# Connectivity Assessment of Random Directed Graphs with Application to Underwater Sensor Networks

Mohammad Mehdi Asadi, Hamid Mahboubi, Jalal Habibi, Amir G. Aghdam, and Stephane Blouin

***Abstract*— In this brief, the problem of connectivity assessment for a random network is investigated. The weighted vertex connectivity (WVC) is introduced as a metric to evaluate the connectivity of the weighted expected graph of a random sensor network, where the elements of the weight matrix characterize the operational probability of their corresponding communication links. The WVC measure extends the notion of vertex connectivity (VC) for random graphs by taking into account the joint effects of path reliability and network robustness to node failure. The problem of computing the WVC measure is transformed into a sequence of iterative deepening depth-first search and maximum weight clique problems. An algorithm is developed accordingly to find the proposed connectivity metric. The approximate WVC measure is defined subsequently as a lower bound on the introduced connectivity metric which can be found by applying a polynomial-time shortest path algorithm in a sequential manner. The performance of the proposed algorithms is validated using an experimental underwater acoustic sensor network.**

***Index Terms*— Connectivity assessment, random directed graphs, underwater sensor networks.**

## I. Introduction

THERE has been a surge of interest in recent years in the implementation of efficient sensor networks for various applications [1]–[3]. As a particular application, an underwater acoustic sensor network consists of a number of fixed or mobile sensors deployed in the underwater environment. These sensors are capable of sending/receiving data using acoustic communication [4], [5]. A typical objective for such networks is to perform data aggregation for a wide range of applications which include underwater exploration, ocean sampling, climate reporting, and disaster prevention [6]–[8]. Unlike the communication channels used in terrestrial sensor networks, there are several sources of uncertainty in underwater acoustic communication, including multipath propagation, temperature and salinity fluctuations, scattering and reverberation, variation of sound speed profile and underwater currents, which may impede data exchange between underwater nodes [9], [10]. These sources of uncertainty vary over time and space, resulting in highly temporal and spatially variable acoustic channels [9]. As a result, an underwater sensor network can be well-represented by a random graph [11], [12].

It is shown in [13]–[15] that various tasks such as consensus, distributed estimation, target localization, and data aggregation over a random sensor network can be achieved cooperatively as long as the expected communication graph of the network remains connected. Moreover, the convergence rate of the cooperative algorithms running over a random network depends heavily on the connectivity degree of the expected graph of the network [16]. Vertex connectivity (VC) is used in [17] and [18] to evaluate the robustness of a sensor network to node failure. A fault-tolerant topology control procedure is subsequently proposed in [17] to minimize the power consumption of the nodes while maintaining a certain degree of VC over the entire network. In [18], efficient algorithms are introduced to improve the VC of a heterogeneous wireless sensor network using relay nodes. Various polynomial-time algorithms have been provided in the literature to measure the VC degree of a graph [19], [20]. The general idea behind these algorithms is that the VC problem for any ordered pair of nonadjacent nodes in a graph can be formulated as the problem of finding multiple vertex-disjoint paths between those nodes [21]. By establishing multiple disjoint paths between a source node and a destination node in an optimal manner, network characteristics such as energy conservation, load balancing, and robustness to failure can be improved. Different procedures are proposed in the literature to obtain a set of disjoint paths between two nodes satisfying certain properties. An algorithm is proposed in [22] to find a prespecified number of vertex-disjoint paths from a source node to a destination node with minimum total weight in a directed graph. Loh *et al.* [23] propose a procedure to find a number of parallel vertex-disjoint paths which can be used to transmit data with maximum reliability.

Motivated by recent applications of data aggregation in underwater acoustic sensor networks, a novel metric of connectivity is developed in this brief to evaluate the connectivity of the expected communication graph of a random sensor network. The notion of weighted VC (WVC) is introduced as an extension of the VC notion reflecting the combined effects of the reliability of the paths and the network robustness to node failure on the connectivity of the expected communication graph [24]. The problem of finding the WVC measure is then transformed into a sequence of iterative deepening depth-first search (IDDFS) and maximum

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M. M. Asadi and A. G. Aghdam are with the Department of Electrical and Computer Engineering, Concordia University, Montréal, QC H3G 1M8, Canada (e-mail: m_asadim@ece.concordia.ca; aghdam@ece.concordia.ca).

H. Mahboubi is with the Harvard John A. Paulson School of Engineering and Applied Sciences, 29 Oxford Street, Cambridge, MA 02138 USA (e-mail: mahboubi@seas.harvard.edu).

J. Habibi is with the Faculty of Engineering, McGill University, Montréal, QC H3A 0C3, Canada (e-mail: jalal.habibi@mcgill.ca).

S. Blouin is with the Defence Research and Development Canada, Atlantic Research Centre, Dartmouth, NS B2Y 3Z7, Canada (e-mail: stephane.blouin@drdc-rddc.gc.ca).

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weight clique (MWC) problems. An approximation of the WVC measure is subsequently proposed which provides a lower bound on the original metric and can be computed by applying a polynomial-time shortest path algorithm sequentially. The proposed connectivity measure and its approximation are computed in a simulation environment, and are then assessed using data from an experimental sea-trial.

The remainder of the brief is organized as follows. In Section II, some important background information on random graphs is given and the problem is formulated. The WVC degree is proposed in Section III and a procedure is developed to find this measure in Section IV. In Section V, an approximation of the WVC metric and a time-efficient algorithm to obtain this measure are presented. The simulation and experimental results are provided in Sections VI and VII, respectively. Finally, concluding remarks are provided in Section VIII.

