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Degrees of Freedom of the Wireless X-Network Assisted by Intelligent Reflecting Surfaces

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ABSTRACT In this paper, we study the DoF of the time-selective $M \times N$ wireless X-network assisted by an IRS. It is well-known that the DoF of the $M \times N$ wireless X-network is $\frac{MN}{M+N-1}$. We show that the maximum DoF of $\min\{M, N\}$ can be achieved when the IRS has enough elements. We consider two kinds of active and passive IRSs. We also consider two different scenarios, where the channel coefficients for IRS elements are either independent or correlated. For the $M \times N$ wireless X-network assisted by an active IRS with independent channel coefficients, we derive the inner and outer bounds on the DoF region and the lower and upper bounds on the sum DoF. We show that the maximum value for the sum DoF, i.e., $\min(M, N)$, is achievable if the number of elements is more than a threshold for the active IRS, which is equal to the approximate capacity of $\min\{M, N\}\log(\rho + 1) + o(\log(\rho))$ for the IRS-assisted X-network, where ρ is the transmission power. For the $M \times N$ wireless X-network assisted by a passive IRS with the assumption of independent and correlated channel coefficients for IRS elements, we introduce probabilistic inner and outer bounds on the DoF region, and the probabilistic lower and upper bounds on the sum DoF and show that the proposed lower bound for the sum DoF asymptotically approaches $\min(M, N)$ with an order of at least $O(\frac{1}{Q})$ for independent channel coefficients (i.e., the sum DoF is $\min\{M, N\}(1 - O(\frac{1}{Q}))$), which is equal to the approximate capacity of $\min\{M, N\}(1 - O(\frac{1}{Q}))\log(\rho + 1) + o(\log(\rho))$ and $O(\frac{1}{\sqrt{Q}})$ for correlated channel coefficients (i.e., the sum DoF is $\min\{M, N\}(1 - O(\frac{1}{\sqrt{Q}}))$, which is equal to the approximate capacity of $\min\{M, N\}(1 - O(\frac{1}{\sqrt{Q}}))\log(\rho + 1) + o(\log(\rho)))$, where Q is the number of IRS elements. Thus, this decrement in the order of convergence shows the performance loss for correlated IRS elements. In addition, we extend the lower bound of the sum DoF proposed for the active IRS with independent channel coefficients to the scenario with correlated channel coefficients, i.e., the sum DoF is the same as independent IRS elements for $\min\{M, N\} \leq 5$ and $Q \leq 20$, and for other cases, the sum DoF converges to $\min\{M, N\}$ with an order of at least $O(\frac{1}{\sqrt{Q}})$.

INDEX TERMS Time-selective $M \times N$ wireless X-network, DoF region, sum DoF, active and passive intelligent reflecting surface, independent and correlated channel coefficients.

I. INTRODUCTION

X-NETWORK is a generalization of an interference channel, in which each transmitter acts as an independent broadcast channel to all receivers, thus, X-network is a combination of interference and broadcast channels. Therefore, transmission schemes introduced for X-network can be useful to achieve better performances in next-generation multiuser wireless networks. On the other hand, intelligent reflecting surfaces (IRSs) are novel types of equipment, which play an important role in the next-generation wireless communication systems [1], [2]. Since the degree of freedom (DoF), is an analytic metric, which is commonly used in the analysis of the performance of multi-user networks, thus, the analysis of the DoF of the IRS-assisted X-network is an important problem.

IRS-aided networks have been studied from various aspects including channel modeling [3], [4], the optimization of IRS elements [5], the system analysis [3], and DoF analysis [6], see [2] for a recent survey. We review some related works to the capacity of IRS-assisted networks. In [5], the authors studied the fundamental limits for the capacity of the IRS-aided multiple-input multiple-output (MIMO) communication by joint optimization of the IRS phase shift matrix and the MIMO transmit covariance matrix. In [8], the authors studied the IRS-assisted communication systems, where the IRS is configured by the transmitter using a finite-rate link. They characterized fundamental limits of the system and showed that the capacity is achievable by jointly encoding the information in the transmitted signal and the IRS operation. In [9], the authors studied the optimization of the channel capacity of IRS-aided millimeter-wave channels without a line-of-sight path. In [10], the authors studied the multi-user downlink communication assisted by an IRS. They maximized the sum rate with an individual constraint for the quality-of-service guarantee by optimizing the IRS phase-shift matrix and the transmit powers. In [11], the authors studied IRS-aided downlink communication in a multi-user multi-input single-output (MISO) system. They assumed discrete phase shifts for the IRS and maximized the weighted sum rate of all users by joint optimization of the beamforming vector of the base station and the phase shifts of the IRS. In [12], the authors studied transmission from a multi-antenna base station to multiple users using an IRS with discrete phase shifts in a downlink system. They proposed a hybrid beamforming scheme considering a reflection-dominated one-hop propagation model between the base station and the users and maximized the sum rate by optimizing the IRS phase-shift matrix. In [13], the authors studied a downlink non-orthogonal multiple access (NOMA) IRS-aided system, where a base station transmits signals to multiple users assisted by multiple IRSs. They maximized the sum rate of the users by jointly optimizing the beamforming vector of the BS and the phase shifts of the IRS, subject to IRS scattering element and successive interference cancellation decoding rate constraints. In [16], the authors used IRS for rank improvement of MIMO communication channels.

From a DoF perspective, it has been proved that the sum DoF of the time-selective K-user interference channel in the absence of an IRS is $\frac{K}{2}$ [17]. In [17], the proof of the achievability of the sum DoF of $\frac{K}{2}$ is asymptotic interference alignment, in which all interference signals are aligned into a specific subspace and the message is aligned into another subspace, such that these subspaces become linearly dependent. Inner and outer bounds for the DoF region and lower and upper bounds for the sum DoF of the time-selective K-user interference channel in the presence of active and passive IRSs have been derived in [6] and it was proved that the maximum K sum DoFs can be achieved by employing a sufficient number of elements for the IRS. In this paper, we study the DoF region and sum DoF of the time-selective $M \times N$ wireless X-network assisted by an IRS. The main difference between the K-user interference channel and the $M \times N$ wireless X-network is that in the X-network, the i-th transmitter, $i \in \{1, \ldots, M\}$, sends N independent messages $w^{[ji]}, j \in \{1, \ldots, N\}$, to each receiver $j \in \{1, \ldots, N\}$, while in the K-user interference channel, the i-th transmitter, $i \in \{1, \ldots, K\}$, sends only one message $w^{[i]}$ to the i-th receiver. In other words, the $M \times N$ wireless X-network is a generalized interference channel. The difference between 2-user interference channel and 2×2 X-network is illustrated in Fig. 1. We see from Fig. 1 that in the 2-user interference channel, the receiver Rx_i can only be served by the transmitter Tx_i, however, in the 2×2 X-network, each receiver can be served by both existing transmitters. Therefore, analyzing the DoF of the $M \times N$ X-network is more important and challenging than analyzing the DoF of the K-user interference channel. The main difficulties in the analysis of $M \times N$ X-network are 1) interference management, because all transmitter's signals cause interference in addition to the message, and 2) IRS interference cancellation design may omit some messages in addition to interferences.

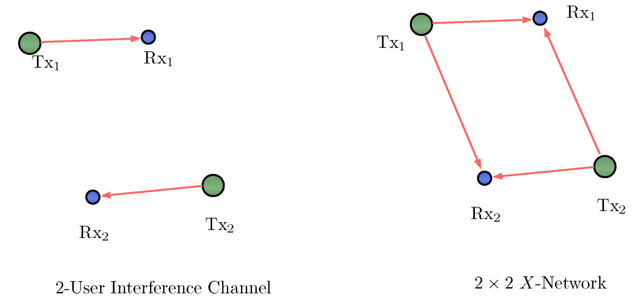

FIGURE 1. Difference between 2-user interference channel and 2×2 X-network.

The sum DoF of the time-selective $M \times N$ wireless X-network without IRS is $\frac{MN}{M+N-1}$ [19]. In addition, an outer bound for the DoF region of the time-selective $M \times N$ wireless X-network was derived in [19]. An achievable DoF region was found for the $M \times N$ wireless X-network with constant channel coefficients over different channel uses [20], which does not necessarily coincides with the outer bound, derived in [19].

We note that IRS and relay seem to be similar. However, they are fundamentally different. For ordinary relays, the output in the t-th time slot is a function of the signals received in time slots $t' \in \{1, \ldots, t-1\}$. However, for the IRS, the output in the t-th time slot is a function of the received signal in the t-th time slot only. It has been proved in [18] that ordinary relays cannot improve the DoF of the time-selective K-user interference channel. Moreover, there exists an ideal kind of relay, called instantaneous relay (IR) [21], whose output in the t-th time slot is a function of its received signals in time slots $t' \in \{1, \ldots, t\}$. Even though the IRS can be considered as a special case of the IR, however, the existing works on IR cannot cover the problem of the DoF of the $M \times N$ X-network assisted by the IRS [22].

In this paper, we study the IRS-assisted X-network for two different types of IRSs. First, we consider active IRSs, which can amplify or attenuate the received signal, in addition to changing its phase, i.e., the u-th IRS element multiplies the received wave by $\rho^{[u]} \exp\{j\theta^{[u]}\}$, $\rho^{[u]} \in \mathbb{R}^+$. Then, we consider passive IRSs, which can only attenuate the received signal, in addition to changing its phase, i.e., the u-th IRS element multiplies the received wave by $\rho^{[u]} \exp\{j\theta^{[u]}\}$, $\rho^{[u]} \in [0, 1]$. We note that active IRSs have been mentioned as a viable technology in some references, see, e.g., [23], [24], [25]. We employ and extend the techniques developed in [6] for characterizing the DoF of the time-selective K-user interference channel. In particular, for active IRSs, the key techniques used in [6] are: 1) interference cancellation by the IRS and 2) interference alignment after interference cancellation for the generated equivalent channel. For passive IRSs, the main idea behind the achievability provided in [6] is to calculate in how many time slots, the channel coefficients are realized in such a manner that active and passive IRSs can operate similarly. Thus, a probabilistic DoF will be derived. In this paper, first, we propose the extension of the methods in [6] such as interference cancellation, interference alignment for the equivalent channel, and probabilistic analysis for the X-network, assisted by either active or passive IRSs. Moreover, we consider each of these two types of IRSs in two different scenarios: 1) the channel coefficients for the elements of the IRS are independent, where the spacing between IRS elements is more than half a wavelength, and 2) the channel coefficients for the elements of the IRS are correlated, where the spacing between IRS elements are less than half a wavelength. We use the proposed framework for the analysis of active IRSs as a basis for the analysis of passive IRSs. As seen in [6], the main contribution of IRS in DoF improvement of the K-user interference channel is the interference cancellation by eliminating cross-links between transmitters and receivers. However, in an $M \times N$ wireless X-network, cross-links do not necessarily play a disruptive role, because each transmitter sends different messages to all receivers. Thus, one of the challenges for a $M \times N$ wireless X-network assisted by IRSs is the interplay between the elimination of cross-links and the sum DoF improvement. We summarize the main results of this paper as follows:

- We define the $M \times N$ network matrix $\mathbf{N}$, which characterizes the topology of the network. Then, for the scenario with independent channel coefficients for IRS elements, which is an appropriate approximate model for more than half-a-wavelength element spacing in rich scattering environments [6], we derive an inner bound for the DoF region of the active IRS-assisted $M \times N$ wireless X-network, when the network matrix $\mathbf{N}$ is forced to be fixed in all time slots. Using this result as a basis, we derive inner and outer bounds for the DoF region of the active IRS-assisted $M \times N$ wireless X-network, when the network matrix is allowed to change by the IRS in different time slots. We also derive lower and upper bounds on the sum DoF and show that the maximum sum DoF of $\min(M, N)$ is achievable if the number of IRS elements is larger than a certain finite value.
- For the passive IRS, where the channel coefficients for different IRS elements are assumed independent, we derive probabilistic[1] inner and outer bounds on the DoF region of the IRS-assisted $M \times N$ wireless X-network. Also, we derive probabilistic lower and upper bounds on the sum DoF of the IRS-aided $M \times N$ wireless X-network and show that the lower bound for the sum DoF asymptotically approaches $\min(M, N)$ with an order of at least $O(\frac{1}{Q})$, where Q is the number of IRS elements.
- Finally, we show that the proposed bounds for the passive IRS are also applicable for the passive IRS-assisted $M \times N$ X-network, where the channel coefficients for different IRS elements are assumed correlated (which is a more accurate model for less than half-a-wavelength element spacing [32]), with the difference in the order of convergence, which is now at least $O(\frac{1}{\sqrt{Q}})$. We also extend the lower bound on the sum DoF for the active IRS.

The remainder of this paper is organized as follows. We present the system model in Section II. In Sections III, IV, and V, the main results for the DoF of the $M \times N$ wireless X-network in the presence of the active IRS, passive IRS with independent elements, and active and passive IRSs with correlated elements are given, respectively. In Section VI, we provide numerical results for DoF derivations and in Section VII, the concluding remarks are presented.

Notations: Sets and vector spaces are denoted by calligraphic upper-case letters. Vectors and matrices are denoted by bold lower-case and upper-case letters, respectively. For a matrix $\mathbf{V}$, $v_{i,j}$ is the element in the i-th row and the j-th column of $\mathbf{V}$. $|\mathcal{A}|$ demonstrates the cardinality of set $\mathcal{A}$. $\mathbf{V}^T$ and $\mathbf{V}^H$ denote the transpose and Hermitian of matrix $\mathbf{V}$, respectively. $\operatorname{diag}(a_1, \ldots, a_m)$ is a main diagonal matrix, whose diagonal elements are $a_1, \ldots, a_m$. $\det(\mathbf{H})$ indicates the determinant of square matrix $\mathbf{H}$. A sequence $a(Q)$ converges to its limit a^* with an order of at least $O(g(Q))$, if we have $\lim_{Q\to\infty} \frac{|a(Q)-a^*|}{|g(Q)|} < \infty$. A sequence $a(n)$ goes to infinity with an order of $O(g(n))$, if $0 < \lim_{n\to\infty} \frac{|a(n)|}{|g(n)|} < \infty$. A function $f(\rho)$ is $o(\log(\rho))$, if $\lim_{\rho\to\infty} \frac{f(\rho)}{\log(\rho)} = 0$. The probability measure of event $\mathcal{A}$ is denoted by $\Pr\{\mathcal{A}\}$. $\mathbb{R}$ and $\mathbb{C}$ are the set of real and complex numbers, respectively. In addition, we have $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$, and $\Phi = \{\}$. We show sets $\{M, M+1, \ldots, N\}$ by $[M : N]$. Moreover, $\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C})$ indicates that $\mathbf{x}$ is a zero-mean complex Gaussian vector with correlation matrix $\mathbf{C}$. We also define the indicator function as $\mathbb{I}(x) = 0$ for $|x| = 0$, otherwise, $\mathbb{I}(x) = 1$.

