

$$Q_{\text{Ordaz}}(x) := 0.6931\, e^{-\left(\frac{9x+8}{14}\right)^2},$$

reported in Table 6, could be presented with *InvLev 6* expanding the square, and in any case the inverse is

$$\frac{2}{9}\left(-4+7\sqrt{\ln\frac{0.6931}{y}}\right)$$

reported in Table 18.

**TABLE 18.** Some new and published inverses of already published *InvLev 7* approximations of $Q(x)$.

| Types of the approximations of $Q(x)$ | Names of the approximations of $Q(x)$ | Formula approximating $Q(x)$ | Inverse |
|---|---|---|---|
| *Type 0* | $Q_{\mathrm{Burr}}$ | $(1+(a+b\,x)^c)^{-d}$ | $\frac{1}{b}\left(\left(y^{-\frac{1}{d}}-1\right)-a\right)$ |
| *Type 2* | $Q_{\mathrm{Ordaz}}$<br>$Q_{\mathrm{Hanandeh-2}}$ [33] | $a\,\mathrm{e}^{-\left(\frac{9x+8}{14}\right)^2}$ | $\frac{2}{9}\left(-4+7\sqrt{\ln\frac{a}{y}}\right)$ |
| Chernoff type | $Q_{\mathrm{Chernoff}}$<br>$Q_{\mathrm{Chernoff-impr.}}$<br>$Q_{\mathrm{Ermolova-2}}$<br>$Q_{\mathrm{Gasull}}$<br>$Q_{\mathrm{Olabiyi-1}}$<br>$Q_{\mathrm{Wu-1}}$<br>$Q_{\mathrm{Ermolova-1}}$<br>$Q_{\mathrm{Chang}}$ | $a\,\mathrm{e}^{-b\,x^2}$ | $\sqrt{-\frac{1}{b}\,\ln\frac{y}{a}} = \sqrt{\frac{1}{b}\,\ln\frac{a}{y}}$ |
| *Type 3* | $Q_{\mathrm{Abderrahmane-1}}$ | $a-b\,\sqrt{1-\mathrm{e}^{-c\,x^2}}$ | $\sqrt{-\frac{1}{c}\,\ln\left(1-\frac{1}{b^2}\,(a-y)^2\right)}$ |
| *Type 3* | $Q_{\mathrm{Polya}}$<br>$Q_{\mathrm{Boiroju-2}}$<br>$Q_{\mathrm{Aludaat}}$ [19] $^{(*)}$<br>$Q_{\mathrm{Eidous}}$ [28]<br>$Q_{\mathrm{Abderrahmane-2}}$<br>$Q_{\mathrm{Hanandeh-4}}$ [34] | $\frac{1}{2}-\frac{1}{2}\,\sqrt{1-\mathrm{e}^{-a\,x^2}}$ | $\sqrt{-\frac{1}{a}\,\ln\left(1-(1-2\,y)^2\right)}$ |
| *Type 5* | $Q_{\mathrm{Abderrahmane-3}}$ | $1-\frac{a}{b+\mathrm{e}^{-c\,x}}$ | $-\frac{1}{c}\,\ln\left(b-\frac{a}{1-y}\right)$ |
| *Type 5* | $Q_{\mathrm{Tocher}}$ [32]<br>$Q_{\mathrm{Hanandeh-1}}$ [33]<br>$Q_{\mathrm{Bowling-1}}$<br>$Q_{\mathrm{Johnson-1}}$ | $\frac{1}{1+\mathrm{e}^{a\,x}}$ | $\frac{1}{a}\,\ln\left(\frac{1}{y}-1\right)$ |
| *Type 6* | $Q_{\mathrm{Hanandeh-3}}$ [33]<br>$Q_{\mathrm{Zogheib-2}}$ | $\frac{1}{2}\,\mathrm{e}^{-a\,x^b}$ | $\left(-\frac{1}{a}\,\ln(2\,y)\right)^{\frac{1}{b}}$ |
| *Type 6* | $Q_{\mathrm{Soranzo-3}}$ [22] | $1-2^{-22^{1-41^{x/10}}}$ | $\frac{10}{\ln 41}\,\ln\left(1-\frac{\ln(-\ln(1-y)/\ln 2)}{\ln 22}\right)$ |
| *Type 6* | $Q_{\mathrm{Soranzo-4}}$ [22] | $1-2^{-54^{1-1.3671^x}}$ | $\frac{1}{\ln 1.3671}\,\ln\left(1-\frac{\ln(-\ln(1-y)/\ln 2)}{\ln 54}\right)$ |
| *Type 7* | $Q_{\mathrm{Lipoth}}$ | $1-\left(1+a\,\left(\ln\left(1+\mathrm{e}^{-\frac{x}{h}+c}\right)\right)^b\right)^{-d}$ | $\left(c-\ln\left(\exp\left(\frac{(1-y)^{-\frac{1}{d}}-1}{a}\right)^{\frac{1}{b}}-1\right)\right)h$ |
| *Type 7* | $Q_{\mathrm{Kundu}}$ | $1-\left(1-\mathrm{e}^{-\mathrm{e}^{a\,x+b}}\right)^c$ | $\frac{1}{a}\left(\ln\left(-\ln\left(1-(1-y)^{\frac{1}{c}}\right)\right)-b\right)$ |
| $(*)$ Published with the oversight $p-0.5$ instead of $2\,p-1$. | | | |

## X. APPROXIMATIONS AND BOUNDS

For the purposes of telecommunications theory, a good quality of an approximation of $Q(x)$ is to be a bound for $x > 0$ and, in the perspective of this research, to be a bound for $x \in I_{\mathrm{significant}}$.

Let us take as reference for instance the BEP (39), $Q(\sqrt{\frac{2E_b}{N_0}})$, holding for the simple binary digital modulation schemes, where $E_b/N_0$ is the signal-to-noise ratio, $E_b$ is the signal energy associated to a bit, and $\sigma^2 = N_0/2$ is the variance of the channel noise.

Given a signal-to-noise ratio $E_b/N_0$, lower bounds of $Q(x)$ are useful for estimating the least possible values of an error detection probability, such as BEP, and thus can be used as a performance measure. Instead, given a signal-to-noise ratio $E_b/N_0$, upper bounds give an estimate of the greatest possible error detection probability, and are even of greater interest in performance analysis because, in this sense, they allow to consider the so-called *worst case*. Notice also that in telecommunications practice it could be more desirable to have a looser upper bound than a tighter approximation, in the sense that it is usual to renounce some precision in favour of having an upper bound, so granting that the BEP calculated using the upper bound is in fact greater than the real one (pessimistic evaluation).

Conversely, since the inverse – if existing – of a lower (or upper bound) of $Q(x)$ is a lower (or upper bound, respectively) of $Q^{-1}(y)$ (as recalled in Section VII-D), given an error detection probability, such as BEP, the lower (or upper) bounds of $Q^{-1}(y)$ are useful for estimating the least (or greatest, respectively) possible values of signal-to-noise ratio $E_b/N_0$ needed to attain that BEP. Somehow similarly as before, it could be more desirable to have a looser upper bound of $Q^{-1}(y)$ than a tighter approximation of it, to grant that the signal-to-noise ratio $E_b/N_0$ – needed to attain a fixed BEP – calculated using the upper bound is in fact greater than that really needed (pessimistic evaluation).

Here below we list all the published (found in this research) *explicitly invertible* upper (Section X-A) and, then, lower bounds (Section X-B), for $x > 0$. They are all to be considered approximations of $Q(x)$, falling into the 8 types considered in Section VI. Moreover, we address some particularly interesting not *explicitly invertible* lower bounds.

For many of those bounds, analytic mathematical (not only graphical) proofs (of the fact that they are bounds) have been published, holding on the whole interval $(0, +\infty)$ or $[0, +\infty)$.

### A. UPPER BOUNDS

We begin with the upper bound $Q_{\text{Wozencraft}}$ (19) holding $\forall x > 0$, which stands out also for its asymptotic equivalence to $Q(x)$ (see Section VII-C) and its *explicit invertibility* (by means of the Lambert $W$-function, see Sections V and IX).

Then we list the most classical and widely used upper bound for $Q(x)$ in telecommunications theory, the Chernoff bound $Q_{\text{Chernoff}}$ (11) and its improvement $Q_{\text{Chernoff–impr.}}$ (10), which in this paper have been proved to be upper bounds on the whole $[0, +\infty)$ in Section II-A3. When considering the whole $[0, +\infty)$, the coefficient $\frac{1}{2}$ multiplying the exponential obviously cannot be reduced because of the value $\frac{1}{2}$ of $Q(x)$ in 0, and also the coefficient $-\frac{1}{2}$ of the monomial $x^2$ cannot [46] be modified to have a tighter upper bound.

For large $x$, $Q_{\text{Chernoff}}$ (11) and $Q_{\text{Chernoff–impr.}}$ (10) have, when intended as approximations of $Q(x)$, quite low absolute errors but high relative errors (see Section VII-C). In particular, in $I_{\text{significant}}$, for $Q_{\text{Chernoff–impr.}}$ it is

$$\varepsilon \approx 1.6 \cdot 10^{-1} \qquad \varepsilon(4.5) \approx 1.7 \cdot 10^{-5}$$
$$\varepsilon_r = \varepsilon_r(4.5) \approx 4.9$$

a $\approx 490\%$ error, with a ratio 4.9: 0.16, more than 30, between $\varepsilon_r$ and $\varepsilon$ (see (58)), and an even more marked ($\approx 2.9 \cdot 10^5$) ratio between $\varepsilon_r(x)$ and $\varepsilon(x)$ in $x = 4.5$. (For the symbols $\varepsilon$, $\varepsilon_r$, $\varepsilon(x)$, and $\varepsilon_r(x)$, see Section VII-A.)

Two other noticeable upper bounds on the whole $[0, +\infty)$, both *explicitly invertible*, originally given in [11] for erfc$(x)$, are $Q_{\text{Chiani–1}}$ (12) and

$$Q_{\text{Chiani–3}} := \frac{1}{6}\,\mathrm{e}^{-2x^2} + \frac{1}{12}\,\mathrm{e}^{-x^2} + \frac{1}{4}\,\mathrm{e}^{-\frac{x^2}{2}},$$

reported in Tables 4 and 5, respectively. (Noticeably the inversion of the latter gives a *depressed* quartic equation.)