1. This means that convergences are in probability.

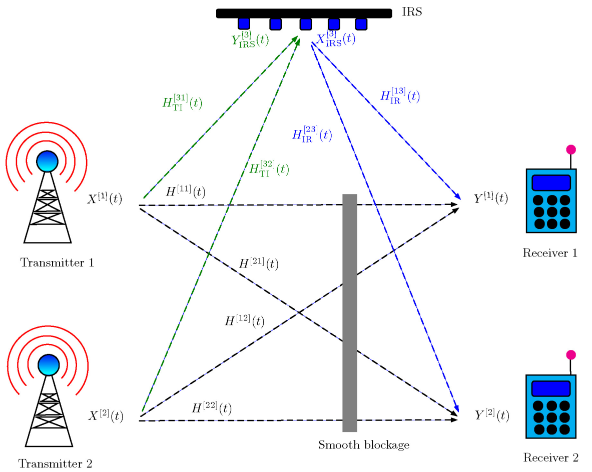

FIGURE 2. **Illustration of the IRS-assisted 2 × 2 wireless *X*-network. The black arrows indicate the direct links between the transmitters and the receivers, the green arrows illustrate the links between the IRS and the transmitters, and the blue arrows denote the links between the IRS and the receivers. The gray rectangle shows a smooth blockage in the direct link.**

II. SYSTEM MODEL AND PRELIMINARIES

A. SYSTEM MODEL

We consider a time-selective[2] $M \times N$ wireless X-network [19] assisted by a Q-element IRS, where M single antenna transmitters send their messages to N single antenna receivers. Each transmitter sends an independent message to each receiver, i.e., the message $w^{[ji]}$ is sent from the i-th transmitter to the j-th receiver for $\forall i \in [1:M], \forall j \in [1:N]$. An illustration of the IRS-aided 2×2 wireless X-network is presented in Fig. 2. We assume that the channel is time-selective. The received signal at the j-th receiver in the t-th time slot is denoted by $Y^{[j]}(t)$, which is as follows:

$$Y^{[j]}(t) = \sum_{i=1}^{M} H^{[ji]}(t)X^{[i]}(t) + \sum_{u=1}^{Q} H^{[ju]}_{\mathrm{IR}}(t)X^{[u]}_{\mathrm{IRS}}(t) + Z^{[j]}(t), \qquad (1)$$

where $X^{[i]}(t)$ indicates the transmitted signal of the i-th transmitter, $H^{[ji]}(t)$ denotes the channel coefficient between the i-th transmitter and the j-th receiver, $X^{[u]}_{\mathrm{IRS}}(t)$ is the transmitted signal of the u-th IRS element $u \in [1:Q]$, $H^{[ju]}_{\mathrm{IR}}(t)$ indicates the channel coefficient between the u-th IRS element and the j-th receiver, $Z^{[j]}(t)$ is an additive white Gaussian noise (AWGN) component at the j-th receiver in the t-th time slot, $t \in [1:T]$, Q is the number of IRS elements, and T is the number of time slots. The variance of $Z^{[j]}(t)$ is denoted by N_0. $Y^{[u]}_{\mathrm{IRS}}(t)$ denotes the received signal at the u-th IRS element in the t-th time slot and can be written as follows[3]:

$$Y^{[u]}_{\mathrm{IRS}}(t) = \sum_{i=1}^{M} H^{[ui]}_{\mathrm{TI}}(t)X^{[i]}(t), \qquad (2)$$

where $H^{[ui]}_{\mathrm{TI}}(t)$ is the channel coefficient between the i-th transmitter and the u-th IRS element $u \in [1:Q]$. $X^{[u]}_{\mathrm{IRS}}(t)$ is written as follows:

$$X^{[u]}_{\mathrm{IRS}}(t) = \tau^{[u]}(t)Y^{[u]}_{\mathrm{IRS}}(t) = \beta^{[u]}(t)e^{j\phi^{[u]}(t)}Y^{[u]}_{\mathrm{IRS}}(t), u \in [1:Q], \quad (3)$$

where $\phi^{[u]}(t) \in [0, 2\pi)$ is the phase shift added to the received signal by the u-th IRS element.[4] The feasible values of $\beta^{[u]}(t)$ realized by the IRS determine the active and passive types of the IRS. For the active IRS, $\beta^{[u]}(t) \in \mathbb{R}^+$ is feasible[5] and for the passive IRS, $\beta^{[u]}(t) \in [0, 1]$ is feasible. The physical meaning of the active IRS is that it is equipped with a controllable amplifier and a phase shifter. Whereas, the passive IRS is equipped with a controllable resistor in addition to a phase shifter [7].

We assume that in the t-th time slot, the channel coefficients of the direct links (i.e., $H^{[ji]}(t')$, $\forall i, j$, $\forall t' \in [1:T]$) and the concatenation of the transmitters-IRS and IRS-receivers channel coefficients (i.e., $H^{[ui]}_{\mathrm{TI}}(t')$ $H^{[ju]}_{\mathrm{IR}}(t'), \forall i, j, u, \forall t' \in [1:T]$) are known causally at the transmitters, the receivers, and the IRS (for channel estimation of links between other nodes and the IRS, see, e.g., [29], [30], [31]).

We also consider two different assumptions for IRS channel coefficients:

1. IRS with independent channel coefficients for the elements: For IRSs with independent channel coefficients for the elements, all channel coefficients $H^{[ji]}(t)$, $H^{[ju]}_{\mathrm{IR}}(t)$, and $H^{[qi]}_{\mathrm{TI}}(t)$ are assumed to be independent random variables for each i, j, u, q, and t, drawn from a continuous cumulative probability distribution. These channel coefficients are complex and the real and imaginary parts of them are independent random variables drawn by a continuous cumulative probability distribution, e.g., complex Gaussian distribution. For the assumption of independent channel coefficients for each element of the active and passive IRSs, the IRS elements must be sufficiently spaced, i.e., by more than half a wavelength [32], and the transmitter to IRS and IRS to receiver communication channels should be rich scattering.

2. IRS with correlated channel coefficients for the elements: In this case, we assume that the IRS has $Q = q^2, q \in \mathbb{N}$, elements, which are arranged into a square array. The IRS consists of $q = \sqrt{Q}$ elements per row and q elements per

2. In a time-selective network, channel coefficients in different time slots are independent.

3. Active IRSs do not have RF chains and the received signal is only amplified [23], [24], [25]. This amplification may cause a low level of noise, which is negligible compared to the noise caused by the RF chains at the receivers [27]. In addition, the considered X-network is analyzed in the high signal-to-noise ratio (SNR) regime and the power of the additive Gaussian noise does not affect the DoF.

4. We note that active IRSs do not contain RF chains and the received wave is only amplified after reflection [23], [24], [25], [26]. Thus, a low level of additive noise may exist, which can be neglected compared to the noise added by the RF chains [27]. In addition, we study the channel in the high signal-to-noise ratio (SNR). Thus, the power of the Gaussian noise added by the active IRS does not affect the DoF results.

5. We do not consider any constraint ($\beta_{\max}$) for the active IRS because of three reasons: 1) the nature of DoF is in high SNR regime and the power constraint will tend to infinity in the DoF definition, so the constraint $\beta_{\max}$ may become meaningless, 2) analysis of the active IRS without amplification constraint forms the basis of the analysis of the passive IRS and is essential in this sense, 3) if we want to consider the constraint $\beta_{\max}$ for the active IRS, we can replace the condition of passive IRS $|\tau^{[u]}(t)| \le 1$ by the constraint $|\tau^{[u]}(t)| \le \beta_{\max}$, then, the derived bounds are also valid, however, the definition of corresponding probability measures will change (we refer to Section IV).

column. The horizontal width and vertical height of each element are d_H and d_V, respectively. For the n-th element of the IRS, we define $\mathbf{u}_n = [0, d_H \bmod (n-1, q), d_V \lfloor (n-1)/q \rfloor]$. It has been proved in [32, Proposition 1] that in an isotropic scattering environment, the distribution of the vector of channel coefficients from the i-th transmitter to the each element of the IRS $\mathbf{h}_{\mathrm{TI}}^{[i]}(t) = [H_{\mathrm{TI}}^{[1i]}(t), H_{\mathrm{TI}}^{[2i]}(t), \ldots, H_{\mathrm{TI}}^{[Qi]}(t)]^T$ is $\mathcal{CN}(\mathbf{0}, \mu_{\mathrm{TI}}^{[i]}\mathbf{R})$, where:

$$[\mathbf{R}]_{m,n} = \frac{\sin\left(\frac{2\pi}{\lambda}\|\mathbf{u}_m - \mathbf{u}_n\|\right)}{\frac{2\pi}{\lambda}\|\mathbf{u}_m - \mathbf{u}_n\|}. \tag{4}$$

We have $\mu_{\mathrm{TI}}^{[i]} = d_H d_V \sigma_{\mathrm{TI}}^2(i)$, where $\sigma_{\mathrm{TI}}^2(i)$ is the average intensity attenuation from the i-th transmitter to the IRS.[6]

Similarly, the distribution of the vector of channel coefficients from each element of the IRS to the j-th receiver $\mathbf{h}_{\mathrm{IR}}^{[j]}(t) = [H_{\mathrm{IR}}^{[j1]}(t), H_{\mathrm{IR}}^{[j2]}(t), \ldots, H_{\mathrm{IR}}^{[jQ]}(t)]^T$ is $\mathcal{CN}(\mathbf{0}, \mu_{\mathrm{IR}}^{[j]}\mathbf{R})$, where $\mathbf{R}$ is given by (4). We have $\mu_{\mathrm{IR}}^{[j]} = d_H d_V \sigma_{\mathrm{IR}}^2(j)$, where $\sigma_{\mathrm{IR}}^2(j)$ is the average intensity attenuation from the IRS to the j-th receiver [32], [37], [38], [39]. In addition, for $\forall i, j$, vectors $\mathbf{h}_{\mathrm{TI}}^{[i]}(t)$ and $\mathbf{h}_{\mathrm{IR}}^{[j]}(t)$ are independent. Moreover, the assumption of time-selectivity can be realized using interleaving technique [33].

$\mathcal{C}(T, \rho, \mathbf{R}) = \{\{X^{[i]}(1), \ldots, X^{[i]}(T)\} | \forall i \in [1:M]\}$ is defined as the set of codes with T time slots, average power constraint ρ, i.e., $\max_i(\frac{1}{T}\sum_{t=1}^{T}|X^{[i]}(t)|^2) \leq \rho$, and rate vector $\mathbf{R}^{M\times N}$, where the message sets are $\mathcal{W}^{[ji]} = [1 : 2^{\lfloor T\times r_{i,j}\rfloor}], i \in [1:M], j \in [1:N]$, and $r_{i,j}$ is the transmission rate from the i-th transmitter to the j-th receiver. In addition, we define $P_e(\mathcal{C}(T, \rho, \mathbf{R}))$ as the probability of error for the code $\mathcal{C}(T, \rho, \mathbf{R})$, i.e., $P_e(\mathcal{C}(T, \rho, \mathbf{R})) = \Pr\{\bigcup_{i,j}\{\hat{w}^{[ji]} \neq w^{[ji]}\}\}$, where $\hat{w}^{[ji]}$ is the estimation of $w^{[ji]}$ at the j-th receiver.

B. PRELIMINARIES

In the following, we introduce some basic definitions that are used throughout the paper.

Capacity Region: The closure of the set of rate vectors $\mathbf{R}$ with power constraint ρ, for which there exists code $\mathcal{C}(T, \rho, \mathbf{R})$ such that $\lim_{T\to\infty} P_e(\mathcal{C}(T, \rho_n, \mathbf{R}(\rho_n))) = 0$ is defined as capacity region and denoted by $\mathcal{CR}(\rho)$.

Sum Capacity: The sum capacity is defined as $C(\rho) = \max \sum_{i=1}^{M}\sum_{j=1}^{N} r_{i,j}$ where rate vector $[r_{i,j}] = \mathbf{R} \in \mathcal{CR}(\rho)$.

Degree of Freedom (DoF): Consider the sequence of power constraints $\rho_n = n\rho_0, n \in \mathbb{N}, \rho_0 > 0$. For a $M \times N$ wireless X-network, we say a DoF matrix $\mathbf{D}$ is achievable in T time slots, if there exist a sequence of codes $\mathcal{C}_n(T, \rho_n, R(\rho_n))$ and an integer $\tilde{n}$, such that for $\forall n > \tilde{n}$, we have the following relations: $r_{i,j}(\rho_n) \geq d_{i,j}\log(\rho_n), \forall i \in [1:M], \forall j \in [1:N]$, and $\lim_{n\to\infty} P_e(\mathcal{C}_n(T, \rho_n, \mathbf{R}(\rho_n))) = 0$. We define set $\mathcal{D}(T)$ as the set of all achievable DoF matrices in T time slots, and the DoF region $\mathcal{D}$ is defined as follows:

$$\mathcal{D} = \left\{\mathbf{D} \in \mathbb{R}_+^{M\times N} \middle| \forall \mathbf{E} \in \mathbb{R}_+^{M\times N} : \sum_{i\in[1:M], j\in[1:N]} e_{i,j}d_{i,j} \leq \lim_{T\to\infty}\sup \sum_{i\in[1:M], j\in[1:N]} e_{i,j}d_{i,j}(T)\right\},$$

which concludes:

$$\mathcal{CR}(\rho) = \{\mathbf{D}\log(\rho) + o(\log(\rho)) : \mathbf{D} \in \mathcal{D}\}.$$

Span: The space spanned by the column vectors of matrix $\mathbf{V}$ is denoted by $\mathrm{span}(\mathbf{V})$.

Dimension: We define the number of dimensions of $\mathrm{span}(\mathbf{V})$ as the dimension of $\mathbf{V}$ and show it by $d(\mathbf{V})$, which is equal to the rank of matrix $\mathbf{V}$.

Normalized Asymptotic Dimension: In the following sections, for a given M, N, and Q, the dimensions of beamforming matrices will have the order of $O(n^l), l, n \in \mathbb{N}$ due to the proposed asymptotic interference alignment scheme. We define the normalized asymptotic dimension (D_N) of a matrix $\mathbf{V}$ as $D_N(\mathbf{V}) = \lim_{n\to\infty}\frac{d(\mathbf{V})}{n^l}$, where l is the minimum integer number, for which we have $\lim_{n\to\infty}\frac{d(\mathbf{V})}{n^l} < \infty$.

We also use these definitions for a vector space $\mathcal{A}$, i.e., $d(\mathcal{A})$ indicates the dimension of $\mathcal{A}$ and $D_N(\mathcal{A})$ denotes the normalized asymptotic dimension of $\mathcal{A}$.

Network Matrix: As seen from (1)-(3), the effective channel coefficient between the i-th transmitter and the j-th receiver is $H^{[ji]}(t) + \sum_{u=1}^{Q} H_{\mathrm{TI}}^{[ui]}(t)H_{\mathrm{IR}}^{[ju]}(t)\tau^{[u]}(t)$. The network matrix $\mathbf{N}$ is an $M \times N$ matrix, which characterizes the connectivity of the X-network, i.e., $n_{i,j} = \mathbb{I}(H^{[ji]}(t) + \sum_{u=1}^{Q} H_{\mathrm{TI}}^{[ui]}(t)H_{\mathrm{IR}}^{[ju]}(t)\tau^{[u]}(t))$.

III. ACTIVE IRS WITH INDEPENDENT CHANNEL COEFFICIENTS FOR ELEMENTS

A. DOF REGION

In this section, we analyze the DoF of $M \times N$ wireless X-network assisted by an active IRS with independent channel coefficients for elements. First of all, we show that a Q-element active IRS can change the connectivity of the network to realize the network matrix $\mathbf{N}$ including Q zero entries with probability equal to 1. This can be achieved by designing the IRS such that the following equations are satisfied:

$$\sum_{u\in[1:Q]} H_{\mathrm{TI}}^{[ui]}(t)H_{\mathrm{IR}}^{[ju]}(t)\tau^{[u]}(t) = -H^{[ji]}(t), \tag{5}$$
$$i \neq j, (i,j) \in \mathcal{B}, |\mathcal{B}| = Q,$$

6. In [32], it has been assumed that there is infinite number of multipath components in an isotropic scattering environment, thus, for the vector of channel coefficients from the i-th transmitter to the IRS ($\mathbf{h}_{\mathrm{TI}}^{[i]}(t)$), we have:

$$\mathbf{h}_{\mathrm{TI}}^{[i]}(t) = \lim_{L\to\infty}\sum_{l=1}^{L}\frac{c_l^j}{\sqrt{L}}\left[\exp\left\{-j\mathbf{k}(\varphi_l,\theta_l)^T\mathbf{u}_1\right\}, \ldots, \exp\left\{-j\mathbf{k}(\varphi_l,\theta_l)^T\mathbf{u}_Q\right\}\right]^T \xrightarrow{Distribution} \mathcal{CN}\left(\mathbf{0}, \mu_{\mathrm{TI}}^{[i]}\mathbf{R}\right),$$

where $\mathbf{k}(\varphi_l, \theta_l)$ is the wave vector of the l-th multipath component and c_l^j is its coefficient (see [32]). Moreover, the convergence is in distribution.

where $\mathcal{B} = \{(i,j)|n_{i,j} = 0\}$. Note that this set of equations has solution almost surely because; if we write these equations in the matrix form, $\mathbf{H}\boldsymbol{\tau} = \mathbf{h}$, then $\det(\mathbf{H})$ will become a nonzero polynomial in terms of $H_{\text{TI}}^{[ui]}$ and $H_{\text{IR}}^{[ju]}$ and by [7, Lemma 1], we have $\Pr\{\det(\mathbf{H}) = 0\} = 0$. Also, we introduce the following notations for the simplicity of presentation. We rewrite Eqs. (1) and (2) in the vector forms:

$$\mathbf{y}^{[j]} = \sum_{i=1}^{M} \mathbf{H}^{[ji]}\mathbf{x}^{[i]} + \sum_{u=1}^{Q} \mathbf{H}_{\text{IR}}^{[ju]}\mathbf{x}_{\text{IRS}}^{[u]} + \mathbf{z}^{[j]},\ \mathbf{y}_{\text{IRS}}^{[q]} = \sum_{i=1}^{M} \mathbf{H}_{\text{TI}}^{[qi]}\mathbf{x}^{[i]}, \quad (6)$$

where $\mathbf{x}^{[i]}$ is a $T \times 1$ column vector containing $X^{[i]}(t)$, i.e., $\mathbf{x}^{[i]} = [\,X^{[i]}(1)\ X^{[i]}(2) \cdots X^{[i]}(T)\,]^T$. Vectors $\mathbf{y}^{[i]}$, $\mathbf{y}_{\text{IRS}}^{[q]}$, $\mathbf{x}_{\text{IRS}}^{[u]}$, and $\mathbf{z}^{[j]}$ are defined similarly. $\mathbf{H}^{[ji]}$ denotes a diagonal matrix defined as $\mathbf{H}^{[ji]} = \text{diag}(H^{[ji]}(1), H^{[ji]}(2), \ldots, H^{[ji]}(T))$. Matrices $\mathbf{H}_{\text{IR}}^{[ju]}$ and $\mathbf{H}_{\text{TI}}^{[qi]}$ are defined similarly. Now, we derive an inner bound on the DoF region of the active IRS-assisted $M \times N$ wireless X-network, when the network matrix is forced to be fixed in all time slots by setting IRS coefficients such that Eqs. (5) are satisfied. This approach is essential, because further inner bounds that we derive on the DoF region of IRS-assisted X-network in the general case (where the network matrix can change via different time slots), are based on the inner bound given for the fixed network matrix. The following achievability theorem, which is based on interference alignment technique, will be the basis of our further achievability theorems for active and passive IRSs in this paper.

Theorem 1: Consider an $M \times N$ X-network assisted by a Q-element active IRS with independent channel coefficients for elements. Assume that based on (5), the IRS is designed such that the network matrix $\mathbf{N}$ is fixed across all T time slots and at most Q entries of it are 0. We define the set $\mathcal{D}_{\mathbf{N}}$ as follows:

$$\mathcal{D}_{\mathbf{N}} = \left\{ \mathbf{D} \in \mathbb{R}_+^{M\times N} \left| \begin{array}{l} \forall j \in [1:N]\colon \sum_{i'=1}^{M} n_{i',j}d_{i',j} + \sum_{j'\neq j} \max_{i'}\{n_{i',j'}d_{i',j'}\} \leq 1 \\ \forall i \in [1:M], \forall j \in [1:N] : 0 \leq d_{i,j} \leq n_{i,j}. \end{array} \right. \right\}, \quad (7)$$

where $n_{i,j}$ is the element in the i-th row and j-th column of network matrix $\mathbf{N}$. Then, we have $\mathcal{D}_{\mathbf{N}} \subseteq \mathcal{D}(\mathbf{N})$, where $\mathcal{D}(\mathbf{N})$ is the DoF region, when the network matrix is $\mathbf{N}$ across all time slots.

Proof: The outline of the proof of this theorem is organized in five steps, which contains: 1) generation of the message stream, 2) designing the interference cancellation method and the channel equalization for the network, 3) designing the interference alignment equations for each receiver in the X-network, 4) designing the beamforming matrices, which satisfy the interference alignment equations, and 5) analysis of the satisfaction of the interference alignment equations, the decodability of message symbols and the calculation of the achieved DoF. The complete proof is provided in Appendix A. ■

Now, we give inner and outer bounds for the DoF region of the active IRS-aided $M \times N$ wireless X-network, when the network matrix can change across T time slots. First, we introduce the inner bound, which is based on time-sharing of the DoF regions provided in Theorem 1.

Theorem 2: Consider a Q-element active IRS-assisted $M \times N$ wireless X-network with independent channel coefficients for elements and define $\mathcal{N}_Q$ as the set of all possible network matrices with at most Q zero elements. Then, we have:

$$\mathcal{D}_{\text{in}} = \bigcup_{\mathbf{a}\in\mathcal{A}} \left\{ \sum_{i=1}^{|\mathcal{N}_Q|} a_i\mathbf{D}_i \middle| \mathbf{D}_i \in \mathcal{D}_{\mathbf{N}_i}, \mathbf{N}_i \in \mathcal{N}_Q \right\} \subseteq \mathcal{D}, \quad (8)$$

where the set $\mathcal{D}_{\mathbf{N}_i}$ is given by (7) and

$$\mathcal{A} = \left\{ \mathbf{a} \middle| 0 \leq a_i \leq 1, \sum_{i=1}^{|\mathcal{N}_Q|} a_i = 1 \right\}. \quad (9)$$

Proof: We use time sharing to prove this theorem. Let us divide T time slots into $|\mathcal{N}_Q|$ groups, such that the i-th group has a_iT sub slots. We also design the IRS such that in the sub slots corresponding to the i-th group, the network matrix is $\mathbf{N}_i$. Thus, the achieved DoF vector in these time slots is a member of the region $D_{\mathbf{N}_i}$ according to Theorem 1. Hence, the region (8) is achievable. ■

Theorem 2 is based on two main facts: 1) An active IRS can null Q elements of the network matrix and 2) the time sharing of the DoFs of all such network matrices with Q non-zero entries is used. In the next step, we present an outer bound for the DoF region of the IRS-assisted $M \times N$ wireless X-network. This theorem indicates that similar to the inner bound given in Theorem 2, the outer bound depends on the network matrix of each time slot and the percentage of their occurrence, but these bounds do not necessarily coincide.

Theorem 3: Consider a Q-element active IRS-assisted $M \times N$ wireless X-network with independent channel coefficients for elements and define $\mathcal{N}_Q$ as the set of all possible network matrices with at most Q zero entries. Assume that each network matrix $\mathbf{N}_k \in \mathcal{N}_Q$, occurs in Ta_k time slots, $k \in [1 : \sum_{l=0}^{Q} \binom{MN}{l}]$, where $\mathbf{a} \in \mathcal{A}$ and $\mathcal{A}$ is given in (9). In addition, for each $\mathbf{a} \in \mathcal{A}$, we define the following sets:

$$\mathcal{D}'(\mathbf{a}) = \bigcap_{i\in[1:M], j\in[1:N]} \left\{ \mathbf{D} \in \mathbb{R}_+^{M\times N} \left| \begin{array}{c} \sum_{i'=1}^{M} d_{i',j} + \sum_{j'=1, j'\neq j}^{N} d_{i,j'} \leq 1 + \sum_{k=1}^{|\mathcal{N}_Q|} a_k\big(1 - [n_k]_{i,j}\big)\Big(\sum_{j''\neq j} [n_k]_{i,j''}\Big) \\ 0 \leq d_{i,j} \leq \sum_{k=1}^{|\mathcal{N}_Q|} a_k[n_k]_{i,j} \end{array} \right. \right\}, \quad (10)$$

where $[n_k]_{i,j}$ is the element of the i-th row and j-th column of network matrix $\mathbf{N}_k$. Then, we have:

$$\mathcal{D} \subseteq \mathcal{D}_{\text{out}} = \bigcup_{\mathbf{a}\in\mathcal{A}} \mathcal{D}'(\mathbf{a}). \quad (11)$$

Proof: The proof is provided in Appendix B. ■

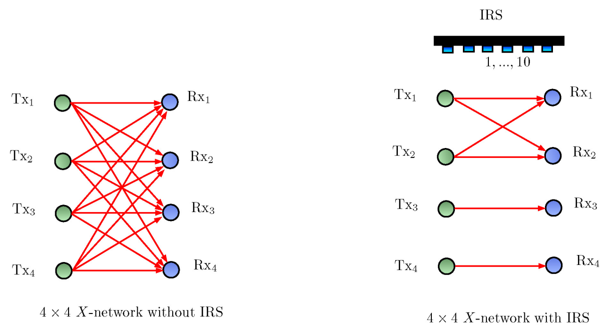

FIGURE 3. An exemplary illustration of the impact of a 10-element active IRS in decomposing a 4 × 4 *X*-network into a 2 × 2 *X*-network and two interference-free transceivers.

Corollary 1: Theorems 2 and 3 indicate that for the approximate capacity region in the presence of active IRS with independent channel coefficients, we obtain the following relation:

$$\{\mathbf{D}\log(\rho+1)+o(\log(\rho)) : \mathbf{D}\in\mathcal{D}_{\text{in}}\}\subseteq\mathcal{CR}(\rho) \subseteq\{\mathbf{D}\log(\rho+1)+o(\log(\rho)) : \mathbf{D}\in\mathcal{D}_{\text{out}}\}. \quad (12)$$

B. SUM DOF

In this subsection, we use the inner and outer bounds given in Theorems 2 and 3 to obtain the lower and upper bounds on the sum DoF of an active IRS-aided $M\times N$ wireless X-network in Theorems 4 and 5.

Theorem 4: With an active IRS with $Q = W(N-1)+W(M-W)$ elements, where W is the number of interference-free receivers caused by the active IRS and $W\in[0 : \min(M,N)]$. Then, the following sum DoF is achievable:

$$DoF_{active-low} = W+\frac{(M-W)(N-W)}{M+N-2W-1}. \quad (13)$$

Proof: The outline of the proof is that we design a proper network matrix by the IRS, which decomposes the $M\times N$ wireless X-network, into two separate networks: 1) a network with W interference-free receivers, which can achieve W sum DoF, and 2) a $(M-W)\times(N-W)$ wireless X-network, which can achieve $\frac{(M-W)(N-W)}{M+N-2W-1}$ sum DoF. This procedure is illustrated in Fig. 3 for the 4 × 4 X-network assisted by a 10-element active IRS. The complete proof is provided in Appendix C. ■

Corollary 2: To achieve the maximum sum DoF of $\min(M,N)$, we require to set $W=\min(M,N)$ in $Q = W(N-1)+W(M-W) = W(M+N-W-1)$.

Theorem 5: With an active IRS with Q elements, the sum DoF is upper bounded by $DoF_{active-up}$ determined as follows:

$$DoF_{active-up} = \min\left\{\frac{MN+(N-1)Q}{M+N-1}, \min\{M,N\}\right\}. \quad (14)$$

Proof: The proof is provided in Appendix D. ■

Corollary 3: Theorems 4 and 5, indicate that the approximate sum capacity of the $M\times N$ X-network assisted by an active IRS with independent channel coefficients for elements using $Q = W(N-1)+W(M-W)$ elements, where $0\le W\le\min(M,N)$, is bounded as follows:

$$\left(W+\frac{(M-W)(N-W)}{M+N-2W-1}-\epsilon\right)\log(1+\rho)+o(log(\rho))\le C(\rho) \le\left(\min\left\{\frac{MN+(N-1)Q}{M+N-1}, \min\{M,N\}\right\}\right)\log(1+\rho)+o(\log(\rho)). \quad (15)$$

IV. PASSIVE IRS WITH INDEPENDENT CHANNEL COEFFICIENTS FOR ELEMENTS

Due to random realization of channel coefficients and lower capabilities of the passive IRS compared to the active IRS (the passive IRS cannot amplify the received signal), probabilistic guarantees are given for the DoF improvement instead of exact guarantees. The main difference between passive and active IRSs is that for a passive IRS, the set of realizable network matrices in a time slot is a function of the realization of the channel coefficients in that time slot, which create randomness in the system, whereby, for an active IRS, all network matrices with at most Q zero entries are realizable in a time slot. The stochastic analysis method introduced for the passive IRS-assisted K-user interference channel in [7] is the basis for the stochastic analysis method of the passive IRS-assisted $M\times N$ wireless X-network.

A. DOF REGION

Define sets $\tilde{\mathcal{N}}_{Q_i}, i\in[1,\ldots,2^{|\mathcal{N}_Q|-1}]$ as all subsets of $\mathcal{N}_Q$ including the following matrix:

$$\mathbf{N}_1 : n_{i,j}=1, \quad \forall i,j. \quad (16)$$

Also, we define event $\mathcal{E}_{Q_i}$ such that the passive IRS can realize network matrices $\mathbf{N}\in\tilde{\mathcal{N}}_{Q_i}$ and cannot realize network matrices $\mathbf{N}\in\tilde{\mathcal{N}}^c_{Q_i}=\mathcal{N}_Q-\tilde{\mathcal{N}}_{Q_i}$, i.e., for $\forall\mathbf{N}\in\tilde{\mathcal{N}}_{Q_i}$, $\exists\tau^{[u]}(t), u\in[1 : Q]$, such that for $\forall u : |\tau^{[u]}(t)|\le 1$, and we have:

$$\forall(i,j) : n_{i,j}=\mathbb{I}\left(H^{[ji]}(t)+\sum_{u=1}^{Q}H^{[ui]}_{\text{TI}}(t)H^{[ju]}_{\text{IR}}(t)\tau^{[u]}(t)\right) \quad (17)$$

and for $\forall\mathbf{N}\in\tilde{\mathcal{N}}^c_{Q_i}$, and for $\forall\tau^{[u]}(t), u\in[1 : Q]$, such that $\forall u : |\tau^{[u]}(t)|\le 1$, we have:

$$\exists(i,j) : n_{i,j}\neq\mathbb{I}\left(H^{[ji]}(t)+\sum_{u=1}^{Q}H^{[ui]}_{\text{TI}}(t)H^{[ju]}_{\text{IR}}(t)\tau^{[u]}(t)\right). \quad (18)$$

Now, we derive a probabilistic outer bound for the DoF region.

Theorem 6: Define the sets $\mathcal{D}''(\mathbf{a}^{[1]},\ldots,\mathbf{a}^{[2^{|\mathcal{N}_Q|-1}]},\delta)$ as follows:

$$\mathcal{D}''\left(\mathbf{a}^{[1]},\ldots,\mathbf{a}^{\left[2^{|\mathcal{N}_Q|-1}\right]},\delta\right)=\bigcap_{i,j}\bigcap_{n=1}^{2^{N-1}}\left\{\mathbf{D}\in\mathbb{R}_+^{M\times N}\right|$$

$$\sum_{i'=1,i'\neq i}^{M}d_{i',j}+\sum_{j'=1}^{N}d_{i,j'}\le\sum_{l=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\{\mathcal{E}_{Q_l}\}+\delta\right)$$

$$\times\left(1+\sum_{k=1}^{\left|\tilde{\mathcal{N}}_{Q_l}\right|}a_k^{[l]}\left(1-\left[n_k^{[l]}\right]_{i,j}\right)\left(\sum_{j''\neq j}\left[n_k^{[l]}\right]_{i,j''}\right)\right),$$

$$0\leq d_{i,j}\leq\sum_{l=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\left\{\mathcal{E}_{Q_l}\right\}+\delta\right)\left(\sum_{k=1}^{\left|\tilde{\mathcal{N}}_{Q_l}\right|}a_k^{[l]}\left[n_k^{[l]}\right]_{i,j}\right)\Bigg\}. \quad (19)$$

In the time slots, where event $\mathcal{E}_{Q_l}$ occurs, $a_k^{[l]}$ is the fraction of time slots, in which the network matrix is $\mathbf{N}_k^{[l]}\in\tilde{\mathcal{N}}_{Q_l}$ and $[n_k^{[l]}]_{i,j}$ is the element in the i-th row and the j-th column of $\mathbf{N}_k^{[l]}$. We also have $\mathbf{a}^{[l]}\in\mathcal{A}_l$, where

$$\mathcal{A}_l=\left\{\mathbf{a}^{[l]}\middle|0\leq a_k^{[l]}\leq 1,\sum_{k=1}^{|\tilde{\mathcal{N}}_{Q_l}|}a_k^{[l]}=1\right\},l\in\left\{1,\ldots,2^{|\mathcal{N}_Q|-1}\right\}. \quad (20)$$

Furthermore, we define the set $\mathcal{D}_{out}(\delta)$ as follows:

$$\mathcal{D}_{out}(\delta)=\bigcup_{\forall l:\mathbf{a}^{[l]}\in\mathcal{A}_l}\mathcal{D}''\left(\mathbf{a}^{[1]},\ldots,\mathbf{a}^{2^{|\mathcal{N}_Q|-1}},\delta\right). \quad (21)$$

Then, if the channel coefficients are drawn independently and identically distributed (i.i.d.) from a continuous cumulative probability distribution across all T time slots, for $\forall\epsilon,\delta>0$, there exists a number T' such that for $\forall T>T'$, we have:

$$\Pr\{\mathcal{D}(T)\subseteq\mathcal{D}_{out}(\delta)\}>1-\epsilon, \quad (22)$$

where $\mathcal{D}(T)$ is the DoF region in T time slots. In other words, we have $\lim_{T\to\infty}\Pr\{\mathcal{D}\subseteq\mathcal{D}_{out}(\delta)\}=1,\forall\delta>0$.