Notice that $Q_{\text{Borjesson–1}}(x) := \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}\,\sqrt{x^2+1}}$ (see Table 10) is not classically considered a bound for $Q(x)$ (for $x \geq 0$) but it is an upper bound for $Q(x)$ in $I_{\text{significant}}$, and in Section VII-C it is shown that it is asymptotically equivalent to $Q(x)$.

The other published upper bounds of $Q(x)$ on the whole $[0, +\infty)$, found in this research, are:

1) $Q_{\text{Chu}}$ reported in Table 3
2) and $Q_{\text{Mastin-2}}$ reported in Table 6.

Finally, we have found 2 further upper bounds for $Q(x)$ in $I_{\text{significant}}$:

1) $Q_{\text{Tocher}}$ reported in Table 11
2) and $Q_{\text{Vedder}}$ reported in the same table.

Notice that $Q_{\text{Tocher}}$ is very likely to be an upper bound on the whole $[0, +\infty)$, but this has never been analytically proven – nor even that $(1 - Q_{\text{Tocher}}(x))$ is a lower bound for $\Phi(x)$ – and a graphical demonstration is not feasible, and then in our Table 11 is classified as an upper bound in $I_{\text{significant}}$, which may be easily seen graphically. Notice also that the ancient (1963) approximation $(1 - Q_{\text{Tocher}}(x))$, essentially of statistical interest, is widely cited: a Google search (July 2023) for "Tocher's approximation" finds about 11300 results.

### B. LOWER BOUNDS

Already we have seen the lower bound on the whole $(0, +\infty)$ $Q^{\diamond}_{\text{Wozencraft–lower}}$, reported in (21), and in Section VII-C we show that it is asymptotically equivalent to $Q(x)$. Also the lower bound on the whole $[0, +\infty)$ $Q^{\diamond}_{\text{Gordon}}$ (20) is asymptotically equivalent to $Q(x)$ for $x \to +\infty$, and this fact is proved in Section VII-C. In [71, Fig. 2.3-2] there are, superimposed, graphs of $Q(x)$, $Q_{\text{Chernoff–impr.}}$, $Q_{\text{Wozencraft}}$ and $Q^{\diamond}_{\text{Gordon}}$.

In 1942 paper [67] the improvement – with respect to $Q^{\diamond}_{\text{Gordon}}$ – $Q^{\diamond}_{\text{Birnbaum}}$ (37), holding on the whole $[0, +\infty)$, is presented, larger than $Q^{\diamond}_{\text{Gordon}}$ but, nevertheless, lower than $Q(x)$. Also this approximation is asymptotically equivalent to $Q(x)$.

Correspondingly to the Chernoff upper bounds considered in Section X-A, *Chernoff type* (see Remark 2 in Section VI) lower bounds, all holding on the whole $[0, +\infty)$, have also been published. In particular, the lower bound $Q_{\text{Gasull}}$ [40] reported in Table 7 and the [46] class of functions $Q_{\text{Chang–class}}$ (14), including the lower bound $Q_{\text{Chang–new}}$ (13), this other recent (2012) [107] class of functions

$$Q_{\text{Cote–class}}(x;\kappa) := \left(\frac{\mathrm{e}^{(\pi\,(\kappa-1)+2)^{-1}}}{2\,\kappa}\cdot \sqrt{\frac{1}{\pi}\,(\kappa-1)\,(\pi\,(\kappa-1)+2)}\right)\mathrm{e}^{-\kappa\,x^2/2} \qquad \kappa \geq 1, \tag{69}$$

and these more recent (2018) classes [29]

$$Q_{\text{Wu–class–1}}(x;c) := \sqrt{\frac{\mathrm{e}}{\pi}}\,\frac{\sqrt{c}}{2\,c+1}\,\mathrm{e}^{-\frac{2c+1}{4c}\,x^2} \qquad c > 0 \tag{70}$$

and [29]

$$Q_{\text{Wu-class-2}}(x;\theta) := \left(\frac{1}{2} - \frac{\theta}{\pi}\right) e^{-\frac{\cot\theta}{\pi-2\theta}x^2} \qquad 0 \le \theta \le \frac{\pi}{2} \quad (71)$$

(in which $\cot 0 = +\infty$ and $e^{-\infty} = 0$). In this latter class (71), with $\theta = \frac{\pi}{4}$, one may find the lower bound [83] $Q_{\text{Wu-1}}$ (15) (see Table 7). The above reported $Q_{\text{Wu-class-2}}$ is a subclass (with $n = 1$ and $\theta_1 = \theta$) of this [83] class of *sum* $S(x)$ *of terms* $b_i\, e^{a_i x^2}$ (*Type 1*, see Section VI) lower bounds

$$Q_{\text{Wu-class-3}}(x; n, \theta_i) := \sum_{i=1}^{n+1} b_i\, e^{a_i x^2}$$
$$a_i := \frac{1}{2}\frac{\cot\theta_i - \cot\theta_{i-1}}{\theta_i - \theta_{i-1}} \qquad b_i := \frac{\theta_i - \theta_{i-1}}{\pi}$$
$$0 = \theta_0 < \theta_1 < \cdots < \theta_{n+1} = \frac{\pi}{2}$$

(in which $\cot 0 = +\infty$ and $e^{-\infty} = 0$), here rewritten following our *Type 1* definition. This class includes also $Q_{\text{Wu-2}}$ (see Table 4) and the not *explicitly invertible* $Q^{\diamond}_{\text{Wu-3}}$ (see Remark 1 in Section VI).

The other published lower bounds on the whole $[0, +\infty)$, found in this research, are:

1) $Q_{\text{Mastin-1}}$ reported in Table 6;
2) $Q_{\text{Chang}}$ reported in Table 7;
3) $Q_{\text{Polya}}$ (the most ancient of these 3, see Section II-G) reported in Table 8.

Finally, we have found that $Q_{\text{Ermolova-1}}$, reported in Table 7, and $Q_{\text{Boiroju-2}}$, reported in Table 8, are lower bounds for $Q(x)$ in $I_{\text{significant}}$. Notice that $Q_{\text{Ermolova-1}}$ is very likely to be a lower bound on the whole $[0, +\infty)$, but this has never been analytically proven and a graphical demonstration is not feasible, and then in our Table 7 is classified as a lower bound in $I_{\text{significant}}$, which may be easily seen graphically.

### C. ON BOUNDS AND INVERSES

Quite obviously, the inverse, if existing, of a positive lower bound $Q^-(x)$ of $Q(x)$ in $[0, +\infty)$ is a lower bound of the inverse $Q^{-1}(y)$ of $Q(x)$ in $(0, Q^-(0)]$, and – in a not completely symmetrical way – the inverse, if existing, of an upper bound $Q^+(x)$ of $Q(x)$ in $[0, +\infty)$ is an upper bound of the inverse $Q^{-1}(y)$ of $Q(x)$ in the whole $(0, Q(0) = 1/2]$ (being, of course, $Q^+(0) \ge Q(0) = 1/2$). Similar things may be said when considering a domain restricted to $(0, +\infty)$ (i.e., $[0, +\infty)$ deprived of 0), as it is necessary, for example, for $Q_{\text{Wozencraft}}$ (60). Analogous things may be said for $\text{erfc}(x)$. On this basis, in [11] two upper bounds have been produced for $\text{erfc}^{-1}(y)$, which here we convert for $Q^{-1}(y)$:

$$Q^{-1}(y) < \sqrt{-2\,\ln 2\,y} \qquad 0 < y \le \frac{1}{2}$$

$$Q^{-1}(y) < \sqrt{-2\,\ln\frac{-1+\sqrt{1+16\,y}}{2}} = $$
$$= \sqrt{2\,\ln\frac{1+\sqrt{1+16\,y}}{8y}} \qquad 0 < y \le \frac{1}{2}$$

corresponding to the inversions of (10) and (12), respectively. Moreover, in [29] a class of lower bounds has been produced directly for $Q^{-1}(y)$:

$$Q^{-1}(y) > \sqrt{-\frac{4\,c}{2\,c+1}\ln\sqrt{\frac{\pi}{e\,c}}\,(2\,c+1)\,y}$$

corresponding to the inversion of (70).

Of course the consideration of $I_{\text{significant}}$ instead of $[0, +\infty)$ or $(0, +\infty)$ as domains of $Q(x)$ and its bounds, makes things quite more complicated: the inverse of a lower bound $Q^-(x)$ of $Q(x)$, when defined on $I_{\text{significant}}$, is not a lower bound of $Q^{-1}(y)$ on the whole $Q(I_{\text{significant}})$, being even not defined in $[Q^-(0.45), Q(0.45))$. Quite better is the situation for a sufficiently tight upper bound $Q^+(x)$ of $Q(x)$, because the lack of definition may hold in the essentially irrelevant interval $[Q(4.5), Q^+(4.5)]$.

## XI. TABLES OF EXPLICITLY INVERTIBLE APPROXIMATIONS OF *Q*(*X*)

In Tables 3–13 we have produced a list, as exhaustive as possible, of published *explicitly invertible* (see Definition in Section V) approximations of $Q(x)$, considering also the approximations which have been originally published for $\Phi(x)$, $\text{erfc}(x)$, $\text{erfc}\sqrt{x}\ldots$ and applying algebra and the rules of conversions reported in Table 1.

Please refer to Section VI for a comprehensive analysis of the categorization of these approximations into various *Types*, which will aid in the understanding of their mathematical properties. Additionally, please consult Section IX for an in-depth examination of their classification based on levels of invertibility (*InvLev*), which will help in comprehending the ease with which they can be inverted.

The classification of the analyzed approximations of $Q(x)$ into *Types* has led to their organization within tables in the following manner:

- *Type 0* in Table 3;
- *Type 1* in Tables 4 and 5;
- *Type 2* in Table 6;
- *Chernoff type* in Table 7 (see Remark 2 in Section VI);
- *Type 3* in Tables 8 and 9;
- *Type 4* in Table 10;
- *Type 5* in Table 11;
- *Type 6* in Table 12;
- *Type 7* in Table 13.

We have also separated the above considered Tables 3–13 in different sections by double horizontal lines, collecting together approximations with the same invertibility level *InvLev* (defined in Section IX).

In each table the approximations have been sorted in this way:

1) by decreasing *InvLev* (defined in Section IX);
2) in the case of equal *InvLev*, by decreasing relative error in absolute value $\varepsilon_r$ (defined in Section VII-A);
3) in the case of equal relative error in absolute value, by decreasing absolute error $\varepsilon$ (defined in Section VII-A);

4) in the case of equal relative error in absolute value and absolute error, by decreasing *total complexity* (defined in Section VIII-D).