Proof: The basis of the proof of this theorem is the law of large numbers. The complete proof is provided in Appendix E. ■

The difference of the outer bound for the passive IRS in Theorem 6 and the outer bound for the active IRS in Theorem 3 is that the coefficients $a_m^{[l]}$ are more restricted for passive IRSs, i.e., the coefficients of the network matrices, which are not realizable in $\mathcal{E}_{Q_l}$ are zero. This difference will cause the outer bound introduced in (11) for the active IRS to contain the outer bound (21).

Next, we introduce a probabilistic inner bound for the DoF region of an $M\times N$ wireless X-network in the presence of a Q-element passive IRS. For a network matrix $\mathbf{N}\in\tilde{\mathcal{N}}_{Q_i}$, set $\mathcal{M}_\mathbf{N}$ is defined as follows:

$$\mathcal{M}_\mathbf{N}=\left\{(i,j)\middle|i\in[1:M],j\in[1:N],n_{i,j}=0\right\}. \quad (23)$$

To realize the network matrix $\mathbf{N}$, the IRS must be designed such that for each $t\in[1:T]$, we have:

$$\sum_{u\in[1:Q]}H_{\mathrm{TI}}^{[ui]}(t)H_{\mathrm{IR}}^{[ju]}(t)\tau^{[u]}(t)=-H^{[ji]}(t),(i,j)\in\mathcal{M}_\mathbf{N}. \quad (24)$$

We rewrite Eqs. (24) in the matrix form ($\mathbf{H_N}\boldsymbol{\tau}_\mathbf{N}=\mathbf{h_N}$), where $\mathbf{H_N}$ is a matrix whose elements are $H_{\mathrm{TI}}^{[ui]}(t)H_{\mathrm{IR}}^{[ju]}(t)$, $\boldsymbol{\tau}_\mathbf{N}$ is a column vector whose elements are $\tau^{[u]}(t)$, and $\mathbf{h_N}$ is a column vector whose elements are $-H^{[ji]}(t)$, $(i,j)\in\mathcal{M}_\mathbf{N}$. We note that in (24), the number of variables can exceed the number of equations ($Q\geq|\mathcal{M}_\mathbf{N}|$), thus, we use pseudo inverse because it is a tractable solution, for which an interference alignment scheme and asymptotic analysis can be provided. to obtain $\boldsymbol{\tau}_\mathbf{N}$, i.e.,

$$\boldsymbol{\tau}_\mathbf{N}^*=\mathbf{H}_\mathbf{N}^H\left(\mathbf{H_N}\mathbf{H}_\mathbf{N}^H\right)^{-1}\mathbf{h_N}. \quad (25)$$

Note that the matrix $\mathbf{H_N}\mathbf{H}_\mathbf{N}^H$ is full rank and invertible almost surely because if we construct a square matrix $\tilde{\mathbf{H}}_\mathbf{N}$ by choosing $|\mathcal{M}_\mathbf{N}|$ columns of the matrix $\mathbf{H_N}$, then $\det(\tilde{\mathbf{H}}_\mathbf{N})$ will be a non-zero polynomial in terms of $H_{\mathrm{TI}}^{[ui]}(t)$ and $H_{\mathrm{IR}}^{[ju]}(t)$ and by [7, Lemma 1], $\Pr\{\det(\tilde{\mathbf{H}}_\mathbf{N})=0\}=0$. Therefore, $\tilde{\mathbf{H}}_\mathbf{N}$ is full rank almost surely. On the other hand, we have $\mathrm{rank}(\mathbf{H_N})=\mathrm{rank}(\tilde{\mathbf{H}}_\mathbf{N})=\mathrm{rank}(\mathbf{H_N}\mathbf{H}_\mathbf{N}^H)$. Note that by increasing the number of IRS elements, the probability of realizability of coefficients $\boldsymbol{\tau}_\mathbf{N}^*$ using a passive IRS is increased. Next, we define event $\mathcal{F}_{Q_i}$ in the t-th time slot as follows:

$$\mathcal{F}_{Q_i}=\left\{\forall u\in[1:Q],\forall\mathbf{N}\in\tilde{\mathcal{N}}_{Q_i}:\left|\tau_\mathbf{N}^{[u]*}(t)\right|\leq 1\right\}$$
$$\bigcap\left\{\forall\mathbf{N}\in\tilde{\mathcal{N}}_{Q_i}^c:\exists u\in[1:Q]\to|\tau_\mathbf{N}^{[u]*}(t)|>1\right\}. \quad (26)$$

We note that similar to events $\mathcal{E}_{Q_i}$, events $\mathcal{F}_{Q_i}$ are disjoint for $\forall i$. Now, we introduce a probabilistic inner bound for the passive IRS-assisted $M\times N$ wireless X-network.

Theorem 7: Consider set $\mathcal{D}_{\tilde{\mathcal{N}}_{Q_i}}$ as follows:

$$\mathcal{D}_{\tilde{\mathcal{N}}_{Q_i}}=\bigcup_{\mathbf{a}\in\mathcal{A}_i}\left\{\sum_{j=1}^{\left|\tilde{\mathcal{N}}_{Q_i}\right|}a_j\mathbf{D}_j\middle|\mathbf{D}_j\in\mathcal{D}_{\mathbf{N}_j},\mathbf{N}_j\in\tilde{\mathcal{N}}_{Q_i}\right\}, \quad (27)$$

where set $\mathcal{D}_{\mathbf{N}_j}$ is given by (7) and set $\mathcal{A}_i$ is given by (20). Also, set $\mathcal{D}_{in}(\delta)$ is defined as follows:

$$\mathcal{D}_{in}(\delta)=\left\{\sum_{i=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\left\{\mathcal{F}_{Q_i}\right\}-\delta\right)\mathbf{D}_i\middle|\mathbf{D}_i\in\mathcal{D}_{\tilde{\mathcal{N}}_{Q_i}},i\in\left[1:2^{|\mathcal{N}_Q|-1}\right]\right\}. \quad (28)$$

Then, if the channel coefficients are drawn i.i.d from a continuous cumulative probability distribution across all T time slots, for $\forall\epsilon,\delta>0$, there exists a number T' such that for $\forall T>T'$, we have:

$$\Pr\{\mathcal{D}_{in}(\delta)\subseteq\mathcal{D}(T)\}>1-\epsilon, \quad (29)$$

where $\mathcal{D}(T)$ is the DoF region in T time slots. In other words, we have $\lim_{T\to\infty}\Pr\{\mathcal{D}_{in}(\delta)\subseteq\mathcal{D}\}=1,\forall\delta>0$.

Proof: The proof of this theorem is based on the law of large numbers and time sharing technique. The complete proof is provided in Appendix F. ■

Similar to the outer bound, the inner bound for the active IRS in Theorem 2 contains the inner bound for the passive IRS in Theorem 7.

Corollary 4: Theorems 6 and 7 show that for the approximate capacity region in the presence of passive IRS with independent channel coefficients, we obtain the following relation:

$$\{\mathbf{D}\log(\rho+1)+o(\log(\rho)) : \mathbf{D}\in\mathcal{D}_{\text{in}}(\delta)\}\subseteq\mathcal{CR}(\rho)$$
$$\subseteq\{\mathbf{D}\log(\rho+1)+o(\log(\rho)) : \mathbf{D}\in\mathcal{D}_{\text{out}}(\delta)\}. \quad (30)$$

The behavior of $\Pr\{\mathcal{F}_{Q_l}\}$ and $\Pr\{\mathcal{E}_{Q_l}\}$ for a K-user interference channel have been studied for large values of Q in [7, Th. 8], where network matrices are $K\times K$. The proof of [7, Th. 8] is also applicable for a $M\times N$ wireless X-network, for which network matrices are $M\times N$. It has been proved in [7, Th. 8] that if the imaginary and real parts of all channel coefficients are zero mean and their probability distributions satisfy [7, Properties (29)–(35)], for $(i,j)\in [1:M]\times[1:N]$, then we have

$$\lim_{Q\to\infty}\Pr\{\mathcal{F}_{Q_1}\}=1,\quad \lim_{Q\to\infty}\Pr\{\mathcal{F}_{Q_i}\}=0,\ i\neq 1,$$
$$\lim_{Q\to\infty}\Pr\{\mathcal{E}_{Q_1}\}=1,\quad \lim_{Q\to\infty}\Pr\{\mathcal{E}_{Q_i}\}=0,\ i\neq 1,\quad (31)$$

where $\tilde{\mathcal{N}}_{Q_1}=\mathcal{N}_Q$. All probability measures $\Pr\{\mathcal{F}_{Q_l}\}$ and $\Pr\{\mathcal{E}_{Q_l}\}$ converges with an order of at least $O(\frac{1}{Q})$.

B. SUM DOF

Now, based on Theorems 6 and 7, we derive probabilistic lower and upper bounds for the sum DoF of the $M\times N$ X-network in the presence of a passive IRS. First, we introduce the upper bound on the sum DoF:

Theorem 8: Consider an $M\times N$ wireless X-network assisted by a Q-element passive IRS with independent channel coefficients for elements. Then, for $\forall\mathbf{D}(T)\in\mathcal{D}(T)$, if the channel coefficients for all T time slots are i.i.d. drawn from a continuous cumulative probability distribution, for $\forall\epsilon,\delta>0$, there exists a number T' such that for $\forall T>T'$, we have:

$$\Pr\left\{\max_{\mathbf{D}(T)\in\mathcal{D}(T)}\sum_{m=1}^{M}\sum_{n=1}^{N}d_{m,n}(T)\leq \min\left\{\min\{M,N\},\sum_{l=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\{\mathcal{E}_{Q_l}\}+\delta\right)\left(\frac{MN+(N-1)\max_{\mathbf{N}_k^{[l]}\in\tilde{\mathcal{N}}_{Q_l}}|\mathcal{M}_{\mathbf{N}_k^{[l]}}|}{M+N-1}\right)\right\}\right\}$$
$$>1-\varepsilon, \quad (32)$$

where $\mathcal{M}_{\mathbf{N}_k^{[l]}}$ is given by (23). In other words, we have:

$$\lim_{T\to\infty}\Pr\left\{\max_{\mathbf{D}(T)\in\mathcal{D}(T)}\sum_{m=1}^{M}\sum_{n=1}^{N}d_{m,n}(T)\leq\min\left\{\min\{M,N\},\sum_{l=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\{\mathcal{E}_{Q_l}\}+\delta\right)\left(\frac{MN+(N-1)\max_{\mathbf{N}_k^{[l]}\in\tilde{\mathcal{N}}_{Q_l}}|\mathcal{M}_{\mathbf{N}_k^{[l]}}|}{M+N-1}\right)\right\}\right\}=1.$$

Proof: The proof is provided in Appendix G. ∎

To introduce the lower bound on the sum DoF, for each set $\tilde{\mathcal{N}}_{Q_i}$, we define the subsets $\tilde{\mathcal{N}}_{Q_i}^W$, $W\in[0:\min(M,N)]$ such that $\tilde{\mathcal{N}}_{Q_i}^W$ contains the network matrices $\mathbf{N}$ of $\tilde{\mathcal{N}}_{Q_i}$, for which there exist sets $\mathcal{B}_\mathbf{N},\mathcal{C}_\mathbf{N}$, $|\mathcal{B}_\mathbf{N}|=|\mathcal{C}_\mathbf{N}|=W$, so that for $\forall i\in\mathcal{B}_\mathbf{N}$, there exists a $j\in\mathcal{C}_\mathbf{N}$, for which $n_{i,j}=1$. In addition, for each $\mathbf{N}\in\tilde{\mathcal{N}}_{Q_i}^W$, if $i\in\mathcal{B}_\mathbf{N}$ and $n_{i,j}=n_{i,j'}=1$, then $j=j'$, if $j\in\mathcal{C}_\mathbf{N}$ and $n_{i,j}=n_{i',j}=1$, then $i=i'$, if $i\notin\mathcal{B}_\mathbf{N}, j\in\mathcal{C}_\mathbf{N}$, then $n_{i,j}=0$, and if $i\notin\mathcal{B}_\mathbf{N}, j\notin\mathcal{C}_\mathbf{N}$, then $n_{i,j}=1$. $\tilde{\mathcal{N}}_{Q_i}^0$ is the set of all network matrices from $\tilde{\mathcal{N}}_{Q_i}$, which do not satisfy the previous conditions for all $W\in[1:\min(M,N)]$. Now, we present a probabilistic lower bound for the sum DoF.

Theorem 9: Consider an $M\times N$ wireless X-network assisted by a Q-element passive IRS with independent channel coefficients for elements. Then, for $\forall\mathbf{D}(T)\in\mathcal{D}(T)$, if the channel coefficients for all T time slots are i.i.d. drawn from a continuous cumulative probability distribution, for $\forall\epsilon,\delta>0$, there exists a number T' such that for $\forall T>T'$, we have:

$$\Pr\left\{\max_{\mathbf{D}(T)\in\mathcal{D}(T)}\sum_{m=1}^{M}\sum_{n=1}^{N}d_{m,n}(T)\geq \sum_{i=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\{\mathcal{F}_{Q_i}\}-\delta\right)\max_{\tilde{\mathcal{N}}_{Q_i}^W\neq\Phi}\left(W+\frac{(M-W)(N-W)}{M+N-2W-1}\right)\right\}>1-\epsilon. \quad (33)$$

In other words, we have:

$$\lim_{T\to\infty}\Pr\left\{\max_{\mathbf{D}(T)\in\mathcal{D}(T)}\sum_{m=1}^{M}\sum_{n=1}^{N}d_{m,n}(T)\geq \sum_{i=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\{\mathcal{F}_{Q_i}\}-\delta\right)\max_{\tilde{\mathcal{N}}_{Q_i}^W\neq\Phi}\left(W+\frac{(M-W)(N-W)}{M+N-2W-1}\right)\right\}=1.$$

Proof: We can see from Theorem 7 that for every $\delta>0$ and for sufficiently large T, with probability higher than $1-\epsilon$, in at least $T(\Pr\{\mathcal{F}_{Q_i}\}-\delta)$ time slots, event $\mathcal{F}_{Q_i}$ occurs. Thus, if we design the IRS such that the network matrix is $\mathbf{N}\in\tilde{\mathcal{N}}_{Q_i}^{W^*}$ in these slots, where $W^*=\max_{\tilde{\mathcal{N}}_{Q_i}^W\neq\Phi}W$, $W^*+\frac{(M-W^*)(N-W^*)}{M+N-2W^*-1}$ sum DoFs can be achieved in these slots, see the proof of Theorem 4. Therefore, the total $\sum_{i=1}^{2^{|\mathcal{N}_Q|-1}}(\Pr\{\mathcal{F}_{Q_i}\}-\delta)\max_{\tilde{\mathcal{N}}_{Q_i}^W\neq\Phi}(W+\frac{(M-W)(N-W)}{M+N-2W-1})$ sum DoFs can be achieved with probability higher than $1-\epsilon'$, by time sharing technique. ∎

Corollary 5: Theorem 9 indicates that the approximate sum capacity of a $M\times N$ X-network assisted by a Q-element passive IRS is lower bounded by $D\log(1+\rho)+o(\log(\rho))$, where

$$D=\sum_{i=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\{\mathcal{F}_{Q_i}\}-\delta\right)\max_{\tilde{\mathcal{N}}_{Q_i}^W\neq\Phi}\left(W+\frac{(M-W)(N-W)}{M+N-2W-1}\right).$$

Moreover, Theorem 8 shows that the approximate sum capacity is upper bounded by $D'\log(1+\rho)+o(\log(\rho))$, where

$$D'=\min\left\{\min\{M,N\},\sum_{l=1}^{2^{|\mathcal{N}_Q|-1}}\left(\Pr\{\mathcal{E}_{Q_l}\}+\delta\right)\left(\frac{MN+(N-1)\max_{\mathbf{N}_k^{[l]}\in\tilde{\mathcal{N}}_{Q_l}}|\mathcal{M}_{\mathbf{N}_k^{[l]}}|}{M+N-1}\right)\right\}. \quad (34)$$

Also, the sum DoF tends to $\min(M, N)$ in probability, because we have $\tilde{\mathcal{N}}_{Q_1}^{\min(M,N)} \neq \Phi$ and by [7, Th. 7], we have $\lim_{Q\to\infty} \Pr\{\mathcal{F}_{Q_1}\} = 1$, i.e., the DoF is lower bounded by $\min(M, N)(1 - O(\frac{1}{Q}))$ for sufficiently large Q. Therefore, the approximate sum capacity of the $M \times N$ wireless X-network assisted by a passive IRS is lower bounded by $(\min(M, N) - \epsilon)\log(1 + \rho) + o(\log(\rho)), \forall\epsilon > 0$, by choosing a sufficiently large Q.

V. ACTIVE AND PASSIVE IRSS WITH CORRELATED CHANNEL COEFFICIENTS FOR ELEMENTS

The outer and inner bounds on the DoF region and upper and lower bounds on the sum DoF of the $M \times N$ wireless X-network in the presence of a passive IRS with correlated channel coefficients for elements, are the same as what have been stated in Theorems 6, 7, 8, and 9, respectively, except three following facts:

Fact 1: Our main achievability theorems (Theorems 1 and 7), were based on the independence of channel coefficients.

Fact 2: For the passive IRS with correlated channel coefficients for elements, we have to ensure that $\mathbf{H_N H_N^H}$ is invertible in (25).

Fact 3: The probability measures $\Pr\{\mathcal{F}_{Q_l}\}$ and $\Pr\{\mathcal{E}_{Q_l}\}$ and the slope of their decay may change.

For Fact 1, in the proof of Theorems 1 and 7, using [7, Lemma 2], we stated that if we choose the variables x_k as $H^{[ji]}(t), (i, j) \notin \mathcal{N}$, and $T^{[ji]}(t), (i, j) \in \mathcal{N}$, y_k as $H^{[ji]}(t), (i, j) \in \mathcal{N}$, and z_k as $H_{\text{IR}}^{[ju]}(t), H_{\text{TI}}^{[u'i]}(t), (i, j) \in [1 : M] \times [1 : N], u, u' \in [1 : Q]$, then by [7, Lemmas 1–3], the subspaces $\tilde{\mathcal{A}}_{k,j}, k \neq j$, and $\tilde{\mathcal{C}}_{i,j}, i \in \mathcal{B}_j$, will be full rank and linearly independent almost surely. In the case of passive IRS with correlated channel coefficients for elements, the variables $\mathbf{h}_{\text{TI}}^{[i]}(t)$ and $\mathbf{h}_{\text{IR}}^{[j]}(t)$ are correlated Gaussian random variables for different values of u (note that these vectors are independent for different values of i and j). Without loss of generality, we assume that matrix $\mathbf{R}$ is full rank because if $\mathbf{R}$ is not full rank, then we can choose some elements of $\mathbf{h}_{\text{TI}}^{[i]}(t)$ and $\mathbf{h}_{\text{IR}}^{[j]}(t)$, for which the covariance matrix is full rank and other elements become a linear combination of the mentioned elements. Then, by [34, Th. 3], there exists complex Gaussian vectors $\hat{\mathbf{h}}_{\text{TI}}^{[i]}(t)$ and $\hat{\mathbf{h}}_{\text{IR}}^{[j]}(t)$ with independent elements (the real and imaginary parts are also independent and $E\{\mathbf{h}_{\text{TI}}^{[i]}(t)(\mathbf{h}_{\text{TI}}^{[i]}(t))^H\} = E\{\mathbf{h}_{\text{IR}}^{[j]}(t)(\mathbf{h}_{\text{IR}}^{[j]}(t))^H\} = \mathbf{I}$), for which $\mathbf{h}_{\text{TI}}^{[i]}(t) = \mathbf{C}\sqrt{\mu_{\text{TI}}^{[i]}}\hat{\mathbf{h}}_{\text{TI}}^{[i]}(t)$ and $\mathbf{h}_{\text{IR}}^{[j]}(t) = C\sqrt{\mu_{\text{IR}}^{[j]}}\hat{\mathbf{h}}_{\text{IR}}^{[j]}(t)$, where $\mathbf{R} = \mathbf{C}\mathbf{C}^H$. Therefore, if we consider the elements of $\hat{\mathbf{h}}_{\text{TI}}^{[i]}(t)$ and $\hat{\mathbf{h}}_{\text{IR}}^{[j]}(t)$ as z_k in [7, Lemma 2], this part of proof remains valid. We study Facts 2 and 3 in the following theorem.

Theorem 10: Assume that the imaginary and real parts of the channel coefficients of the direct links are zero mean and their probability distributions have the following properties:

$$\int |h|^n f_{H_r^{[ji']}(t)}(h)dh < \infty, \int |h|^n f_{H_i^{[ji']}(t)}(h)dh < \infty, \quad (35)$$
$$(i', j) \in [1 : M] \times [1 : N], 0 \le n \le 2.$$

where indices r and i denote the real and imaginary parts of the channel coefficients, respectively. In addition, without loss of generality, assume that $\tilde{\mathcal{N}}_{Q_1} = \mathcal{N}_Q$. Then, we have:

$$\lim_{Q\to\infty} \Pr\left\{\det\left(\mathbf{H_N H_N^H}\right) \neq 0\right\} = 1, \quad (36)$$
$$\lim_{Q\to\infty} \Pr\{\mathcal{F}_{Q_1}\} = 1, \quad \lim_{Q\to\infty} \Pr\{\mathcal{F}_{Q_i}\} = 0, i \neq 1, \quad (37)$$

where the order of convergences is at least $O(\frac{1}{\sqrt{Q}})$.

Proof: The proof is provided in Appendix H. ∎

Remark 1: We can see $1 - O(\frac{1}{\sqrt{Q}}) \le \Pr\{\mathcal{E}_{Q_1}\} \le 1$, because $\mathcal{F}_{Q_1} \subseteq \mathcal{E}_{Q_1}$. Hence, we have:

$$\lim_{Q\to\infty} \Pr\{\mathcal{E}_{Q_1}\} = 1, \quad \lim_{Q\to\infty} \Pr\{\mathcal{E}_{Q_i}\} = 0, i \neq 1, \quad (38)$$

where the order of convergence is at least $O(\frac{1}{\sqrt{Q}})$.

Corollary 6: Theorem 10 is applicable to the K-user interference channel studied in [7]. Therefore, using [7, Th. 10], the sum DoF of the K-user interference channel assisted by a passive IRS with correlated channel coefficients for elements will be $K(1 - O(\frac{1}{\sqrt{Q}}))$.

Remark 2: For the $M \times N$ wireless X-network assisted by an active IRS with correlated channel coefficients for the elements, from the proof of [7, Lemma 1], we can see that the assumption of independence of random variables $X_1, \ldots, X_k$ with a continuous cumulative probability distribution can be replaced by the assumption of the continuity of the conditional cumulative probability functions, defined as follows:

$$f_{i,\mathcal{A}}(h) = \Pr\{X_i \le h | \{X_j, \forall j \in \mathcal{A}\} : \mathcal{A} \subseteq \{1, \ldots, k\} - \{i\}\}, \forall\mathcal{A}, i. \quad (39)$$

For $\min\{M, N\} \le 5$ and $Q \le 20$, we can numerically make sure that $\det(\mathbf{R}) \neq 0$ in (4). Therefore, conditions (39) are satisfied and the sum DoF in (13) is achievable. For higher values of M and N, for which $\det(\mathbf{R}) \neq 0$ cannot be guaranteed, the statement (36) of Theorem 10 can be used, which shows an order of at least $O(\frac{1}{\sqrt{Q}})$ for the convergence of sum DoF to $\min(M, N)$, i.e., for $\forall\epsilon, \delta > 0$ and $Q \ge MN - \min\{M, N\}$, there exists a number T' such that for $\forall T > T'$, we have:

$$\Pr\left\{\max \sum_{i=1}^{M}\sum_{j=1}^{N} d_{i,j} \ge \min\{M, N\} \times \left(\Pr\left\{\det\left(\mathbf{H}_{\tilde{\mathbf{N}}}\mathbf{H}_{\tilde{\mathbf{N}}}^H\right) \neq 0\right\} - \delta\right)\right\} > 1 - \epsilon,$$

where

$$\left[\tilde{\mathbf{N}}\right]_{i,j} = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases}.$$

We remind that in this case, we must use the pseudo inverse in (25).

VI. NUMERICAL RESULT

In this section, we present numerical results to quantify the proposed bounds. We have used a path loss model for channel coefficients. All channel coefficients are drawn from a zero-mean complex Gaussian distribution. For the independent channel coefficients scenario, the variance of channel

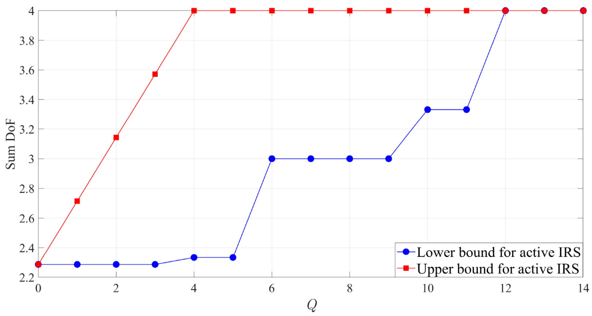

FIGURE 4. Lower and upper bounds on the sum DoF of the 4×4 wireless *X*-network assisted by an active IRS.

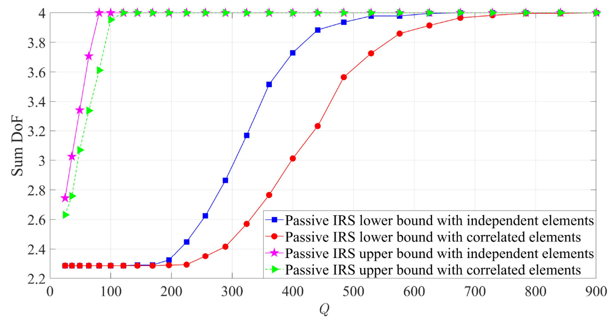

FIGURE 5. The comparison of the lower and upper bounds on the sum DoF of the 4×4 wireless *X*-network assisted by passive IRSs with independent and correlated channel coefficients for elements, where $d_H = d_V = \frac{\lambda}{4}$.

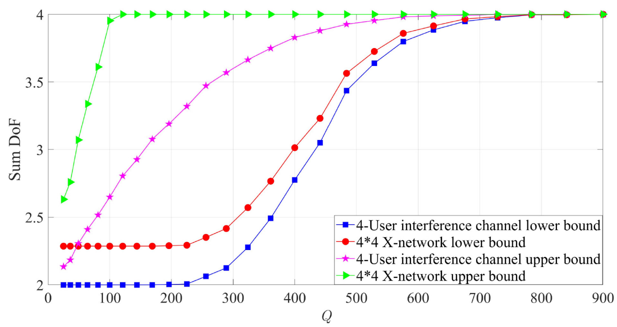

FIGURE 6. The comparison of lower and upper bounds on the sum DoF of the 4-user interference channel and 4×4 wireless *X*-network assisted by a passive IRS with correlated channel coefficients for elements, where $d_H = d_V = \frac{\lambda}{4}$.

coefficients from the transmitters to the IRS and from the IRS to the receivers is $\sigma_1^2 = (\frac{\lambda}{4\pi\rho_1})^2$ and the variance of the direct links between each transmitter and each receiver is $\sigma_2^2 = (\frac{\lambda}{4\pi\rho_2})^2\hat{h}$, where ρ_1 denotes the distance between the IRS and users, ρ_2 represents the distance between each transmitter and each receiver, and $\hat{h}$ characterizes a blockage in the direct links between each transmitter and each receiver. For the correlated channel coefficients scenario, the variance of the direct links between each transmitter and each receiver will change, which will be discussed later. We assumed that the carrier frequency is 5 GHz, i.e., $\lambda \approx 0.06m$, and $\rho_1 = \rho_2 = 5\sqrt{2}m$. In these simulations, we evaluate: 1) the impact of the number of IRS elements Q on the asymptotic behavior of DoF (for independent IRS elements the order of convergence is $O(\frac{1}{Q})$ and for correlated IRS elements the order of convergence is $O(\frac{1}{\sqrt{Q}})$), 2) the gap between the upper and lower bounds for different values of Q, and 3) the impact of distance of the IRS between other nodes.

In Fig. 4, we plot lower and upper bounds on the sum DoF for 4×4 wireless X-network assisted by an active IRS. We note that in this figure, independent and correlated channel coefficients for the elements of the IRS do not change the curves. This figure demonstrates that the proposed lower bound grows stepwise until it approaches the maximum sum DoF, however, the upper bound grows linearly. This behavior follows from the fact that the proposed lower bound does not grow for the values of Q in the interval $[W(N-1) + W(M-W) : (W+1)(N-1) + (W+1)(M-W-1) - 1]$. We note that the sum DoF plotted in this figure does not depend on the value of parameters ρ_1, ρ_2, and $\hat{h}$, because the achievable sum DoF for active IRSs does not depend on the realization of channel coefficients.

In Fig. 5, we present the lower and upper bounds on the sum DoF of 4×4 wireless X-network assisted by a passive IRS with independent and correlated channel coefficients for elements. For the correlated IRS, we consider $d_H = d_V = \frac{\lambda}{4}$. In addition, for the IRS with correlated elements, to have a fair comparison, we assume $\mu_{\text{TI}}^{[i]} = \mu_{\text{IR}}^{[i]} = \frac{d_H d_V}{4\pi\rho^2}$ because the physical surface (upper bound of the effective surface) of each element is equal to $d_H d_V$ and the received power at the IRS position is $\frac{1}{4\pi\rho^2}$. Moreover, we assume $\hat{h} = 10^{-5}$, which shows a considerable blockage in direct links between each transmitter and each receiver. We can see the performance loss between the independent model for the IRS (which is an approximate model for element spacing more than $\frac{\lambda}{2}$) and the correlated model for the IRS (which is a more accurate model for element spacing less than $\frac{\lambda}{2}$ and in this case $d_H = d_V = \frac{\lambda}{4}$). Also, we observe the gap between lower and upper bounds.

As we mentioned in Corollary 2, Theorem 10 can be used for the K-user interference channel assisted by a passive IRS with correlated channel coefficients for IRS elements [7]. In Fig. 6, we compare lower and upper bounds on the sum DoF of the 4-user interference channel and 4×4 wireless X-network assisted by a passive IRS with correlated channel coefficients for elements, where $d_H = d_V = \frac{\lambda}{4}$ and $\hat{h} = 10^{-5}$. In addition, as we mentioned in the previous paragraph, we assume $\mu_{\text{TI}}^{[i]} = \mu_{\text{IR}}^{[i]} = \frac{d_H d_V}{4\pi\rho^2}$. In this figure, we observe that the achievable sum DoF for both systems approaches the maximum value of 4 when the number of elements grows large. In addition, we can see that the achievable sum DoF for the 4×4 wireless X-network is higher than that for the 4-user interference channel.