The first column of each table reports the bibliographic reference in which the approximation has been published together with the number of the formula as it appears in the cited reference and the approximation name (the choice of which has been explained in Section IV-D). In case of lacking of the formula number, we have reported the approximation name as it appears in the cited reference preceded by the word "called" (as "called $\Phi_5(x)$" in Table 8). Moreover, in case of lacking of both the formula number and the approximation name we have reported the page number at which the approximation may be found in the cited reference (as "p. 32" in Table 11) or the number of the table of the reference in which it may be found (as "Table I" in Table 12). Finally, if the published approximation was not meant for $Q(x)$ we have reported in parentheses the function it was originally published for (as "Origin. for $\Phi(x)$" for the first approximation reported in Table 3, where "Origin." stands for "Originally").

In the second column of each table we have reported a possible reasonable analytical expression of the approximation which is considered in that line of the table; with "possible reasonable" we mean that, for example, in all cases we have chosen to write $\sqrt{A}$ instead of $A^{0.5}$ or $A^{\frac{1}{2}}$ and in all the other recurrences of the constant 0.5 we have written $\frac{1}{2}$ (which has the same typographic complexity). Moreover, we have reported the decimal constants, if more than 1, outside the analytical expression, calling them $a$, $b$, $c$, $d$, $h$, $k$, and choosing always the plus sign.

In Columns 3-8 we evaluate the worth of any considered approximation in these ways:

- 3-rd column: a reasonable majorization of the absolute error in $I_{\text{significant}} = [0.45, 4.5]$ (see Section VII-A);
- 4-th column: a reasonable majorization of the relative error in absolute value in the same $I_{\text{significant}} = [0.45, 4.5]$ (see Section VII-A);
- 5-th column: the typographic complexity (see Section VIII-A), the computational complexity (see Section VIII-B), and the decimal complexity (see Section VIII-C) all together;
- 6-th column: the *total complexity* (see Section VIII-D), summarizing the 3 above said complexities;
- 7-th column: the invertibility level *InvLev* (see Section IX) and eventually the table in which the *explicit inverse* is reported;
- 8-th column: the remark if the approximation is a bound or not for $Q(x)$ in $I_{\text{significant}} = [0.45, 4.5]$; in the first case we have reported if it is an "upper" or a "lower" bound (see Section X), in the second case the symbol "N" denotes the fact that the approximation is not a bound in $I_{\text{significant}}$. Moreover, we have also reported if the bound eventually has been proven to hold on the whole $(0, +\infty)$ or even on the whole $[0, +\infty)$.

We recall here that the struggle of the scientific researches considered in this survey is exactly to achieve high precision with low complexity, both defined in any reasonable way. In this sense, to compare the various approximations, we have made these 2 choices:

- as a measure of precision we have chosen (a majorization of) the relative error in absolute value $\varepsilon_r$, defined in (56) and, secondarily, (a majorization of) the absolute error $\varepsilon$, defined in (54);
- as a measure of complexity – or conversely of simplicity – we have chosen the *total complexity*, defined in Section VIII-D, which in turn is based on 3 different complexity indexes (see Section VIII).

## XII. TABLES OF PUBLISHED AND NEW APPROXIMATIONS OF THE INVERSE OF $Q(X)$

Though this is not the main aim of this paper, in Tables 14–18 we have listed some examples of approximations of the inverse of $Q(x)$, some of which have been originally published for $Q^{-1}(y)$ itself or $\phi_y = \Phi^{-1}(y)$ or $\text{InvErf}(y) = \text{erf}^{-1}(y)$ or $\text{InvErfc}(y) = \text{erfc}^{-1}(y)$, which are all equivalent by the 12 conversion formulas of Table 2, among which

$$Q^{-1}(y) = \Phi^{-1}(1 - y).$$

Please refer to Section IX for a detailed analysis of the categorization of the approximate inverses according to their degrees of invertibility (*InvLev*). This section will aid in understanding the degree of ease with which the approximations can be inverted. Furthermore, it is recommended to consult Section VI for a thorough examination of the classification of the approximations into different *Types*, as this will facilitate comprehension of their mathematical characteristics.

In Tables 14–18 have been reported new (computed in this research) and published *explicit inverses* of known approximations of $Q(x)$ with *InvLev 4*, *InvLev 5*, *InvLev 6*, *InvLev 6.5*, and *InvLev 7*, respectively, starting with the least easily invertible approximations (*InvLev 4*). These tables are organized as follows:

- the first column reports the approximation *Type* (defined in Section IX);
- the second column reports the approximation name (see Section IV-D) together with the reference number containing the published *explicit inverse* of that approximation (if no reference number is reported, this means that the inverse has been originally computed in this research);
- the third column recalls the analytic expression of the approximation;
- the last column reports the *explicit inverse* of the approximation (please consult Section IX for the step-by-step instructions required to execute the inversion process).

Inside each table, the approximations have been sorted by increasing number of *Type*. Each table has been eventually separated in different sections by double horizontal lines, collecting together approximations with the same *Type*. The approximations sharing the same analytic expression have been collected together with no horizontal line separation.

## XIII. SUMMARY OF THE MAIN RESULTS AND RESEARCH DIRECTIONS

The following 3 sections summarize the main findings of this research, including

1) the precision of the approximations, as defined in Section VII (Section XIII-A),
2) the characteristics of the *Types*, as defined in Section VI (Section XIII-B),
3) and the complexity of the approximations, as defined in Section VIII (Section XIII-C),

also suggesting intriguing research directions.

### A. MAIN RESULTS ABOUT PRECISION OF THE APPROXIMATIONS

Table 19 reports the best published approximations for each *InvLev* and each *Type*, from the viewpoint of minimizing (the majorization of) the relative error in absolute value $\varepsilon_r$ on $I_{\text{significant}}$, secondarily, of minimizing (a majorization of) the absolute error $\varepsilon$ on $I_{\text{significant}}$, and thirdly of minimizing the *total complexity*. Only once it happened that 2 approximations of the same *Type* and *InvLev* ($Q_{\text{Abderrahmane-2}}$ and $Q_{\text{Hanandeh-4}}$ in Table 8) have presented the same (majorization of the) relative error in absolute value and the same (majorization of the) absolute error: in this case, the approximation with the least *total complexity* ($Q_{\text{Hanandeh-4}}$) was chosen.

For each selected approximation, the (majorization of) the relative error in absolute value $\varepsilon_r$ on $I_{\text{significant}}$ has been reported right under the approximation name. Furthermore, the (majorization of the) absolute error and the total complexity have been indicated in the following 2 rows.

For each *InvLev* (namely, for each column), the best approximation has been highlighted.

The table has been organized in 2 sections, separated by a double vertical line, collecting *InvLev* from 1 to 4, and from 5 to 7, respectively: in the Definition of Section IX, the approximations with *InvLev* from 1 to 4 (whose inversion requires the solution of cubic or quartic equations) have been defined *explicitly invertible, not simply*, whereas the approximations with *InvLev* from 5 to 7 have been defined *simply explicitly invertible*.

The best approximation of each of the above said 2 sections (*explicitly invertible, not simply* and *simply explicitly invertible*, vertically separated) has been reported in boldface:

1) the (*Type 2*) approximation (the most precise *explicitly invertible* approximation of $Q(x)$ ever published, as far as we know)

$$Q_{\text{Derenzo}}(x) := \frac{1}{2}\,\mathrm{e}^{-\frac{(83\,x+351)\,x+562}{703/x+165}}$$

(reported in Table 6), with $\varepsilon_r < 4.2 \cdot 10^{-4}$ over $I_{\text{significant}} = [0.45, 4.5]$, is the best *InvLev 3* one and also the best of the section (approximations which are *explicitly invertible, not simply)* and of the whole table, but unluckily requiring the solution of a cubic equation in the inversion;

2) the (*Type 3*) approximation (less precise but more easily invertible than previous)

$$Q_{\text{Soranzo-2}}(x) := \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mathrm{e}^{-x^2\,\frac{17+x^2}{26.694+2\,x^2}}}$$

(reported in Table 9), with $\varepsilon_r < 2.2 \cdot 10^{-2}$ over $I_{\text{significant}} = [0.45, 4.5]$, is the best (and the only) *InvLev 5* one and also the best of the section (*simply explicitly invertible* approximations): its inversion requires the solution of a biquadratic equation, proceeds like the inversion of $Q_{\text{Winitzki}}$ [23], and the inverse is reported in Table 15.

### B. MAIN RESULTS ABOUT THE TYPES OF THE APPROXIMATIONS

In this research, a noticeable work has been done in classifying the *explicitly invertible* published approximations of $Q(x)$ identifying 7 (from *Type 0* to *Type 6*) specific types or classes of functions, leaving in a supplementary class (*Type 7*) some more approximations of miscellaneous types. These 8 types are treated in Section VI.

Some concluding words about the types as revealed in this research, following Table 19.

Let us begin with the ancient *Type 2* (*exponentials of polynomials* $A \cdot \mathrm{e}^{P(x)}$, or even *exponentials of rational functions* $A \cdot \mathrm{e}^{R(x)}$) and *Type 3* (*Pólya type approximations* of the form $\frac{1}{2} - \frac{1}{2}\sqrt{1 - \mathrm{e}^{R(x)}}$, being $R(x)$ a rational function), developments of Chernoff bound (11) and $Q_{\text{Polya}}$ (see Table 8), respectively, because the approximations of these 2 types turned out to be the best from the viewpoint of the precision (see the previous section),

In this analysis, *Type 2* has emerged to be very promising, as one *Type 2* approximation, $Q_{\text{Derenzo}}$, is the best of Table 19 (as explained in details in the previous section) and of the section including *explicitly invertible* approximations, even if *not simply*.

In *Type 3*, including essentially developments of the 1945 Pólya approximation, we have found 13 already published *explicitly invertible* approximations (see Tables 8 and 9), the maximum amount for any type. In this type, $Q_{\text{Soranzo-2}}$ presents the lowest (majorization of the) relative error in absolute value on $I_{\text{significant}}$ of the section including *simply explicitly invertible* approximations (as explained in details in the previous section). However, their worth resides essentially in being good approximations in the sense of the absolute error for $\Phi(x)$ (and $Q(x)$, the same), being the best achieved (majorization of the) absolute error $1.2 \cdot 10^{-5}$ (achieved by $Q_{\text{Soranzo-1}}$ (with the good *InvLev 5*), see Table 9).

Let us consider the other types, different from 2 and 3.

**TABLE 19.** The most precise, for any *Type* and *InvLev*, new and published approximations of $Q(x)$ on $I_{\text{significant}} = [0.45, 4.5]$.

| | *InvLev* 1 | *InvLev* 2 | *InvLev* 3 | *InvLev* 4 | *InvLev* 5 | *InvLev* 6 and 6.5 | *InvLev* 7 |
|---|---|---|---|---|---|---|---|
| *Type 0* | $Q_{\text{Hastings}-1}$ | | $Q_{\text{Boiroju}-1}$ | | | $Q_{\text{Chu}}$ | $Q_{\text{Burr}}$ |
| $\varepsilon_r$ | $5.6 \cdot 10^{0}$ | | $1.8 \cdot 10^{-1}$ | | | $5.4 \cdot 10^{3}$ | $5.6 \cdot 10^{0}$ |
| $\varepsilon$ | $2.4 \cdot 10^{-4}$ | | $2.2 \cdot 10^{-5}$ | | | $5.4 \cdot 10^{-2}$ | $2.3 \cdot 10^{-3}$ |
| Tot. compl. | 6.34 | | 5.19 | | | 3.11 | 7.49 |
| *Type 1* | $Q_{\text{Chiani}-2}$ | $Q_{\text{Chiani}-3}$ | $Q_{\text{Olabiyi}-3}$ | | | $Q_{\text{Olabiyi}-2}$ | |
| $\varepsilon_r$ | $2.7 \cdot 10^{-1}$ | $2.0 \cdot 10^{0}$ | $9.0 \cdot 10^{-2}$ | | | $6.6 \cdot 10^{-2}$ | |
| $\varepsilon$ | $3.3 \cdot 10^{-2}$ | $7.9 \cdot 10^{-2}$ | $3.0 \cdot 10^{-2}$ | | | $1.4 \cdot 10^{-2}$ | |
| Tot. compl. | 3.98 | 4.76 | 10.51 | | | 7.56 | |
| *Type 2* | | | $\mathbf{Q_{Derenzo}}$ | | | $Q_{\text{Benitez}-2}$ | $Q_{\text{Hanandeh}-2}$ |
| $\varepsilon_r$ | | | $4.2 \cdot 10^{-4}$ | | | $8.1 \cdot 10^{-2}$ | $2.5 \cdot 10^{-1}$ |
| $\varepsilon$ | | | $7.2 \cdot 10^{-5}$ | | | $2.7 \cdot 10^{-2}$ | $1.9 \cdot 10^{-3}$ |
| Tot. compl. | | | 3.87 | | | 5.85 | 4.31 |
| *Chernoff type* | | | | | | | $Q_{\text{Chang}}$ |
| $\varepsilon_r$ | | | | | | | $5.4 \cdot 10^{-1}$ |
| $\varepsilon$ | | | | | | | $1.8 \cdot 10^{-1}$ |
| Tot. compl. | | | | | | | 4.76 |
| *Type 3* | | | $Q_{\text{Brophy}}$ | | $\mathbf{Q_{Soranzo-2}}$ | $Q_{\text{Lin}-2}$ | $Q_{\text{Hanandeh}-4}$ |
| $\varepsilon_r$ | | | $6.7 \cdot 10^{-1}$ | | $2.2 \cdot 10^{-2}$ | $8.3 \cdot 10^{-2}$ | $7.6 \cdot 10^{-1}$ |
| $\varepsilon$ | | | $3.3 \cdot 10^{-5}$ | | $4.0 \cdot 10^{-5}$ | $9.2 \cdot 10^{-4}$ | $1.7 \cdot 10^{-3}$ |
| Tot. compl. | | | 8.76 | | 5.65 | 6.46 | 3.91 |
| *Type 4* | | | | | | $Q_{\text{Borjesson}-1}$ | |
| $\varepsilon_r$ | | | | | | $8.1 \cdot 10^{-2}$ | |
| $\varepsilon$ | | | | | | $1.5 \cdot 10^{-2}$ | |
| Tot. compl. | | | | | | 3.56 | |
| *Type 5* | | | | $Q_{\text{Zogheib}-1}$ | | $Q_{\text{Divgi}}$ | $Q_{\text{Johnson}-1}$ |
| $\varepsilon_r$ | | | | $5.5 \cdot 10^{-1}$ | | $1.2 \cdot 10^{1}$ | $8.3 \cdot 10^{1}$ |
| $\varepsilon$ | | | | $5.8 \cdot 10^{-4}$ | | $2.2 \cdot 10^{-3}$ | $2.3 \cdot 10^{-2}$ |
| Tot. compl. | | | | 6.14 | | 5.09 | 2.71 |
| *Type 6* | | | | | | | $Q_{\text{Soranzo}-4}$ |
| $\varepsilon_r$ | | | | | | | $7.7 \cdot 10^{-2}$ |
| $\varepsilon$ | | | | | | | $1.3 \cdot 10^{-2}$ |
| Tot. compl. | | | | | | | 4.93 |
| *Type 7* | | | | | | | $Q_{\text{Kundu}}$ |
| $\varepsilon_r$ | | | | | | | $7.3 \cdot 10^{-1}$ |
| $\varepsilon$ | | | | | | | $3.2 \cdot 10^{-4}$ |
| Tot. compl. | | | | | | | 8.00 |
| | *Explicitly invertible, not simply.* **Best in boldface.** | | | | *Simply explicitly invertible.* **Best in boldface.** | | |
| | Highlighted, the best approximations for each *InvLev*. | | | | | | |

From *Type 0*, *rational and irrational* approximations, if one requires both *explicit invertibility* and reasonable simplicity, little may be expected for a good precision on $I_{\text{significant}}$. Thus, no approximation of this type (though we have identified 4 interesting approximations, from the ancient $Q_{\text{Chu}}$ to the newest $Q_{\text{Boiroju-1}}$, evolving also in terms of (the majorization of) the relative error in absolute value) wins the comparison in any *InvLev*, in the sense that in this type we do not find any approximation presenting the minimum relative error $\varepsilon_r$ for any *InvLev*.

The best worth of *Type 1* approximations, *sum* $S(x)$ *of terms* $b_i\, e^{a_i x^2}$, is not the precision if *explicit invertibility* is required, but the fact that they allow the expression in closed form – eventually by means of hypergeometric functions – of the integrals (46), involved in the error probabilities computation. As for *Type 1*, the already classical approximation $Q_{\text{Chiani–2}}$ is the best approximation having *InvLev 1*, $Q_{\text{Chiani–3}}$ is the best approximation having *InvLev 2*, and $Q_{\text{Olabiyi–2}}$ is the best approximation having *InvLev 6*.

As far as *Chernoff type* is concerned, at the intersection of *Type 1* and *Type 2*, its worth, not lying in precision, remains in

- classicism;
- simplicity;
- high level (*InvLev 7*) and easiness of invertibility;
- to be in several cases an upper or a lower bound (even for $x \geq 0$).

The only approximation ($Q_{\text{Chang}}$) of this type, worth of a place in Table 19, does not win the comparison for its invertibility level, though it is quite interesting because it reduces to 54% the relative error in absolute value on $I_{\text{significant}}$, overcoming the 490% of the original (upper bounds) $Q_{\text{Chernoff–impr.}}$ and the 1100% of the ancient $Q_{\text{Chernoff}}$ (see Table 7), but becoming a lower bound.

The only *Type 4* approximation – of the type $\mathrm{e}^{-\frac{x^2}{2}} A(x)$ being $A(x)$ a (non constant) rational or irrational function – worth of a place in Table 19 is $Q_{\text{Borjesson–1}}$, neither winning the comparison for its invertibility level, although *Type 4* has emerged as a promising type to search for accurate approximations, since on unbounded intervals (differently from $I_{\text{significant}}$) only *Type 4* approximations may have small relative error in absolute value (for their asymptotic equivalence to $Q(x)$, see Section VII-C and, in particular, the last Remark on the bad behaviour in $+\infty$ of almost all approximations).

In *Type 5*, including essentially *(complementary) logistic-type* approximations of $Q(x)$, the best result in terms of relative error in absolute value on $I_{\text{significant}}$ is obtained (55%) by $Q_{\text{Zogheib-1}}$, winning also the comparison for its invertibility level. A noticeable worth of this type, however, is to allow a good approximation, at least in the sense of the absolute error, holding on the whole real axis, especially useful for the statistical use of $\Phi(x)$. In this research we have found quite many, 10, of such approximations, all originally published for $\Phi(x)$, *explicitly invertible* (see Table 11).

*Type 6*, including *nested exponentials* approximations of $Q(x)$, is a particular type, allowing to obtain a moderately good precision, 7.7% in $Q_{\text{Soranzo–4}}$, with the maximum invertibility level, *InvLev7*, winning also the comparison for this level. However, if the purpose is to obtain a high precision, very likely this is not the right research direction, at least at reasonable levels of total complexity (and of oddity).

In *Type 7*, including a *miscellanea* of approximations not falling in the previously defined types, among *explicitly invertible* approximations only 2 published approximations (see Table 13) have been found in this research, not allowing, at the moment, to identify reasonable other types. $Q_{\text{Kundu}}$ is the best one, in terms of accuracy, as far as its relative error in absolute value is concerned. However, this may be considered an interesting field for future researches (see the last Remark on the bad behaviour in $+\infty$ of almost all approximations in Section VII-C).

### C. MAIN RESULTS ABOUT THE COMPLEXITY OF THE APPROXIMATIONS

In this research, a noticeable work has been done to classify the complexity of the approximations, in particular of $Q(x)$, defining

- the typographic complexity in Section VIII-A,
- the computational complexity in Section VIII-B,
- the decimal complexity in Section VIII-C,

all summarized by

- the total complexity in Section VIII-D.

Though the chosen criteria are quite arbitrary, the result is pretty convincing, and here we report 3 examples:

1) $Q_{\text{Chernoff}}$ (see Table 7) with the minimum (so far found) total complexity 2.29:

$$\mathrm{e}^{-\frac{x^2}{2}}$$

2) $Q_{\text{Olabiyi–2}}$ (see Table 4) with an intermediate total complexity 7.56:

$$0.15085\,\mathrm{e}^{-0.5255\,x^2} + 0.21945\,\mathrm{e}^{-2\,0.5255\,x^2}$$

3) $Q_{\text{Lipoth}}$ (see Table 13) with the highest (so far found) total complexity 13.66:

$$1 - \left(1 + a\left(\ln\left(1 + \mathrm{e}^{-\frac{x}{h}+c}\right)\right)^{b}\right)^{-d}$$

(for the long decimal constants, determining this high total complexity, please refer to Table 13).