In Fig. 7, we compare the achievable sum DoF for both independent and correlated IRS elements, for $\rho_{\text{TI}} = \rho_{\text{IR}} = 5\sqrt{2}m$, and $\rho_{\text{TI}} = 5\sqrt{2}m$, $\rho_{\text{IR}} = 3\sqrt{2}m$, where ρ_{TI} and ρ_{IR} are distances between transmitter–IRS and IRS–receiver, respectively. We observe that when the IRS is nearer to the

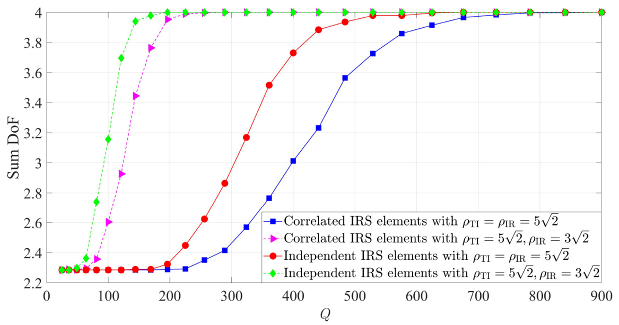

FIGURE 7. The comparison of the achievable sum DoF of the 4 × 4 wireless *X*-network assisted by a passive IRS with both independent and correlated channel coefficients for elements, where $d_H = d_V = \frac{\lambda}{4}$, for different values of ρ_{IR}.

receivers (ρ_{IR} decreases), the achievable sum DoF increases. This phenomenon is symmetric, thus, if ρ_{IR} is constant and ρ_{TI} decreases, the same observation will be seen.

VII. CONCLUSION

In this paper, we studied the DoF region and sum DoF of the time-selective $M \times N$ wireless X-network assisted by active and passive IRSs. We obtained inner and outer bounds on the DoF region and lower and upper bounds on the sum DoF of the $M \times N$ wireless X-network in the presence of active and passive IRSs. For the active IRS case, we proved that by choosing the number of IRS elements more than a certain finite value, the maximum $\min(M, N)$ sum DoFs can be achieved. For the passive IRS case, we proved that by employing a sufficiently large number of elements for the IRS, any value less than $\min(M, N)$ is achievable for the sum DoF. Our future research directions are summarized as follows: 1) finding tighter bounds for both active and passive IRSs, 2) analyzing more physically-motivated models for IRSs, and 3) considering imperfect CSI in DoF analysis.

APPENDIX A

The basis of the proof of this theorem is the achievability proof of [7, Th. 1]. However, interference alignment scheme for the X-network and analysis of the inteplay between interference cancellation and the achieved DoF for each $w^{[ji]}$, is more complicated, which is the subject of this proof. We prove this theorem in five steps.

Step 1 (Message Stream Generation): For each transmitter $i \in [1 : M]$, we provide N vectors of symbol streams $\tilde{\mathbf{x}}^{[ji]} \in \mathbb{C}^{d_{\tilde{\mathbf{x}}^{[ji]}} \times 1}$ for each receiver (we use the notation $d_{\tilde{\mathbf{x}}^{[ji]}}$ because this parameter is unknown in this step and we will determine it in step 5), and $T \times d_{\tilde{\mathbf{x}}^{[ji]}}$ matrix $\tilde{\mathbf{V}}^{[ji]}$ as the beamforming matrix, whose columns are the beamforming vectors corresponding to each element of $\tilde{\mathbf{x}}^{[ji]}$. Therefore, we have:

$$\mathbf{x}^{[i]} = \sum_{j=1}^{N} \tilde{\mathbf{V}}^{[ji]} \tilde{\mathbf{x}}^{[ji]}.$$

Step 2 (Interference Cancellation Method and Channel Equalization): First, we consider the set $\mathcal{N}$ as follows:

$$\mathcal{N} = \left\{(i, j) \middle| i \in [1 : M], j \in [1 : N], n_{i,j} = 0\right\}, \quad (40)$$

then, we design the IRS such that for each $t \in [1 : T]$, we have:

$$\sum_{u \in [1:Q]} H_{\text{TI}}^{[ui]}(t) H_{\text{IR}}^{[ju]}(t) \tau^{[u]}(t) = -H^{[ji]}(t), (i, j) \in \mathcal{N}. \quad (41)$$

This procedure eliminates the links from the i-th transmitter to the j-th receiver, for $(i, j) \in \mathcal{N}$. If we rewrite Eqs. (41) in the matrix form, $\mathbf{H}\boldsymbol{\tau} = \mathbf{h}$, we have $\Pr(\det(\mathbf{H}) = 0) = 0$ by [7, Lemma 1], because $\det(\mathbf{H})$ is a non-zero polynomial in terms of $H_{\text{TI}}^{[ui]}(t)$ and $H_{\text{IR}}^{[ju]}(t)$. Thus, set of equations (41) are solvable almost surely. Therefore, $\tau^{[u]}(t)$ has the following form:

$$\tau^{[u]}(t) = \sum_{(i',j') \in \mathcal{N}} H^{[j'i']}(t) P^{[uj'i']} \times \left(\left\{H_{\text{TI}}^{[u'i'']}(t), H_{\text{IR}}^{[j''u']}(t) : u' \in [1 : Q], (i'', j'') \in \mathcal{N}\right\}\right), \quad (42)$$

where $P^{[uj'i']}(\mathcal{X})$ are fractional polynomials formed by variables $x \in \mathcal{X}$. Thus, the equivalent channel becomes into the following form:

$$\begin{aligned} Y^{[j]}(t) &= \sum_{i \in [1:M]} H^{[ji]}(t) X^{[i]}(t) \\ &\quad + \sum_{i \in [1:M]} \sum_{u \in [1:Q]} X^{[i]}(t) H_{\text{TI}}^{[ui]}(t) H_{\text{IR}}^{[ju]}(t) \tau^{[u]}(t) + Z^{[j]}(t) \\ &= \sum_{i \in [1:M]} \tilde{H}^{[ji]}(t) X^{[i]}(t) + Z^{[j]}(t). \end{aligned}$$

In our interference alignment analysis, we will need the matrix $\tilde{\mathbf{H}}^{[ji]}$, which is defined as follows:

$$\tilde{\mathbf{H}}^{[ji]} = \text{diag}\left(\tilde{H}^{[ji]}(1), \tilde{H}^{[ji]}(2), \ldots, \tilde{H}^{[ji]}(T)\right).$$

Step 3 (Interference Alignment Equations for the j-th Receiver): In this step, we determine the interference alignment equations for each receiver. For the j-th receiver and for the set of transmitters $\mathcal{B}_j = \{i | i \in [1 : M], n_{i,j} = 1.\}$, we have the following interference alignment equations:

$$\forall k \neq j \rightarrow \text{span}\left(\tilde{\mathbf{H}}^{[ji]} \tilde{\mathbf{V}}^{[ki]}\right) \subseteq \tilde{\mathcal{A}}_{k,j}, \forall i \in \mathcal{B}_j, \quad (43)$$

where $\tilde{\mathcal{A}}_{k,j}$ is a subspace, for which we have:

$$\max_{i \in \mathcal{B}_j} D_N\left(\text{span}\left(\tilde{\mathbf{H}}^{[ji]} \tilde{\mathbf{V}}^{[ki]}\right)\right) = D_N\left(\tilde{\mathcal{A}}_{k,j}\right). \quad (44)$$

We also define the message subspaces as $\tilde{\mathcal{C}}_{i,j} = \text{span}(\tilde{\mathbf{H}}^{[ji]} \tilde{\mathbf{V}}^{[ji]})$ and we want subspaces $\tilde{\mathcal{C}}_{i,j}$ and $\tilde{\mathcal{A}}_{k,j}, \forall k \neq j, \forall i \in \mathcal{B}_j$, to be full rank and linearly independent. Therefore, we can ensure that the message streams $\tilde{\mathbf{x}}^{[ji]}, \forall i \in \mathcal{B}_j$ can be decoded by zero forcing at the j-th receiver.

Step 4 (Beamforming Matrix Design): The beamforming matrix $\tilde{\mathbf{V}}^{[ji]}$, which corresponds to the symbol stream $\tilde{\mathbf{x}}^{[ji]}$, is designed as follows:

$$\tilde{\mathbf{V}}^{[ji]} = \left\{ \left[\prod_{(i',j')\in\tilde{\mathcal{S}}_j} \left(\tilde{\tilde{\mathbf{H}}}^{[j'i']} \right)^{\alpha_{j'i'}} \right] \mathbf{w} : \alpha_{j'i'} \in \left[1 : t_{i,j}n\right] \right\}, \quad (45)$$

where

$$\begin{aligned} \tilde{\tilde{\mathbf{H}}}^{[ji]} &= \begin{cases} \tilde{\mathbf{H}}^{[ji]}, n_{i,j} = 1 \\ \mathbf{T}^{[ji]}, n_{i,j} = 0 \end{cases}, \\ \tilde{\mathcal{S}}_j &= \left\{ \left(i', j'\right) \middle| i' \in [1:M],\ j' \in [1:N], j' \neq j \right\}, \quad (46) \\ \mathbf{w} &= \left[1 \ \cdots \ 1\right]^H. \end{aligned}$$

Moreover, $\mathbf{T}^{[ji]}$ are independent diagonal matrices with independent diagonal entries drawn from a continuous cumulative probability distribution. $n \in \mathbb{N}$ is an auxiliary variable, which can go to infinity and $t_{i,j}$ is a parameter, which controls the dimension of $\tilde{\mathbf{V}}^{[ji]}$, i.e., $d(\tilde{\mathbf{V}}^{[ji]})$. Note that if we have $n_{i,j} = 0$ for the pair of (i, j), then we must have $t_{i,j} = 0$, because there would not be any link between that pair of transmitter and receiver. Equation (45) indicates that any value of set $[1 : t_{i,j}n]$ can be assumed for $\alpha_{j'i'}$, thus, the number of columns of $\tilde{\mathbf{V}}^{[ji]}$ will be $(t_{i,j}n)^{MN-M}$.

Step 5 (Satisfaction of the Interference Alignment Equations, Decodability of Message Symbols and DoF Analysis: We derive the message subspace $\tilde{\mathcal{C}}_{i,j}, i \in \mathcal{B}_j$ and the interference subspaces $\tilde{\mathcal{A}}_{k,j}, k \neq j$ as:

$$\tilde{\mathcal{C}}_{i,j} = \text{span}\left\{ \left[\prod_{(i',j')\in\tilde{\mathcal{S}}^{\mathcal{C}}} \left(\tilde{\tilde{\mathbf{H}}}^{[j'i']} \right)^{\alpha_{j'i'}} \right] \mathbf{w} : \alpha_{j'i'} \in \tilde{\mathcal{S}}^{\mathcal{C}}_{j'i'ji} \right\}, \quad (47)$$

$$\tilde{\mathcal{A}}_{k,j} = \text{span}\left\{ \left[\prod_{(i',j')\in\tilde{\mathcal{S}}_j} \left(\tilde{\tilde{\mathbf{H}}}^{[j'i']} \right)^{\alpha_{j'i'}} \right] \mathbf{w} : \alpha_{j'i'} \in \tilde{\mathcal{S}}^{\mathcal{A}}_{j'i'jk} \right\}, \quad (48)$$

where $\tilde{\mathcal{S}}_j$ is given by (46), and sets $\tilde{\mathcal{S}}^{\mathcal{C}}$, $\tilde{\mathcal{S}}^{\mathcal{C}}_{j'i'ji}$, and $\tilde{\mathcal{S}}^{\mathcal{A}}_{j'i'jk}$ are given as follows:

$$\tilde{\mathcal{S}}^{\mathcal{C}} = [1:M] \times [1:N], \quad \tilde{\mathcal{S}}^{\mathcal{C}}_{j'i'ji} = \begin{cases} [1 : t_{i,j}n],\ j' \neq j \\ \{0\}, j' = j,\ i' \neq i \\ \{1\}, j' = j,\ i' = i \end{cases}, \quad (49)$$

$$\tilde{\mathcal{S}}^{\mathcal{A}}_{j'i'jk} = \left[1 : \left(\max_{i'', n_{i'',j}=1} t_{i'',k} \right) n + 1 \right]. \quad (50)$$

Considering $\tilde{\mathcal{C}}_{i,j}, i \in \mathcal{B}_j$ in (47) and $\tilde{\mathcal{A}}_{k,j}, k \neq j$ in (48), from the statement of [7, Lemma 2], we can see that if we choose the variables x_k as $H^{[ji]}(t), (i, j) \notin \mathcal{N}$, and $T^{[ji]}(t), (i, j) \in \mathcal{N}$, y_k as $H^{[ji]}(t), (i, j) \in \mathcal{N}$, and z_k as $H^{[ju]}_{\text{IR}}(t), H^{[u'i]}_{\text{TI}}(t), i, j \in [1 : M] \times [1 : N], u, u' \in [1 : Q]$, then by [7, Lemmas 1–3], the subspaces $\tilde{\mathcal{A}}_{k,j}, k \neq j$, and $\tilde{\mathcal{C}}_{i,j}, i \in \mathcal{B}_j$, will be full rank and linearly independent almost surely because if we put the column vectors of $\tilde{\mathcal{C}}_{i,j}, i \in \mathcal{B}_j$, and $\tilde{\mathcal{A}}_{k,j}, k \neq j$, into a matrix and construct a square matrix by eliminating some of its rows, then, by [7, Lemmas 2–3], its determinant will be a non-zero polynomial constructed by independent random variables and by [7, Lemmas 1], its determinant will be non-zero almost surely. Note that, in the first step, we assumed that the parameter T is sufficiently large. In this step, we determine the value of T. For more clarity, we review [7, Lemmas 1–3] as follows.

Reference [7, Lemma 1]: Consider k independent random variables $X_1, \ldots, X_k$, each constructed from a continuous cumulative probability distribution. The probability of the event that a nonzero polynomial $P_k(X_1, \ldots, X_k)$ constructed from $X_1, \ldots, X_k$ with finite degree assumes the value zero is zero, i.e., $\Pr\{P_k(X_1, \ldots, X_k) = 0\} = 0$.

Reference [7, Lemma 2]: Consider three sets of variables $\{x_i, i \in \mathcal{A}_x, |\mathcal{A}_x| < \infty\}$, $\{y_i, i \in \mathcal{A}_y, |\mathcal{A}_y| < \infty\}$, and $\{z_i, i \in \mathcal{A}_z, |\mathcal{A}_z| < \infty\}$. Consider the following functions:

$$f_j = \prod_{i=1}^{|\mathcal{A}_x|} \left(x_i + \sum_{i' \in \mathcal{C}_j, i'' \in \mathcal{D}_j} x_{i'} y_{i''} P_1{}^{[i'i''j]}(z_k : k \in \mathcal{A}_z) + y_{i''} P_2{}^{[i'i''j]}(z_k : k \in \mathcal{A}_z) \right)^{a_i^j}, \quad (51)$$

$$\left(a_1^j, \ldots, a_{|\mathcal{A}_x|}^j\right) \in \mathbb{W}^{|\mathcal{A}_x|}, j \in \{1, \ldots, J\},$$

where $P_1^{[i'i''j]}(\cdot)$ and $P_2^{[i'i''j]}(\cdot)$ are fractional polynomials and for $\forall j$, we have $|\mathcal{C}_j|, |\mathcal{D}_j| < \infty$. If for $\forall j, j'$ with $j \neq j'$, $(a_1^j, \ldots, a_{|\mathcal{A}_x|}^j) \neq (a_1^{j'}, \ldots, a_{|\mathcal{A}_x|}^{j'})$, then the functions f_j will be linearly independent.