The highest total complexity (13.66) reached by $Q_{\text{Lipoth}}$ grants the very modest 74% precision, in the sense of the relative error, but the exceptional $2.4 \cdot 10^{-5}$ absolute error for $Q(x)$, and $\Phi(x)$ for which it was produced. Its total complexity could be reduced carefully rounding its 5 very long decimal constants.

In the summarizing Table 19, the (approximation of the) total complexity has been indicated, under each approximation name, in the third line.

In Table 19, $Q_{\text{Johnson–1}}$ presents the lowest (best) total complexity 2.71 (see also Table 11) and $Q_{\text{Olabiyi–3}}$ presents the highest (worst) total complexity 10.51 (see also Table 5).

It should be noticed that, in general, the approximations having a high complexity – such as, for instance, the above mentioned $Q_{\text{Olabiyi–3}}$ – present a low relative error (of the order of $10^{-2}$), whereas approximations presenting a low complexity – such as, e.g., the above mentioned $Q_{\text{Johnson–1}}$ – present a high relative error (of the order of $10^{1}$).

A notable advantage of the total complexity defined in this research is that it is not at all linked to the approximations of the function $Q(x)$, but can be applied to the evaluation of the complexity of any function, at least if defined in closed form (without integrals and so on) by means of elementary functions.

## XIV. CONCLUSION

This paper is devoted to the reviewing of (all, as far as possible) published *explicitly invertible* approximations of the function $Q(x)$ – of particular interest in information and telecommunications theory – even if originally published for the other 4 related functions $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and Mills' ratio $m(x)$ (and $\mathrm{erfc}\sqrt{x}\cdots$ ), from which $Q(x)$ may be immediately obtained.

The argument has been introduced presenting more than a dozen equivalent published definitions of the function $Q(x)$, the different names under which it is known, the other 4 *special functions* ($\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and Mills' ratio $m(x)$) strictly related to it, their probabilistic meaning and general behaviour (including that of $Q(x)$), the symmetry formulas (which can be useful to simplify the approximation of these functions), and some historical issues. Moreover, we have produced and listed in a table all the 20 mutual relations among the strictly related functions $Q(x)$, $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and $m(x)$, and similarly in another table all the 12 mutual relations among the inverses of $Q(x)$, $\Phi(x)$, $\mathrm{erf}(x)$, and $\mathrm{erfc}(x)$.

We have mainly focused on the approximation of $Q(x)$ in the interval [0.45, 4.5], named $I_{\text{significant}}$, which already in [22] has been recognized as a range of major practical interest in telecommunications theory, due to the BEP holding for the simple binary digital modulation schemes, recalled in (39). Moreover we recalled some notable values of $Q(x)$, and of the 4 classical *special functions* related to it, in the domain of interest.

Considerable attention has been devoted to the critical importance of approximating the function $Q(x)$. Additionally, this discourse included instances of practical applications of the function $Q(x)$, as well as a compilation of the essential attributes that an approximation of $Q(x)$ should possess in order to be included in this investigation. This subject has been progressively explored, taking into account a broader range of significant merits, focusing on a specific criterion that an approximation of $Q(x)$ should possess, which is the property of being easily and *explicitly invertible*. Additionally, this study also explored the issue of approximating $Q(x)$ by inverting a given estimate of $Q^{-1}(y)$.

The concept of *explicitly invertible* approximation – by means of elementary functions having standard names in mathematics and of the Lambert $W$-function – has been defined.

Then, having fixed so a large basis of requirements, a large survey of approximations of the Gaussian $Q$-function (also not *explicitly invertible* and also not directly published for $Q(x)$) found in the literature has been proposed and, successively, the analysis has been in details – referring the Reader to several tables – restricted to 60 *explicitly invertible* approximations (the most extensive list published so far, as far as we know), gathering those originally published for any of the mutually related functions $Q(x)$, $\Phi(x)$, $\mathrm{erf}(x)$, and $\mathrm{erfc}(x)$. The approximations have been classified defining 8 fundamental types (7 well defined ones and 1 miscellaneous one).

An extensive dissertation about the worth of the (majorizations of the) absolute and relative errors in measuring the accuracy of an approximation of $Q(x)$ and also of the other mutually related functions has been presented. Then, the topic of the tightness of an approximation of $Q(x)$ and of the inverse of an approximation of $Q(x)$ has been addressed, together with a wide dissertation about the concepts of asymptoticity and asymptotic equivalence of an approximation of $Q(x)$.

We have defined 4 kinds of complexity of an approximation of $Q(x)$ (3 specific and 1, the total complexity, summarizing the previous 3) which could be applied to a large range of functions, not necessarily related to the $Q$-function.

We have greatly expanded a previously published classification of the easiness of inversion of an *explicitly invertible* approximation of $Q(x)$, now spanning from *InvLev 0* (corresponding to not *explicit invertibility*) to *InvLev 7* (corresponding to the easiest inversion), with 1 special intermediate case, *InvLev 6.5*, taking into account the necessity of the (non elementary) Lambert $W$-function to perform the inversion. Also this classification could be applied to a large range of functions, not necessarily related to the $Q$-function.

Then, we have addressed the topic of bounds of $Q(x)$, treating separately upper bounds, related to the issue of the so-called *worst case* in performance analysis, and lower bounds. Moreover we have added some comments on bounds and inverses.

The approximations have been classified in 11 tables on the basis of:

1) their *Type*, determining their organization in tables;
2) their accuracy on $I_{\text{significant}}$, measured in 2 ways, the absolute error and the relative error;
3) their simplicity, defined by the 4 defined kinds of complexity;
4) their easiness of inversion, defined by the *InvLev*;
5) a remark about the 3 cases of upper bound, lower bound, or not a bound on $I_{\text{significant}}$, and eventually on the whole $(0, +\infty)$ or on the whole $[0, +\infty)$.

Then, the subsequent 5 tables present a compilation of the inverses for numerous approximations, ranging from *InvLev 4* to *InvLev 7*. These inverses encompass both previously published inverses as well as newly discovered ones within the scope of this research.

In conclusion, we have conducted a comparison of the published approximations for each *Type* and *InvLev* (excluding *InvLev 0*, which corresponds to non-*explicit invertibility*). The comparison was based on the relative errors of the approximations. The results of this comparison, including the most accurate approximations, their relative and absolute errors, and total complexities, are presented in Table 19. In this table, we have classified the approximations into 2 distinct categories: those that are *explicitly invertible* but *not simply* (denoted by *InvLev 1* to *InvLev 4*), and those that are simply and *explicitly invertible* (denoted by *InvLev 5* to *InvLev 7*). The first group involves the solution of cubic or quartic equations, which are generally regarded as complex.

The most precise – presenting the lowest (majorization of) the relative error (in absolute value) – *explicitly invertible* approximation of $Q(x)$ ever published has resulted to be the *Type 2* approximation $Q_{\text{Derenzo}}(x) := \frac{1}{2}\,\mathrm{e}^{-\frac{(83\,x+351)\,x+562}{703/x+165}}$ (originally published for $2\,Q(x)$) reported in Tables 6 and 19 with $\varepsilon < 7.2 \cdot 10^{-5}$, $\varepsilon_r < 4.2 \cdot 10^{-4}$, total complexity 3.86, and *InvLev 3*, since requiring the solution of a cubic equation in the inversion (thus being *explicitly invertible, not simply*).

The most precise – presenting the lowest (majorization of) the relative error (in absolute value) – *simply explicitly invertible* approximation of $Q(x)$ ever published has resulted to be the *Type 3* approximation $Q_{\text{Soranzo-2}}(x) := \frac{1}{2} - \frac{1}{2}\sqrt{1-\mathrm{e}^{-x^2\frac{17+x^2}{26.694+2\,x^2}}}$ (originally published for $\Phi(x)$) reported in Tables 9 and 19 with $\varepsilon < 4.0 \cdot 10^{-5}$, $\varepsilon_r < 2.2 \cdot 10^{-2}$, total complexity 5.65, and *InvLev 5*, since its inversion requires the solution of a biquadratic equation and proceeds like the inversion (see Section IX) of $Q_{\text{Winitzki}}$ (the inverse is reported in Table 15). Clearly $Q_{\text{Soranzo-2}}$ is less precise than $Q_{\text{Derenzo}}$ but more easily invertible.

In conclusion, we present a concise overview of the research results pertaining to the mathematical categorization and the total complexity of the examined approximations. Additionally, we highlight the most promising avenues for further exploration in the future.

## APPENDIX A THE HIDDEN POLYNOMIALS IN THE APPROXIMATIONS OF *Q*(*X*)

In this appendix we are going to show the frequent recurrence of hidden polynomials in the approximations of $Q(x)$. This short dissertation is useful to obtain the explicit inversion of many published approximations of $Q(x)$, using the methods exposed in the following Appendix B.

Considering a *polynomial* of *degree* $N$, with *coefficients* $a_k$,

$$P(x) := \sum_{k=0}^{N} a_k\, x^k,$$

the function

$$f(x) := P(x)$$

is a *polynomial function* and the *roots* of the polynomial are the *solutions* of the *eqnarray*

$$f(x) = 0.$$

If, for $x \geq 0$, $f(x)$ is *monotonic* (increasing or decreasing), so it is for the polynomial functions of the same degree

$$g(x) := f(x) - y \qquad y \in \mathbb{R}$$

each of which has 0 or 1 roots, considered as a polynomial. In the case of existence of 1 root $\xi$,

$$g(\xi) = f(\xi) - y = 0$$

or

$$f(\xi) = y$$

or

$$\xi = f^{-1}(y)$$

which is to say that to find the root $\xi$ of $g(x)$, depending on the value of $y$, is equivalent to the *explicit inversion* of the polynomial $f(x)$, obtaining $x(y)$.

Let's show all that using a very classical scholastic approximation of $Q(x)$, Shah's approximation (44) when restricted to $[0, 2.2]$:

$$f(x) = \frac{1}{2} - \frac{x\,(4.4 - x)}{10}.$$

We have the 2-nd degree equation in $x$ with *parameter* $y \in \mathbb{R}$

$$\frac{1}{2} - \frac{x\,(4.4 - x)}{10} - y = 0 \qquad 0 \leq x \leq 2.2$$

which gives, before considering the limitations on the domain,

$$\xi_{1,2} = \frac{11 \pm \sqrt{250\,y - 4}}{5},$$

and choosing the correct branch and the limitations:

$$x(y) = \frac{11 - \sqrt{250\,y - 4}}{5} \qquad 0.016 \leq y \leq 0.5.$$

In literature we have found no approximations of $Q(x)$ by polynomial functions, of any degree, holding (at least) on $I_{\text{significant}}$, and clearly the particular behaviour of $Q(x)$ – see the last Remark of Section VII-C – makes it impossible to approximate it with a reasonable precision in the sense of the relative error using polynomial functions unless with huge degrees, obtained for example truncating the series (23).