Reference [7, Lemma 3]: Consider the set of nonzero linearly independent fractional polynomials $\{P^{[j]}(\cdot), j \in \{1, \ldots, J\}\}$ and consider J sets of variables $\mathcal{X}_j = \{x_i^j : i \in \mathcal{I}, \mathcal{I} \subseteq \mathbb{N}, |\mathcal{I}| < \infty\}, j \in \{1, \ldots, J\}$. The determinant of the following matrix will be a nonzero fractional polynomial:

$$\mathbf{A} = \begin{bmatrix} P^{[1]}(\mathcal{X}_1) & P^{[2]}(\mathcal{X}_1) & \cdots & P^{[J]}(\mathcal{X}_1) \\ P^{[1]}(\mathcal{X}_2) & P^{[2]}(\mathcal{X}_2) & \cdots & P^{[J]}(\mathcal{X}_2) \\ \vdots & \vdots & \ddots & \vdots \\ P^{[1]}(\mathcal{X}_J) & P^{[2]}(\mathcal{X}_J) & \cdots & P^{[J]}(\mathcal{X}_J) \end{bmatrix}. \quad (52)$$

Finaly, we analyze the dimensions of the message and interference subspaces. Hence, for subspaces $\tilde{\mathcal{C}}_{i,j}, i \in \mathcal{B}_j$ and $\tilde{\mathcal{A}}_{k,j}$ at the j-th receiver, we have:

$$\begin{aligned} d\left(\tilde{\mathcal{C}}_{i,j}\right) &= \left(t_{i,j}n\right)^{MN-M}, \\ d\left(\tilde{\mathcal{A}}_{k,j}\right) &= \left(\left(\max_{i'', n_{i'',j}=1} t_{i'',k} \right) n + 1 \right)^{MN-M}, \end{aligned} \quad (53)$$

From the definition of normalized asymptotic dimension, we have $l = MN - M$. Thus, the normalized asymptotic dimension of $\tilde{\mathcal{C}}_{i,j}$ and $\tilde{\mathcal{A}}_{k,j}$ are:

$$\begin{aligned} D_N\left(\tilde{\mathcal{C}}_{i,j}\right) &= t_j^{MN-M}, \\ D_N\left(\tilde{\mathcal{A}}_{i,j}\right) &= \left(\max_{i'', n_{i'',j}=1} t_{i'',k} \right)^{MN-M} = \max_{i'', n_{i'',j}=1} \left(t_{i'',k}{}^{MN-M} \right). \end{aligned} \quad (54)$$

Now, we consider T as following:

$$T = (n+1)^{MN-M}, \quad \lim_{n\to\infty} \frac{T}{n^{MN-M}} = 1. \tag{55}$$

By (55), for interference alignment equations (43) and (44) to be satisfied, we must have the following conditions for the j-th receiver:

$$\sum_{i\in\mathcal{B}_j} D_N\left(\tilde{\mathcal{C}}_{i,j}\right) + \sum_{k\neq j} D_N\left(\tilde{\mathcal{A}}_{k,j}\right) = \sum_{i\in\mathcal{B}_j} t_{i,j}{}^{MN-M} + \sum_{k\neq j} \max_{i'',n_{i'',j}=1} \left(t_{i'',k}{}^{MN-M}\right) \leq 1, \tag{56}$$

$$D_N\left(\tilde{\mathcal{C}}_{i,j}\right) = t_{i,j}{}^{MN-M} \leq 1, \quad D_N\left(\tilde{\mathcal{A}}_{k,j}\right) = \max_{i'',n_{i'',j}=1} \left(t_{i'',k}{}^{MN-M}\right) \leq 1. \tag{57}$$

In addition, (55) concludes that the DoF achieved from the i-th transmitter for the j-th receiver will be:

$$d_{i,j} = \frac{D_N\left(\tilde{\mathcal{C}}_{i,j}\right)}{1} = t_{i,j}{}^{MN-M}. \tag{58}$$

Therefore, by Eqs. (56)-(58), for each $j \in \{1, \ldots, N\}$, we obtain region (7).

APPENDIX B

By Fano's inequality [35, Th. 2.10.1], we have $r_{i,j} \leq \frac{1}{T} I(w^{[ji]}; \mathbf{y}^{[j]}) + \epsilon$. Thus, we obtain:

$$\begin{aligned}
\sum_{i'=1}^{M} r_{i',j} &\leq \frac{1}{T}\sum_{i'=1}^{M} I\left(w^{[ji']}; \mathbf{y}^{[j]}\right) + \epsilon \\
&\leq \frac{1}{T}\sum_{i'=1}^{M} I\left(w^{[ji']}; \mathbf{y}^{[j]} \middle| \left\{w^{[ji'']} \middle| i'' \in \left[1 : i'-1\right]\right\}\right) + \epsilon \\
&= \frac{1}{T} I\left(\left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}; \mathbf{y}^{[j]}\right) + \epsilon \\
&\leq \frac{1}{T} I\left(\left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}; \mathbf{y}^{[j]} \middle| \mathcal{W}_{-i,-j}\right) + \epsilon
\end{aligned} \tag{59}$$

which follows from the independence of messages $w^{[ji]}$ and we have:

$$\mathcal{W}_{-i,-j} = \left\{w^{[j''i'']} \middle| i'' \neq i, j'' \neq j\right\}. \tag{60}$$

On the other hand, we have:

$$\begin{aligned}
\sum_{j'\neq j} r_{i,j'} &\leq \frac{1}{T}\sum_{j'\neq j} I\left(w^{[j'i]}; \mathbf{y}^{[j']}\right) + \epsilon \\
&\leq \frac{1}{T}\sum_{j'\neq j} I\left(w^{[j'i]}; \mathbf{y}^{[j']}, \mathbf{y}^{[j]}\right) + \epsilon \\
&\leq \frac{1}{T}\sum_{j'\neq j} I\left(w^{[j'i]}; \mathbf{y}^{[j']}, \mathbf{y}^{[j]} \middle| \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}, \left\{w^{[j''i]} \middle| j'' \in \left[1 : j'\right], j'' \neq j\right\}, \mathcal{W}_{-i,-j}\right) + \epsilon \qquad (61) \\
&= \frac{1}{T}\sum_{j'\neq j} I\left(w^{[j'i]}; \mathbf{y}^{[j]} \middle| \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}, \left\{w^{[j''i]} \middle| j'' \in \left[1 : j'\right], j'' \neq j\right\}, \mathcal{W}_{-i,-j}\right) \\
&\quad + \frac{1}{T}\sum_{j'\neq j} I\left(w^{[j'i]}; \mathbf{y}^{[j']} \middle| \mathbf{y}^{[j]}, \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}, \left\{w^{[j''i]} \middle| j'' \in \left[1 : j'\right], j'' \neq j\right\}, \mathcal{W}_{-i,-j}\right) + \epsilon \\
&= \frac{1}{T} I\left(\left\{w^{[j''i]} \middle| j'' \in [1 : N], j'' \neq j\right\}; \mathbf{y}^{[j]} \middle| \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}, \mathcal{W}_{-i,-j}\right) \\
&\quad + \frac{1}{T}\sum_{j'\neq j} I\left(w^{[j'i]}; \mathbf{y}^{[j']} \middle| \mathbf{y}^{[j]}, \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}, \left\{w^{[j''i]} \middle| j'' \in \left[1 : j'\right], j'' \neq j\right\}, \mathcal{W}_{-i,-j}\right) + \epsilon, \qquad (62)
\end{aligned}$$

where (61) and (62) follows from the independence of messages $w^{[ji]}$. Therefore, by combining (60) and (62), we will have:

$$\begin{aligned}
\sum_{i'=1}^{M} r_{i',j} + \sum_{j'\neq j} r_{i,j'} &= \frac{1}{T} I\left(\left\{w^{[j''i]} \middle| j'' \in [1 : N], j'' \neq j\right\}, \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}; \mathbf{y}^{[j]} \middle| \mathcal{W}_{-i,-j}\right) \\
&\quad + \frac{1}{T}\sum_{j'\neq j} I\left(w^{[j'i]}; \mathbf{y}^{[j']} \middle| \mathbf{y}^{[j]}, \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}, \left\{w^{[j''i]} \middle| j'' \in \left[1 : j'\right], j'' \neq j\right\}, \mathcal{W}_{-i,-j}\right) + \epsilon \\
&= \frac{1}{T} I\left(\left\{w^{[j''i]} \middle| j'' \in [1 : N], j'' \neq j\right\}, \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}; \mathbf{y}^{[j]} \middle| \mathcal{W}_{-i,-j}\right) \\
&\quad + \frac{1}{T}\sum_{j'\neq j} I\left(w^{[j'i]}; \tilde{\mathbf{H}}^{[j'i]}\mathbf{x}^{[i]} + \mathbf{z}^{[j']} \middle| \tilde{\mathbf{H}}^{[ji]}\mathbf{x}^{[i]} + \mathbf{z}^{[j]}, \left\{w^{[ji'']} \middle| i'' \in [1 : M]\right\}, \left\{w^{[j''i]} \middle| j'' \in \left[1 : j'\right], j'' \neq j\right\}, \mathcal{W}_{-i,-j}\right) + \epsilon \qquad (63) \\
&\leq \log(\rho)\left(1 + \frac{\sum_{j'\neq j}\left|\left\{t \middle| \tilde{H}^{[j'i]}(t) \neq 0, \tilde{H}^{[ji]}(t) = 0\right\}\right|}{T}\right) + o(\log(\rho)) \\
&= \log(\rho)\left(1 + \frac{\sum_{t=1}^{T}\sum_{j'\neq j} n_{i,j'}(t)\left(1 - n_{i,j}(t)\right)}{T}\right) + o(\log(\rho)), \qquad (64)
\end{aligned}$$

where $n_{i,j}(t)$ is the element of the i-th row and the j-th column of the network matrix $\mathbf{N}$ in the t-th time slot, $\tilde{\mathbf{H}}^{[ji]}$ is the diagonal matrix of equivalent channel coefficients in the presence of the IRS, and (63) follows from the fact that conditioned on $\{w^{[ji'']}|i'' \in [1 : M]\}$, $\{w^{[j''i]}|j'' \in [1 : j'], j'' \neq j\}$, and $\mathcal{W}_{-i,-j}$, variables $\mathbf{x}^{[i']}, i' \neq i$ will be determined and equivalent channel coefficients are known. Now, if we assume that each network matrix $\mathbf{N}_k \in \mathcal{N}_Q$ occurs in Ta_k time slots, by (64), we will have:

$$\sum_{i'=1}^{M} r_{i',j} + \sum_{j'\neq j} r_{i,j'} \leq \log(\rho) + \log(\rho) \times \sum_{k=1}^{|\mathcal{N}_Q|}\left[a_k\left(1 - [n_k]_{i,j}\right)\left(\sum_{j''\neq j} [n_k]_{i,j''}\right)\right] + o(\log(\rho)). \tag{65}$$

Other inequalities, which follow from the independence of messages $w^{[ji]}$ can be derived as:

$$\begin{aligned} r_{i,j} &\le \frac{1}{T} I\left(w^{[ji]}; \mathbf{y}^{[j]}\right) + \epsilon \\ &\le \frac{1}{T} I\left(w^{[ji]}; \mathbf{y}^{[j]} \middle| \left\{ w^{[j'i']} \middle| (i', j') \neq (i,j) \right\}\right) + \epsilon \qquad (66) \\ &\le \frac{1}{T} I\left(w^{[ji]}; \tilde{\mathbf{H}}^{[ji]}\mathbf{x}^{[i]} + \mathbf{z}^{[j]} \middle| \left\{ w^{[j'i']} \middle| (i', j') \neq (i,j) \right\}\right) + \epsilon \\ &\le \frac{1}{T} I\left(\left\{ w^{[j'i']} \middle| i' \in [1:M], j' \in [1:N] \right\}; \tilde{\mathbf{H}}^{[ji]}\mathbf{x}^{[i]} + \mathbf{z}^{[j]}\right) + \epsilon \\ &= \frac{1}{T} I\left(\left\{ \mathbf{x}^{[i']} \middle| i' \in [1:M] \right\}; \tilde{\mathbf{H}}^{[ji]}\mathbf{x}^{[i]} + \mathbf{z}^{[j]}\right) + \epsilon \\ &\le \sum_{k:[n_k]_{i,j}=1} a_k \log(\rho) + o(\log(\rho)), \qquad (67) \end{aligned}$$

APPENDIX C

By Theorems 1 and 2, the following DoF matrix is achievable, which results $W + \frac{(M-W)(N-W)}{M+N-2W-1}$ sum DoFs:

$$\begin{cases} d_{i,i} = 1, i \in [1:W] \\ d_{i,j} = \frac{1}{M+N-2W-1}, i \in [W+1:M], j \in [W+1:N] \\ d_{i,j} = 0, \{i \in [1:W], j \neq i\} \text{ or } \{i \in [W+1:M], j \in [1:W]\}, \end{cases} \qquad (68)$$

APPENDIX D

For the first term of (14), by Theorem 3, we have:

$$\begin{aligned} &\sum_{i,j} \left(\sum_{i'=1}^{M} d_{i',j} + \sum_{j' \neq j} d_{i,j'} \right) \\ &= (M+N-1) \sum_{i,j} d_{i,j} \\ &\le \sum_{i,j} \sum_{k=1}^{|N_Q|} \left[a_k \left(1 - [n_k]_{i,j}\right) \left(\sum_{j'' \neq j} [n_k]_{i,j''} \right) \right] \\ &\le \sum_{k=1}^{|N_Q|} a_k \sum_{i,j} \left[1 + \left(1 - [n_k]_{i,j}\right) \left(\sum_{j'' \neq j} [n_k]_{i,j''} \right) \right] \\ &\le \max \sum_{i,j} \left[1 + \left(1 - [n_k]_{i,j}\right) \left(\sum_{j'' \neq j} [n_k]_{i,j''} \right) \right] \\ &\le \sum_{i,j} \left[1 + (N-1)\left(1 - [n_k]_{i,j}\right) \right] \\ &\le MN^2 - (N-1)(MN-Q) = MN + (N-1)Q. \qquad (69) \end{aligned}$$

The second term of (14) is obvious because the sum DoF cannot be more than the sum DoF of the $M \times N$ MIMO channel.

APPENDIX E

Let X be a discrete random variable with possible events $\mathcal{E}_{Q_i}, i \in [1 : 2^{|\mathcal{N}_Q|-1}]$ and let X^T be T i.i.d. realizations of X. We denote $\pi(\mathcal{E}_{Q_i}|X^T)$ as the fraction of T, in which event $\mathcal{E}_{Q_i}$ occurs. By the law of large numbers, for each event $\mathcal{E}_{Q_i}$ and for each $\delta > 0$, there exists a sequence $\epsilon(\delta, T)$ such that:

$$\Pr\left\{ \left| \pi\left(\mathcal{E}_{Q_i} \middle| X^T\right) - \Pr\left\{\mathcal{E}_{Q_i}\right\} \right| > \delta \right\} < \epsilon(\delta, T), \qquad (70)$$

where $\forall \delta > 0 \rightarrow \lim_{T \to \infty} \epsilon(\delta, T) = 0$. Note that inequalities (65) and (67) obtained in the proof of Theorem 3 are valid for both active and passive IRSs. The only difference is that for passive IRSs the realizable network matrices in inequalities (65) and (67) are constrained and depends on the realization of channel coefficients. Hence, the region (21) will be an outer bound for the DoF region because in at most $\Pr\{\mathcal{E}_{Q_i}\} + \delta$ time slots, event $\mathcal{E}_{Q_i}$ occurs for each $i \in [1 : 2^{|\mathcal{N}_Q|-1}]$ with a probability higher than $1 - \epsilon$ for a sufficiently large T (by (70)).

APPENDIX F

The proof of this theorem is similar to the proof of Theorem 6. Let X be a discrete random variable with possible events $\mathcal{F}_{Q_i}, i \in [1 : 2^{|\mathcal{N}_Q|-1}]$ and let X^T be T i.i.d. realizations of X. By the law of large numbers, for each event $\mathcal{F}_{Q_i}$ and for each $\delta > 0$, there exists a sequence $\epsilon(\delta, T)$ such that:

$$\Pr\left\{ \left| \pi\left(\mathcal{F}_{Q_i} \middle| X^T\right) - \Pr\left\{\mathcal{F}_{Q_i}\right\} \right| > \delta \right\} < \epsilon(\delta, T), \qquad (71)$$

where $\lim_{T\to\infty}\epsilon(\delta, T) = 0, \forall \delta > 0$. We also have the following lemma.

Lemma 1: In time slots, where $\mathcal{F}_{Q_i}$ occurs, the DoF region (27) can be achieved.

Proof: Proof of this lemma is the same as proof of Theorems 1 and 2, except that we must use pseudo inverse instead of regular inverse in Eqs. (41), but this will not change the arguments made in the proof. ∎

By Lemma 1 and inequality (71), for sufficiently large T and with probability higher than $1-\epsilon$, each event $\mathcal{F}_{Q_i}$ occurs in at least $T(\Pr\{\mathcal{F}_{Q_i}\} - \delta)$ time slots. Therefore, region (28) can be achieved with probability higher than $1 - \epsilon'$. This completes the proof.