Nevertheless, in partial contradiction with the previous statement, the inversion of polynomial functions is of great relevance for the topic of this paper because there are dozens of approximations of $Q(x)$ whose *explicit inversion* requires solving polynomial equations *after* making obvious substitutions, as illustrated below.

As a first example, let us consider the approximation $Q_{\text{Chiani–2}}$ of $Q(x)$ reported in Table 5,

$$f(x) = \frac{1}{12}\,\mathrm{e}^{-\frac{x^2}{2}} + \frac{1}{4}\,\mathrm{e}^{-2\,\frac{x^2}{3}}$$

which, with the obvious substitution $z := \mathrm{e}^{-\frac{x^2}{6}}$, gives the 4-th degree equation

$$y = \frac{z^4}{4} + \frac{z^3}{12}$$

in the unknown $z$ with parameter $y = f(x)$. (For the resolution of a 4-th degree equation see Appendix B.)

A different example is the approximation $Q_{\text{Benitez–2}}$ of $Q(x)$ reported in Table 6,

$$f(x) = e^{-0.4774\,x^2 - 0.4484\,x - 0.9049}$$

which, with the obvious substitution $y := \ln f(x)$, gives the second degree equation

$$y = -0.4774\,x^2 - 0.4484\,x - 0.9049$$

in the unknown $x$ with parameter $y$.

An example of another type is the approximation $Q_{\text{Hamaker}}$ of $Q(x)$ reported in Table 8,

$$f(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mathrm{e}^{-(0.806\,x\,(1-0.018\,x))^2}},$$

whose inversion is made first obtaining

$$2\left(\frac{1}{2} - f(x)\right) = \sqrt{1 - \mathrm{e}^{-(0.806\,x\,(1-0.018\,x))^2}},$$

then squaring

$$\left(2\left(\frac{1}{2} - f(x)\right)\right)^2 = 1 - \mathrm{e}^{-(0.806\,x\,(1-0.018\,x))^2},$$

then taking logarithms

$$\ln\left(1 - \left(2\left(\frac{1}{2} - y\right)\right)^2\right) = -(0.806\,x\,(1 - 0.018\,x))^2$$

which is a 4-th degree equation in $x$ with parameter $y = f(x)$ and the solution of such equations will be treated in the following Appendix B.

## APPENDIX B
## A SHORT BRIEF ON THE ROOTS OF POLYNOMIALS UP TO THE 4-TH DEGREE

For the degree $N \le 4$, there is a complete theory for finding the eventual roots of polynomials, which we are going to treat shortly.

*Remark:* The solutions of a *generic* equation with degree $N \ge 5$ cannot be expressed by roots and rational functions applied to the coefficients $a_0, \ldots, a_N$ of the equation (it is the ancient and very classical Abel–Ruffini Theorem in [108]). In this paper, with a more than extremely high probability, all polynomials of degree $N \ge 5$ have been considered not *explicitly invertible*: strictly, one may only state that, so far, no one has explicitly inverted them (which *very probably* is impossible by radicals).

Let's begin with the first degree equation

$$a_1\,x + a_0 = 0 \tag{72}$$

with the obvious solution

$$\xi = -\frac{a_0}{a_1}. \tag{73}$$

In particular, consider the easiest first degree approximation of $Q(x)$ around 0, holding with good precision only on an interval $[0, b]$ with small $b$:

$$f(x) = -\frac{1}{\sqrt{2\pi}}\,x + \frac{1}{2}$$

which gives the equation

$$-\frac{1}{\sqrt{2\pi}}\,x + \frac{1}{2} = y,$$

that is (72) with $a_1 = -1/\sqrt{2\pi}$ and $a_0 = 1/2 - y$, from which the inverse, following (73), is

$$x(y) = -\frac{a_0}{a_1} = -\frac{\frac{1}{2} - y}{-\frac{1}{\sqrt{2\pi}}}.$$

The second degree equation

$$a_2\,x^2 + a_1\,x + a_0 = 0$$

has – under conditions on coefficients – solutions

$$\xi_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4\,a_2\,a_0}}{2\,a_2}.$$

For the 3-rd degree equation

$$a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0 = 0$$

there is a method due to Niccolò Tartaglia (XVI century, illustrated in [109]). First of all the equation can be transformed into a simplified so called *depressed* 3-rd degree equation

$$g(u) = u^3 + p\,u + q = 0 \tag{74}$$

with this change of variable

$$x = u - \frac{a_2}{3\,a_3}$$

and with

$$u = x + \frac{a_2}{3\,a_3}$$
$$p = \frac{3\,a_3\,a_1 - a_2^2}{3\,a_3^2}$$
$$q = \frac{2\,a_2^3 - 9\,a_3\,a_2\,a_1 + 27\,a_3^2\,a_0}{27\,a_3^3}.$$

The roots $\xi_1$, $\xi_2$, and $\xi_3$ of the original equation are related to the roots $u_1$, $u_2$, and $u_3$ of the *depressed* equation by

$$\xi_k = u_k - \frac{a_2}{3\,a_3} \quad \text{for} \quad k = 1, 2, 3,$$

being

$$u_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{d}} + \sqrt[3]{-\frac{q}{2} - \sqrt{d}}$$
$$u_2 = \frac{-1 + i\,\sqrt{3}}{2}\,u_1$$
$$u_3 = \frac{-1 - i\,\sqrt{3}}{2}\,u_1,$$

where $i$ is the imaginary unit and $d$ is given by

$$d = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2.$$

Notice that at least 1 of the 3 previous listed solutions of the *depressed* cubic equation is a real number, and the other 2 roots are both real or (non real) conjugate complex.

Moreover, the *depressed* cubic equation – and then the *generic* cubic equation – may be also solved [110] by means of the hyperbolic functions and their inverses. Besides the solution formulas previously shown for (74), if the latter has only 1 real solution $u_1$ – which is the case generally recurring in the approximation of $Q(x)$, usually decreasing – for this real solution $u_1$ there exists also an expression simply involving sinh and $\sinh^{-1}$ and no complex numbers (see [27] and Table 14).

For the 4-th degree equation

$$a_4\, x^4 + a_3\, x^3 + a_2\, x^2 + a_1\, x + a_0 = 0$$

Lodovico Ferrari found a method of solution (XVI century, illustrated in [109]) based on similar concepts. First of all the latter equation can be transformed into a simplified so called "depressed" quartic equation

$$g(u) = u^4 + p\, u^2 + q\, u + r = 0$$

with this change of variable

$$x = u - \frac{a_3}{4\, a_4}$$

and with

$$\begin{aligned} u &= x + \frac{a_3}{4\, a_4} \\ p &= \frac{8\, a_2\, a_4 - 3\, a_3^2}{8\, a_4^2} \\ q &= \frac{a_3^3 - 4\, a_2\, a_3\, a_4 + 8\, a_1\, a_4^2}{8\, a_4^3} \\ r &= \frac{-3\, a_3^4 + 256\, a_0\, a_4^3 - 64\, a_1\, a_3\, a_4^2 + 16\, a_2\, a_3^2\, a_4}{256\, a_4^4}. \end{aligned}$$

The so called "resolvent cubic" of the "depressed" quartic equation is

$$z^3 + \frac{5}{2}\, p\, z^2 + \left(2\, p^2 - r\right) z + \left(\frac{p^3}{2} - \frac{p\, r}{2} - \frac{q^2}{8}\right) = 0,$$

one of which roots $z_1 \neq 0$ determines the roots of the "depressed" quartic as

$$u_k = \frac{\pm_d \sqrt{p + 2\, z_1} \pm \sqrt{-\left(3\, p + 2\, z_1 \pm_d \frac{2\, q}{\sqrt{p + 2\, z_1}}\right)}}{2},$$

for $k = 1, 2, 3, 4$, where $\pm_d$ are both $+$ or both $-$ giving altogether 4 cases because of the $\pm$'s. The (real or complex) roots $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$ of the original quartic equation are related to the roots $u_1$, $u_2$, $u_3$, and $u_4$ of the "depressed" equation by

$$\begin{aligned} \xi_k = u_k - \frac{a_3}{4\, a_4} &= -\frac{a_3}{4\, a_4} \\ &+ \frac{\pm_d \sqrt{p + 2\, z_1} \pm \sqrt{-\left(3\, p + 2\, z_1 \pm_d \frac{2\, q}{\sqrt{p + 2\, z_1}}\right)}}{2} \end{aligned}$$

for $k = 1, 2, 3, 4$.

## REFERENCES


[1] M. K. Simon and M.-S. Alouini, *Digital Communication Over Fading Channels*. New York, NY, USA: Wiley, 2000.

[2] F. H. Qureshi, S. A. Sheikh, Q. U. Khan, and F. M. Malik, "SEP performance of triangular QAM with MRC spatial diversity over fading channels," *EURASIP J. Wireless Commun. Netw.*, Jan. 2016, Art. no. 5.

[3] G. T. F. de Abreu, "Jensen–Cotes upper and lower bounds on the Gaussian *Q*-function and related functions," *IEEE Trans. Commun.*, vol. 57, no. 11, pp. 3328–3338, Nov. 2009.

[4] G. K. Karagiannidis and A. S. Lioumpas, "An improved approximation for the Gaussian *Q*-function," *IEEE Commun. Lett.*, vol. 11, no. 8, pp. 644–646, Aug. 2007.

[5] M. K. Simon, "Single integral representations of certain integer powers of the Gaussian *Q*-function and their application," *IEEE Commun. Lett.*, vol. 6, no. 12, pp. 532–534, Dec. 2002.

[6] O. Olabiyi and A. Annamalai, "New exponential-type approximations for the ERFC(.) and ERFC$^p$(.) functions with applications," in *Proc. IWCMC*, Limassol, Cyprus, 2012, pp. 1221–1226.

[7] Y. Isukapalli and B. D. Rao, "An analytically tractable approximation for the Gaussian *Q*-function," *IEEE Commun. Lett.*, vol. 12, no. 9, pp. 669–671, Sep. 2008.

[8] P. Loskot and N. C. Beaulieu, "Prony and polynomial approximations for evaluation of the average probability of error over slow-fading channels," *IEEE Trans. Veh. Technol.*, vol. 58, no. 3, pp. 1269–1280, Mar. 2009.

[9] O. Olabiyi and A. Annamalai, "Invertible exponential-type approximations for the Gaussian probability integral $Q(x)$ with applications," *IEEE Wireless Commun. Lett.*, vol. 1, no. 5, pp. 544–547, Oct. 2012.

[10] Q. Shi and Y. Karasawa, "An accurate and efficient approximation to the Gaussian *Q*-function and its applications in performance analysis in Nakagami-*m* fading," *IEEE Commun. Lett.*, vol. 15, no. 5, pp. 479–481, May 2011.

[11] M. Chiani, D. Dardari, and M. K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 4, pp. 840–845, Jul. 2003.

[12] D. Sadhwani, R. N. Yadav, and S. Aggarwal, "Tighter bounds on the Gaussian *Q* function and its application in Nakagami-*m* fading channel," *IEEE Wireless Commun. Lett.*, vol. 6, no. 5, pp. 574–577, Oct. 2017.

[13] M. Benitez and F. Casadevall, "Versatile, accurate, and analytically tractable approximation for the Gaussian *Q*-function," *IEEE Trans. Commun.*, vol. 59, no. 4, pp. 917–922, Apr. 2011.

[14] G. R. Cooper and C. D. McGillem, *Probabilistic Methods of Signal and System Analysis*, 3rd ed. Oxford, U.K.: Oxford Univ. Press, 1998.

[15] Y. Polyanskiy, H. V. Poor, and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2307–2359, May 2010.

[16] A. Soranzo and E. Epure, "Very simply explicitly invertible approximations of normal cumulative and normal quantile function," *Appl. Math. Sci.*, vol. 8, no. 87, pp. 4323–4341, 2014.

[17] H. C. Hamaker, "Approximating the cumulative normal distribution and its inverse," *Appl. Stat.*, vol. 27, no. 1, pp. 76–77, 1978.

[18] J. T. Lin, "Approximating the normal tail probability and its inverse for use on a pocket calculator," *J. Roy. Stat. Soc. C Appl. Stat.*, vol. 38, pp. 69–70, Mar. 1989.

[19] K. M. Aludaat and M. T. Alodat, "A note on approximating the normal distribution function," *Appl. Math. Sci.*, vol. 2, no. 9, pp. 425–429, 2008.

[20] J. T. Lin, "Alternatives to Hamaker's approximations to the cumulative normal distribution and its inverse," *J. Roy. Stat. Soc. D Stat.*, vol. 37, nos. 4–5, pp. 413–414, 1988.

[21] J. T. Lin, "A simpler logistic approximation to the normal tail probability and its inverse," *J. Roy. Stat. Soc. C Appl. Stat.*, vol. 39, no. 2, pp. 255–257, 1990.

[22] A. Soranzo, F. Vatta, M. Comisso, G. Buttazzoni, and F. Babich, "New very simply explicitly invertible approximation of the Gaussian *Q*-function," in *Proc. SOFTCOM*, Split, Croatia, 2019, pp. 1–5.

[23] S. Winitzki. "A handy approximation for the error function and its inverse." Feb. 2008. Accessed: Aug. 23, 2023. [Online]. Available: https://scholar.google.com/citations?user=Q9U40gUAAAAJ&hl=en

[24] A. Soranzo and E. Epure. "Simply explicitly invertible approximations to 4 decimals of error function and normal cumulative distribution function." Jan. 2012. Accessed: Aug. 23, 2023. [Online]. Available: https://www.researchgate.net/publication/51978788

[25] A. Soranzo and E. Epure. "Practical explicitly invertible approximation to 4 decimals of normal cumulative distribution function modifying Winitzki's approximation of ERF." Nov. 2012. [Online]. Available: https://arxiv.org/abs/1211.6403

[26] E. Page, "Approximation to the cumulative normal function and its inverse for use on a pocket calculator," *Appl. Stat.*, vol. 26, no. 1, pp. 75–76, Mar. 1977.

[27] J. D. Vedder, "An invertible approximation to the normal distribution function," *Comput. Stat. Data Anal.*, vol. 16, no. 1, pp. 119–123, Jun. 1993.

[28] O. Eidous and S. Al-Salman, "One-term approximation for normal distribution function," *Math. Stat.*, vol. 4, no. 1, pp. 15–18, 2016.

[29] M.-W. Wu, Y. Li, M. Gurusamy, and P.-Y. Kam, "A tight lower bound on the Gaussian $Q$-function with a simple inversion algorithm, and an application to coherent optical communications," *IEEE Commun. Lett.*, vol. 22, no. 7, pp. 1358–1361, Jul. 2018.

[30] J. Lipoth, Y. Tereda, S. M. Papalexiou, and R. J. Spiteri, "A new very simply explicitly invertible approximation for the standard normal cumulative distribution function," *AIMS Math.*, vol. 7, no. 7, pp. 1–9, 2022.

[31] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions. With Formulas, Graphs, and Mathematical Tables*, vol. 55, 10th ed. Washington, DC, USA: Nat. Bureau Stand. Appl. Math. Series, 1972.

[32] K. D. Tocher, *The Art of Simulation*. London, U.K.: English Univ. Press, 1963.

[33] A. A. Hanandeh and O. M. Eidous, "Some improvements for existing simple approximation of the normal distribution function," *Pakistan J. Stat. Oper. Res.*, vol. 18, no. 3, pp. 555–559, 2022.

[34] A. A. Hanandeh and O. M. Eidous, "A new one-term approximation to the standard normal distribution," *Pakistan J. Stat. Oper. Res.*, vol. 17, no. 2, pp. 381–385, 2021.

[35] B. W. Schmeiser, "Approximations to the inverse cumulative normal function for use on hand calculators," *J. Roy. Stat. Soc. C Appl. Stat.*, vol. 28, no. 2, pp. 175–176, 1979.

[36] H. Shore, "Simple approximations for the inverse cumulative function, the density function and the loss integral of the normal distribution," *J. Roy. Stat. Soc. C Appl. Stat.*, vol. 31, no. 2, pp. 108–114, 1982.

[37] G. W. Hill and A. W. Davis, "Algorithm 442: Normal deviate," *Commun. ACM*, vol. 16, no. 1, pp. 51–52, Jan. 1973.

[38] R. E. Odeh and J. O. Evans, "Algorithm AS 70: The percentage points of the normal distribution," *J. Roy. Stat. Soc. C Appl. Stat.*, vol. 23, no. 1, pp. 96–97, 1974.

[39] J. K. Patel and C. B. Read, *Handbook of the Normal Distribution*. New York, NY, USA: CRC Press, 1996.

[40] A. Gasull and F. Utzet, "Approximating mills ratio," *J. Math. Anal. Appl.*, vol. 420, no. 2, pp. 1832–1853, 2014.

[41] J. P. Mills, "Table of the ratio: Area to bounding ordinate, for any portion of normal curve," *Biometrika*, vol. 18, pp. 395–400, Nov. 1926.

[42] J. W. Craig, "A new, simple and exact result for calculating the probability of error for two-dimensional signal constellations," in *Proc. MILCOM*, Boston, MA, USA, 1991, pp. 571–575.

[43] N. M. Temme. "Error functions, Dawson's and Fresnel integrals." Accessed: Aug. 23, 2023. [Online]. Available: https://dlmf.nist.gov/

[44] A. S. Gvozdarev, "The novel approach to the closed-form average bit error rate calculation for the Nakagami-$m$ fading channel," *Digit. Signal Process.*, vol. 127, Jul. 2022, Art. no. 103563.

[45] S. Aggarwal, "A survey-cum-tutorial on approximations to Gaussian $Q$ function for symbol error probability analysis over Nakagami-$m$ fading channels," *IEEE Commun. Surveys Tuts.*, vol. 21, no. 3, pp. 2195–2223, 3rd Quart., 2019.

[46] S.-H. Chang, P. C. Cosman, and L. B. Milstein, "Chernoff-type bounds for the Gaussian error function," *IEEE Trans. Commun.*, vol. 59, no. 11, pp. 2939–2944, Nov. 2011.

[47] H. Chernoff, "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations," *Ann. Math. Stat.*, vol. 23, no. 4, pp. 493–507, Dec. 1952.

[48] J. G. Proakis, *Digital Communications*, 4th ed. New York, NY, USA: McGraw-Hill, 2000.

[49] P. S. de Laplace, *Théorie Analytique des Probabilités*. Paris, France: Courcier, 2022.

[50] J. M. Wozencraft and I. M. Jacobs, *Principles of Communication Engineering*. New York, NY, USA: Wiley, 1965.

[51] R. D. Gordon, "Values of mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument," *Ann. Math. Stat.*, vol. 12, no. 3, pp. 364–366, Sep. 1941.

[52] D. F. Kerridge and G. W. Cook, "Yet another series for the normal integral," *Biometrika*, vol. 63, no. 2, pp. 401–403, Aug. 1976.

[53] N. C. Beaulieu, "A simple series for personal computer computation of the error function $Q(\cdot)$," *IEEE Trans. Commun.*, vol. 37, no. 9, pp. 989–991, Sep. 1989.

[54] Á. Baricz, "Mills' ratio: Monotonicity patterns and functional inequalities," *J. Math. Anal. Appl.*, vol. 340, no. 2, pp. 1362–1370, 2008.

[55] P. Y. Kam and R. Li, "A new geometric view of the first-order Marcum $Q$-function and some simple tight ERFC-bounds," in *Proc. VTC*, 2006, pp. 2553–2557.

[56] R. Li and P. Y. Kam, "Computing and bounding the generalized Marcum $Q$-function via a geometric approach," in *Proc. ISIT*, Seattle, WA, USA, 2006, pp. 1–28.

[57] A. H. Nuttall, "Some integrals involving the $Q_m$ function," *IEEE Trans. Inf. Theory*, vol. IT-21, no. 1, pp. 95–96, Jan. 1975.

[58] H. Cramér, "Sur un nouveau théorème-limite de la théorie des probabilités," in *Les Sommes et Les Fonctions de Variables Aléatoires*. Paris, France: Hermann & C, 1938.

[59] H. Cramér, "On a new limit theorem in probability theory (Sur un nouveau théorème-limite de la théorie des probabilités)," in *Collected Works*, vol. 2. Berlin, Germany: Springer, 1994, pp. 895–913. [Online]. Available: https://arxiv.org/pdf/1802.05988v4.pdf

[60] J. D. Williams, "An approximation to the probability integral," *Ann. Math. Stat.*, vol. 17, no. 3, pp. 363–365, Sep. 1946.