APPENDIX G

The first term of upper bound given in (32) is obvious, thus, we prove the second term. From Theorem 6, we have:

$$\begin{aligned} &\sum_{i,j} \left(\sum_{i'=1}^{M} d_{i',j}(T) + \sum_{j' \neq j} d_{i,j'}(T) \right) \\ &= (M+N-1) \sum_{i,j} d_{i,j}(T) \\ &\le \sum_{i,j} \sum_{l=1}^{2^{|\mathcal{N}_Q|-1}} \left(\Pr\left\{\mathcal{E}_{Q_l}\right\} + \delta\right) \\ &\quad \times \left(1 + \sum_{k=1}^{\left|\tilde{N}_{Q_l}\right|} a_k^{[l]} \left(1 - \left[n_k^{[l]}\right]_{i,j}\right) \left(\sum_{j'' \neq j} \left[n_k^{[l]}\right]_{i,j''} \right) \right) \end{aligned}$$

$$\leq \sum_{l=1}^{2^{|\mathcal{N}_Q|-1}} \left(\Pr\{\mathcal{E}_{Q_l}\}+\delta\right) \times \sum_{i,j}\left(1+\sum_{k=1}^{\left|\tilde{\mathcal{N}}_{Q_l}\right|} a_k^{[l]}\left(1-\left[n_k^{[l]}\right]_{i,j}\right)(N-1)\right)$$

$$\leq \sum_{l=1}^{2^{|\mathcal{N}_Q|-1}} \left(\Pr\{\mathcal{E}_{Q_l}\}+\delta\right)\left(MN+(N-1)\max_{\mathbf{N}_k^{[l]}\in\tilde{\mathcal{N}}_{Q_l}}|\mathcal{M}_{\mathbf{N}_k^{[l]}}|\right).$$

APPENDIX H

All steps of the proof of are the same as [7, Appendix H], except [7, Lemma 5], which can be modified as the following lemma.

Lemma 2: Assume network matrix $\mathbf{N}_i^*$ such that $\mathbf{N}_i^* \in \mathcal{N}_Q, \mathbf{N}_i^* \notin \tilde{\mathcal{N}}_{Q_i}, i \neq 1$. We rewrite (24) for $\mathbf{N}_i^*$ in matrix form $\mathbf{H}_{\mathbf{N}_i^*}\tau_{\mathbf{N}_i^*} = \mathbf{h}_{\mathbf{N}_i^*}$. If we assume $Q > MN$ and define $\mathbf{L} = \mathbf{H}_{\mathbf{N}_i^*}\mathbf{H}_{\mathbf{N}_i^*}^H$, then we obtain:

$$\Pr\left\{\frac{1}{Q}|l_{n,n}-\sum_{u=1}^{Q}E\left\{|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)|^2\right\}|>\delta\right\}<\varepsilon_1(Q),$$

$$\Pr\left\{\frac{1}{Q}\left|l_{n,m}\right|>\delta\right\}<\varepsilon_2(Q), n\neq m,$$

where $\lim_{Q\to\infty}\varepsilon_1(Q) = \lim_{Q\to\infty}\varepsilon_2(Q) = 0$, and $[\,H_{\mathrm{TI}}^{[1i_n]}(t)H_{\mathrm{IR}}^{[j_n1]}(t)\cdots H_{\mathrm{TI}}^{[Qi_n]}(t)H_{\mathrm{IR}}^{[j_nQ]}(t)\,]$ is the n-th row of $\mathbf{H}_{\mathbf{N}_i^*}$. The order of convergence of $\varepsilon_1(Q)$ and $\varepsilon_2(Q)$ is at least $O(\frac{1}{\sqrt{Q}})$.

Proof: By Markov inequality, we have:

$$\begin{aligned}
&\Pr\left\{\frac{1}{Q}\left|l_{n,n}-\sum_{u=1}^{Q}E\left\{\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2\right\}\right|>\delta\right\}\\
&=\Pr\left\{\frac{1}{Q^2}\left(l_{n,n}-\sum_{u=1}^{Q}E\left\{\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2\right\}\right)^2>\delta^2\right\}\\
&\leq\frac{E\left\{\left(l_{n,n}-\sum_{u=1}^{Q}E\left\{\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2\right\}\right)^2\right\}}{Q^2\delta^2}
\end{aligned}\tag{72}$$

To bound this expression, by [34, Th. 3], we have:

$$\begin{bmatrix}H_{\mathrm{TI}}^{[ui_n]}(t)\\H_{\mathrm{TI}}^{[u'i_n]}(t)\end{bmatrix}=\begin{bmatrix}\sqrt{\frac{\mu_{\mathrm{TI}}^{[i_n]}(1+[\mathbf{R}]_{u,u'})}{2}} & \sqrt{\frac{\mu_{\mathrm{TI}}^{[i_n]}(1-[\mathbf{R}]_{u,u'})}{2}}\\ \sqrt{\frac{\mu_{\mathrm{TI}}^{[i_n]}(1+[\mathbf{R}]_{u,u'})}{2}} & -\sqrt{\frac{\mu_{\mathrm{TI}}^{[i_n]}(1-[\mathbf{R}]_{u,u'})}{2}}\end{bmatrix}\times\begin{bmatrix}\eta_{\mathrm{r}}^{[ui_n]}(t)+\eta_i^{[ui_n]}(t)\sqrt{-1}\\ \eta_{\mathrm{r}}^{[u'i_n]}(t)+\eta_i^{[u'i_n]}(t)\sqrt{-1}\end{bmatrix},\tag{73}$$

$$\begin{bmatrix}H_{\mathrm{IR}}^{[j_nu]}(t)\\H_{\mathrm{IR}}^{[j_nu']}(t)\end{bmatrix}=\begin{bmatrix}\sqrt{\frac{\mu_{\mathrm{IR}}^{[j_n]}(1+[\mathbf{R}]_{u,u'})}{2}} & \sqrt{\frac{\mu_{\mathrm{IR}}^{[j_n]}(1-[\mathbf{R}]_{u,u'})}{2}}\\ \sqrt{\frac{\mu_{\mathrm{IR}}^{[j_n]}(1+[\mathbf{R}]_{u,u'})}{2}} & -\sqrt{\frac{\mu_{\mathrm{IR}}^{[j_n]}(1-[\mathbf{R}]_{u,u'})}{2}}\end{bmatrix}\times\begin{bmatrix}\xi_{\mathrm{r}}^{[j_nu]}(t)+\xi_i^{[j_nu]}(t)\sqrt{-1}\\ \xi_{\mathrm{r}}^{[j_nu']}(t)+\xi_i^{[j_nu']}(t)\sqrt{-1}\end{bmatrix},\tag{74}$$

where $\eta_{\mathrm{r}}^{[ui_n]}(t)$, $\eta_i^{[ui_n]}(t)$, $\eta_{\mathrm{r}}^{[u'i_n]}(t)$, $\eta_i^{[u'i_n]}(t)$, $\xi_{\mathrm{r}}^{[j_nu]}(t)$, $\xi_i^{[j_nu]}(t)$, $\xi_{\mathrm{r}}^{[j_nu']}(t)$, and $\xi_i^{[j_nu']}(t)$ are real-valued independent zero-mean Gaussian random variables with variance equal to $\frac{1}{2}$ and $\mathbf{R}$ is given by (4). Using (73) and (74), we have:

$$E\left\{\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2\right\}=E\left\{\left|H_{\mathrm{TI}}^{[u'i_n]}(t)H_{\mathrm{IR}}^{[j_nu']}(t)\right|^2\right\}=\mu_{\mathrm{TI}}^{[i_n]}\mu_{\mathrm{IR}}^{[j_n]},$$

thus, for $u\neq u'$, we obtain

$$\begin{aligned}
&E\left\{\left(\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2-E\left\{\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2\right\}\right)\right.\\
&\left.\times\left(\left|H_{\mathrm{TI}}^{[u'i_n]}(t)H_{\mathrm{IR}}^{[j_nu']}(t)\right|^2-E\left\{\left|H_{\mathrm{TI}}^{[u'i_n]}(t)H_{\mathrm{IR}}^{[j_nu']}(t)\right|^2\right\}\right)\right\}\\
&=E\left\{\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2\left|H_{\mathrm{TI}}^{[u'i_n]}(t)H_{\mathrm{IR}}^{[j_nu']}(t)\right|^2\right\}-\left(\mu_{\mathrm{TI}}^{[i_n]}\mu_{\mathrm{IR}}^{[j_n]}\right)^2\\
&=\left(\mu_{\mathrm{TI}}^{[i_n]}\mu_{\mathrm{IR}}^{[j_n]}\right)^2\left(2([\mathbf{R}]_{u,u'})^2+([\mathbf{R}]_{u,u'})^4\right).
\end{aligned}\tag{75}$$

Using (72), (75) can be bounded as follows:

$$\begin{aligned}
&\frac{E\left\{\left(l_{n,n}-\sum_{u=1}^{Q}E\left\{\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2\right\}\right)^2\right\}}{Q^2\delta^2}=\frac{1}{Q^2\delta^2}\\
&\times E\left\{\sum_{u'=1}^{Q}\sum_{u=1}^{Q}\left(\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2-E\left\{\left|H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\right|^2\right\}\right)\right.\\
&\left.\times\left(\left|H_{\mathrm{TI}}^{[u'i_n]}(t)H_{\mathrm{IR}}^{[j_nu']}(t)\right|^2-E\left\{\left|H_{\mathrm{TI}}^{[u'i_n]}(t)H_{\mathrm{IR}}^{[j_nu']}(t)\right|^2\right\}\right)\right\}\\
&\leq\frac{3Q\sqrt{Q}\left(1+2\left(1+\left(1-\frac{1}{\sqrt{Q}}\right)\right)\right)\left(\mu_{\mathrm{TI}}^{[i_n]}\mu_{\mathrm{IR}}^{[j_n]}\right)^2}{Q^2\delta^2}\\
&\times\max\left\{\frac{1}{\left(\frac{2\pi}{\lambda}\min\{d_H,d_V\}\right)^2},\frac{1}{\left(\frac{2\pi}{\lambda}\min\{d_H,d_V\}\right)^4}\right\}=\varepsilon_1(Q),
\end{aligned}\tag{76}$$

which follows from the fact $\sum_{i=1}^{u}\frac{1}{i^2}\leq 1+(1-\frac{1}{u})$. Then, we can see that $\varepsilon_1(Q)$ goes to zero with an order of at least $O(\frac{1}{\sqrt{Q}})$. Next, we analyze the following probability:

$$\begin{aligned}
&\Pr\left\{\frac{1}{Q}|l_{n,m}|>\delta\right\}\\
&=\Pr\left\{\frac{1}{Q^2}|l_{n,m}|^2>\delta^2\right\}\leq\frac{E\{|l_{n,m}|^2\}}{Q^2\delta^2}\\
&=\frac{E\left\{\left|\sum_{u=1}^{Q}H_{\mathrm{TI}}^{[ui_n]}(t)H_{\mathrm{IR}}^{[j_nu]}(t)\left(H_{\mathrm{TI}}^{[ui_m]}(t)H_{\mathrm{IR}}^{[j_mu]}(t)\right)^*\right|^2\right\}}{Q^2\delta^2}.
\end{aligned}\tag{77}$$

In the above inequality, three cases may occur: 1) $i_n = i_m, j_n \neq j_m$, 2) $i_n \neq i_m, j_n = j_m$, and 3) $i_n \neq i_m, j_n \neq j_m$. Cases 1 and 2 are the same, thus, we study cases 1 and 3. Using (73) and (74), we have:

Case 1:

$$E\Big\{H_{\text{TI}}^{[ui_n]}(t)H_{\text{IR}}^{[j_nu]}(t)H_{\text{TI}}^{[u'i_m]}(t)H_{\text{IR}}^{[j_mu']}(t) \times \Big(H_{\text{TI}}^{[ui_m]}(t)H_{\text{IR}}^{[j_mu]}(t)H_{\text{TI}}^{[u'i_n]}(t)H_{\text{IR}}^{[j_nu']}(t)\Big)^*\Big\} = \Big(\mu_{\text{TI}}^{[i_n]}\Big)^2\Big(1+([\mathbf{R}]_{u,u'})^2\Big)\mu_{\text{IR}}^{[j_n]}\mu_{\text{IR}}^{[j_m]}([\mathbf{R}]_{u,u'})^2 \quad (78)$$

Case 3:

$$E\Big\{H_{\text{TI}}^{[ui_n]}(t)H_{\text{IR}}^{[j_nu]}(t)H_{\text{TI}}^{[u'i_m]}(t)H_{\text{IR}}^{[j_mu']}(t) \times \Big(H_{\text{TI}}^{[ui_m]}(t)H_{\text{IR}}^{[j_mu]}(t)H_{\text{TI}}^{[u'i_n]}(t)H_{\text{IR}}^{[j_nu']}(t)\Big)^*\Big\} = \mu_{\text{TI}}^{[i_n]}\mu_{\text{TI}}^{[i_m]}\mu_{\text{IR}}^{[j_n]}\mu_{\text{IR}}^{[j_m]}([\mathbf{R}]_{u,u'})^4. \quad (79)$$

By (78) and (79), (77) can be bounded as follows:

$$\frac{E\Big\{\Big|\sum_{u=1}^{Q} H_{\text{TI}}^{[ui_n]}(t)H_{\text{IR}}^{[j_nu]}(t)\Big(H_{\text{TI}}^{[ui_m]}(t)H_{\text{IR}}^{[j_mu]}(t)\Big)^*\Big|^2\Big\}}{Q^2\delta^2} = \frac{1}{Q^2\delta^2}\times E\Bigg\{\sum_{u'=1}^{Q}\sum_{u=1}^{Q} H_{\text{TI}}^{[ui_n]}(t)H_{\text{IR}}^{[j_nu]}(t)H_{\text{TI}}^{[u'i_m]}(t)H_{\text{IR}}^{[j_mu']}(t) \times\Big(H_{\text{TI}}^{[ui_m]}(t)H_{\text{IR}}^{[j_mu]}(t)H_{\text{TI}}^{[u'i_n]}(t)H_{\text{IR}}^{[j_nu']}(t)\Big)^*\Bigg\}$$
$$\le \frac{2Q\sqrt{Q}\Big(1+2\Big(1+\Big(1-\frac{1}{\sqrt{Q}}\Big)\Big)\Big)\Big(\mu_{\text{TI}}^{[i_n]}\mu_{\text{TI}}^{[i_m]}\mu_{\text{IR}}^{[j_n]}\mu_{\text{IR}}^{[j_m]}\Big)}{Q^2\delta^2} \times\max\Bigg\{\frac{1}{\Big(\frac{2\pi}{\lambda}\min\{d_H,d_V\}\Big)^2}, \frac{1}{\Big(\frac{2\pi}{\lambda}\min\{d_H,d_V\}\Big)^4}\Bigg\} = \varepsilon_2(Q). \quad (80)$$

Similarly this inequality follows from the fact $\sum_{i=1}^{u}\frac{1}{i^2} \le 1+(1-\frac{1}{u})$. Therefore, $\varepsilon_2(Q)$ goes to zero with an order of at least $O(\frac{1}{\sqrt{Q}})$. ■

This lemma reveals that the n-th diagonal element $l_{n,n}$ tends to $Q(E\{|H_{\text{TI}}^{[ui_n]}(t)H_{\text{IR}}^{[j_nu]}(t)|^2\}\pm\delta)$, and the absolute value of the other non-diagonal elements of the n-th row $l_{n,n'}, n' \neq n$ are less than $Q\delta$, with a probability, which tends to 1. Therefore, By Gershgorin Circle Theorem [36, Th. 7.2.1], we have:

$$\lambda_i\Big(\mathbf{H}_{\mathbf{N}_i^*}\mathbf{H}_{\mathbf{N}_i^*}^H\Big) \in \bigcup_n \Bigg\{x\in\mathbb{C} : \Bigg|x - Q\Bigg(E\Bigg\{\Big|H_{\text{TI}}^{[ui_n]}(t)H_{\text{IR}}^{[j_nu]}(t)\Big|^2\Bigg\}\pm\delta\Bigg)\Bigg| \le \Big|\mathcal{M}_{\mathbf{N}_i^*}\Big|Q\delta\Bigg\},$$

where $\lambda_i(\mathbf{H}_{\mathbf{N}_i^*}\mathbf{H}_{\mathbf{N}_i^*}^H)$ is the i-th eigenvalue of $\mathbf{H}_{\mathbf{N}_i^*}\mathbf{H}_{\mathbf{N}_i^*}^H$. This shows that if we choose a sufficiently small δ, then, $\mathbf{H}_{\mathbf{N}_i^*}\mathbf{H}_{\mathbf{N}_i^*}^H$ will be invertible and (36) will be proved.

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