[61] G. Pólya, "Remarks on computing the probability integral in one and two dimensions," in *Proc. Symp. Math. Stat. Probab.*, 1945, p. 12.

[62] J. H. Cadwell, "The bivariate normal integral," *Biometrika*, vol. 38, nos. 3–4, pp. 475–479, Dec. 1951.

[63] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, 4th ed. Cambridge, U.K.: Cambridge Univ. Press, 1990.

[64] C. Hastings, *Approximations for Digital Computers*. Princeton, NJ, USA: Princeton Univ. Press, 1955.

[65] J. Burnet, *Platonis Opera*. Oxford, U.K.: Oxford Univ. Press, 1903.

[66] C. F. Gauss, *Theoria Motus Corporum Celestium*. Hamburg, Germany: Perthes & Besser, 2023, p. 213. [Online]. Available: https://www.e-rara.ch/zut/doi/10.3931/e-rara-522

[67] Z. W. Birnbaum, "An inequality for mill's [SiC] ratio," *Ann. Math. Stat.*, vol. 13, no. 2, pp. 245–246, Jun. 1942.

[68] J. T. Chu, "On bounds for the normal integral," *Biometrika*, vol. 42, nos. 1–2, pp. 263–265, 1955.

[69] X. Lin, C. Genest, D. L. Banks, G. Molenberghs, D. W. Scott, and J.-L. Wang, *Past, Present, and Future of Statistical Science*. New York, NY, USA: CRC Press, 2014, p. 35.

[70] H. Zhang, Z. Liu, H. Huang, and L. Wang, "FTSGD: An adaptive stochastic gradient descent algorithm for Spark MLlib," in *Proc. DASC PiCom DataCom CyberSciTech*, 2018, p. 12.

[71] J. G. Proakis and M. Salehi, *Digital Communications*, 5th ed. New York, NY, USA: McGraw-Hill, 2008.

[72] A. K. Shah, "A simpler approximation for areas under the standard normal curve," *Amer. Stat.*, vol. 39, no. 1, p. 80, Feb. 1985.

[73] Z. H. Peric, A. V. Markovic, N. Z. Kontrec, S. R. Panic, and P. C. Spalevic, "Novel composite approximation for the Gaussian $Q$-function," *Elektronika ir Elektrotechnika*, vol. 26, no. 5, pp. 33–38, 2020.

[74] F. Vatta, A. Soranzo, M. Comisso, G. Buttazzoni, and F. Babich, "New explicitly invertible approximation of the function involved in LDPC codes density evolution analysis using a Gaussian approximation," *Electron. Lett.*, vol. 55, no. 22, pp. 1183–1186, Oct. 2019.

[75] F. Vatta, A. Soranzo, M. Comisso, G. Buttazzoni, and F. Babich, "A survey on old and new approximations to the function $\phi(x)$ involved in LDPC codes density evolution analysis using a Gaussian approximation," *Information*, vol. 12, no. 5, p. 212, May 2021.

[76] F. Vatta, A. Soranzo, and F. Babich, “More accurate analysis of sum-product decoding of LDPC codes using a Gaussian approximation,” *IEEE Commun. Lett.*, vol. 23, no. 2, pp. 230–233, Feb. 2019.

[77] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, “On the Lambert W function,” *Adv. Comput. Math.*, vol. 5, pp. 329–359, Dec. 1996.

[78] M. Bronstein, R. M. Corless, J. H. Davenport, and D. J. Jeffrey, “Algebraic properties of the Lambert W function from a result of Rosenlicht and of Liouville,” *Integral Transf. Special Functions*, vol. 19, no. 10, pp. 709–712, 2008.

[79] F. Vatta, A. Soranzo, and F. Babich, “Low-complexity bound on irregular LDPC belief-propagation decoding thresholds using a Gaussian approximation,” *Electron. Lett.*, vol. 54, no. 17, pp. 1038–1040, Aug. 2018.

[80] I. W. Burr, “A useful approximation to the normal distribution function, with application to simulation,” *Technometrics*, vol. 9, no. 4, pp. 647–651, Nov. 1967.

[81] N. K. Boiroju and K. R. Rao. “Simple approximations to Gaussian $Q$-function.” Apr. 2015. [Online]. Available: https://www.researchgate.net/profile/Naveen-Boiroju

[82] B. Zogheib and M. Hlynka. “Approximations of the standard normal distribution.” Jan. 2009. [Online]. Available: https://www.researchgate.net/publication/242100205

[83] M. Wu, X. Lin, and P.-Y. Kam, “New exponential lower bounds on the Gaussian $Q$-function via Jensen’s inequality,” in *Proc. VTC-Spring*, Budapest, Hungary, 2011, pp. 1–5.

[84] A. Powari, D. Sadhwani, L. Gupta, and R. N. Yadav, “Novel Romberg approximation of the Gaussian $Q$ function and its application over versatile $\kappa-\mu$ shadowed fading channel,” *Digit. Signal Process.*, vol. 132, Jan. 2023, Art. no. 103800.

[85] M. Ordaz, “A simple approximation to the Gaussian distribution,” *Struct. Safety*, vol. 9, no. 4, pp. 315–318, Jun. 1991.

[86] A. Mastin and P. Jaillet. “Log-quadratic bounds for the Gaussian $Q$-function.” Apr. 2013. [Online]. Available: https://www.researchgate.net/publication/236136696

[87] P. C. Sofotasios and S. Freear, “Novel expressions for the Marcum and one dimensional $Q$-functions,” in *Proc. ICWITS*, Honolulu, HI, USA, 2010, pp. 736–740.

[88] S. E. Derenzo, “Approximations for hand calculators using small integral coefficients,” *Math. Comput.*, vol. 31, no. 137, pp. 214–222, Jan. 1977.

[89] D. N. Phong, N. X. Hoai, R. I. McKay, C. Siriteanu, N. Q. Uy, and N. Park, “Evolving the best known approximation to the $Q$ function,” in *Proc. Conf. Genet. Evol. Comput.*, 2012, pp. 807–814.

[90] N. Ermolova and S.-G. Haggman, “Simplified bounds for the complementary error function; application to the performance evaluation of signal-processing systems,” in *Proc. EUSIPCO*, Vienna, Austria, 2004, pp. 1087–1090.

[91] M. Abderrahmane and B. Kamel, “A new approximation to standard normal distribution function,” *J. Appl. Comput. Math.*, vol. 6, no. 2, pp. 1–18, 2017.

[92] M. Abderrahmane and B. Kamel, “Two new approximations to standard normal distribution function,” *J. Appl. Comput. Math.*, vol. 5, no. 5, pp. 1–18, Jan. 2016.

[93] A. L. Brophy, “Accuracy and speed of seven approximations of the normal distribution function,” *Behav. Res. Methods Instrument.*, vol. 15, no. 6, pp. 604–605, 1983.

[94] P. O. Borjesson and C.-E. W. Sundberg, “Simple approximations of the error function $Q(x)$ for communications applications,” *IEEE Trans. Commun.*, vol. C-27, no. 3, pp. 639–643, Mar. 1979.

[95] S. R. Bowling, M. T. Khasawneh, S. Kaewkuekool, and B. R. Cho, “A logistic approximation to the cumulative normal distribution,” *J. Ind. Eng. Manag.*, vol. 2, no. 1, pp. 114–127, 2009.

[96] N. L. Johnson and S. Kotz, *Continuous Univariate Distributions*, vol. 2, 2nd ed. New York, NY, USA: Wiley, 1995.

[97] D. R. Divgi, “Approximations for three statistical functions,” Center Naval Anal., Hudson Inst., Washington, DC, USA, Rep. CRM 89-253, Apr. 1990.

[98] O. Eidous and R. Abu-Shareefa, “New approximations for standard normal distribution function,” *Commun. Stat. Theory Methods*, vol. 49, no. 6, pp. 1357–1374, 2020.

[99] D. Kundu, R. Gupta, and A. Manglick, “A convenient way of generating normal random variables using generalized exponential distribution,” *J. Mod. Appl. Stat. Methods*, vol. 5, no. 1, pp. 300–306, 2006.

[100] R. J. Bagby, “Calculating normal probabilities,” *Amer. Math. Month.*, vol. 102, no. 1, pp. 46–49, 1995.

[101] P. A. P. Moran, “Calculation of the normal distribution function,” *Biometrika*, vol. 67, no. 3, pp. 675–676, Dec. 1980.

[102] H. V. Leal, R. C. Sheissa, V. F. Nino, A. S. Reyes, and J. S. Orea, “High accurate simple approximation of normal distribution integral,” *Math. Probl. Eng.*, Feb. 2012, Art. no. 124029.

[103] W. Byrc, “A uniform approximation to the right normal tail integral,” *Appl. Math. Comput.*, vol. 127, nos. 2–3, pp. 365–374, Apr. 2002.

[104] A. Soranzo, F. Vatta, M. Comisso, G. Buttazzoni, and F. Babich, “A new accurate approximation of the Gaussian $Q$-function with relative error less than 1 thousandth in a significant domain,” in *Proc. SOFTCOM*, 2021, pp. 1–5.

[105] V. N. Q. Bao, L. P. Tuyen, and H. H. Tue, “A survey on approximations of one-dimensional Gaussian $Q$-function,” *REV J. Electron. Commun.*, vol. 5, nos. 1–2, p. 20, Jun. 2015.

[106] A. L. Brophy, “Approximation of the inverse normal distribution function,” *Behav. Res. Methods Instrum. Comput.*, vol. 17, no. 3, pp. 415–417, 1985.

[107] F. D. Côté, I. N. Psaromiligkos, and W. J. Gross. “A Chernoff-type lower bound for the Gaussian $Q$-function.” Mar. 2012. [Online]. Available: https://arxiv.org/abs/1202.6483

[108] P. Ruffini, *Teoria Generale Delle Equazioni, In Cui Si Dimostra Impossibile La Soluzione Algebrica Delle Equazioni Generali Di Grado Superiore Al Quarto*. Bologna, Italy: Stamperia di S. Tommaso d’Aquino, 1799.

[109] J. E. Thompson, *Algebra for the Practical Man*. New York, NY, USA: D. Van Nostrand Company, 1968.

[110] G. C. Holmes, “The use of hyperbolic cosines in solving cubic polynomials,” *Math. Gazette*, vol. 86, no. 507, pp. 473–477, Nov. 2002.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