## II. Preliminaries and Problem Formulation

Throughout this brief, the set of positive and nonnegative real numbers are denoted by $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$, respectively. Also, $\mathbb{N}_n := \{1, 2, \ldots, n\}$, $|\mathcal{S}|$ is the cardinality of the finite set $\mathcal{S}$, and $s^i$ represents the $i$th element of the set $\mathcal{S}$. Moreover, the power set of the finite set $\mathcal{S}$, denoted by $\mathcal{P}(\mathcal{S})$, is the set of all subsets of $\mathcal{S}$.

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ denote a random directed graph (digraph) composed of a set of nodes $\mathscr{V}$ and a set of edges $\mathscr{E}$. Let also $p_{ij} \in [0, 1]$ be the probability of the existence of the edge $(j, i) \in \mathscr{E}$, and define the probability matrix $\mathbf{P} = [p_{ij}]$ for the random digraph $\mathscr{G}$ accordingly. Define also $\mathbf{A} = [a_{ij}]$ as the adjacency matrix of $\mathscr{G}$, where $a_{ij}$ is a binary random variable such that

$$a_{ij} = \begin{cases} 1, & \text{with probability } p_{ij} \\ 0, & \text{with probability } 1 - p_{ij} \end{cases}$$

and $(j, i) \in \mathscr{E}$ if and only if $a_{ij} = 1$. Consider $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}}, \mathbf{P})$ as the expected graph of the random digraph $\mathscr{G}$, which is a weighted deterministic digraph with node set $\hat{\mathscr{V}}$, edge set $\hat{\mathscr{E}}$, and weight matrix $\mathbf{P}$. Moreover, $(j, i) \in \hat{\mathscr{E}}$ if and only if $p_{ij} \neq 0$.

*Definition 1:* Let the communication graph of a network composed of $n$ sensors be specified by a random digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the probability matrix $\mathbf{P} = [p_{ij}]$, where its node and edge sets are defined as

$$\begin{aligned} \mathscr{V} &= \{1, 2, \ldots, n\} \\ \mathscr{E} &= \{(i, j) \in \mathscr{V} \times \mathscr{V} | a_{ji} = 1\}. \end{aligned}$$

Let also $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}}, \mathbf{P})$ be the expected communication graph of the network, where $\hat{\mathscr{V}} = \mathscr{V}$ and

$$\hat{\mathscr{E}} = \{(i, j) \in \hat{\mathscr{V}} \times \hat{\mathscr{V}} | p_{ji} \neq 0\}.$$

Due to the importance of the connectivity of the expected communication graph in performing any cooperative task in a random sensor network, the main objective of this brief is to introduce appropriate global measures for the connectivity of an expected communication graph, and develop efficient algorithms to evaluate them. Note that a digraph is said to be strongly connected if a directed path exists between any ordered pair of distinct nodes.

## III. Weighted Vertex Connectivity Metric

Consider a group of sensors represented by the node set of a random digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where every directed edge in $\mathscr{E}$ is characterized by a binary random variable noted earlier, and let these binary random variables be mutually independent. Let also a weighted deterministic digraph $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}}, \mathbf{P})$ represent the expected communication graph associated with $\mathscr{G}$. The VC degree of $\hat{\mathscr{G}}$ is defined as the minimum number of nodes that should be removed in order for $\hat{\mathscr{G}}$ to lose strong connectivity [21]. This is, in fact, a measure of the global connectivity of a digraph, and is mathematically expressed as

$$\kappa(\hat{\mathscr{G}}) = \min_{i, j \in \hat{\mathscr{V}},\, i \neq j} \kappa_{i,j}(\hat{\mathscr{G}})$$

where

$$\kappa_{i,j}(\hat{\mathscr{G}}) = \begin{cases} N_{i,j}(\hat{\mathscr{G}}), & \text{if } (i, j) \notin \hat{\mathscr{E}} \\ |\hat{\mathscr{V}}| - 1, & \text{if } (i, j) \in \hat{\mathscr{E}} \end{cases}$$

and $N_{i,j}(\hat{\mathscr{G}})$ denotes the maximum number of vertex-disjoint directed paths connecting $i$ to $j$ in $\hat{\mathscr{G}}$. The general idea behind the algorithms provided in the literature to compute the VC degree of a graph is that the minimum number of nodes whose removal disconnects any ordered pair of nonadjacent nodes is equal to the maximum number of mutually vertex-disjoint simple paths between them (see Menger's Theorem [21]). However, this measure does not account for the probability matrix of random networks, and merely reflects the robustness of the network to node failure. This calls for a generalized measure of connectivity to capture the probabilistic nature of the communication links. The weighted vertex connectivity (WVC) measure is introduced here to extend the notion of the VC degree to the case of weighted digraphs, where the elements of the weight matrix denote the operational probability of their corresponding communication links. This measure is strictly positive for a strongly connected digraph, and the larger it is, the "stronger" the connectivity of the graph is, taking into account the combined effects of the operational probability of the paths and the network robustness to node failure. To further clarify this new notion, the multiplicative weight of a path is subsequently defined, based on the assumption of the mutual independence of the binary random variables describing the probabilistic nature of the edges of the network.

*Definition 2:* Let $\Pi_{i,j}$ denote the set of all simple paths from node $i$ to node $j$, whose lengths are greater than one in the expected communication graph $\hat{\mathscr{G}}$. Let also $\pi_{i,j}^k \in \Pi_{i,j}$ represent the $k$th element of $\Pi_{i,j}$, defined as the node set $\pi_{i,j}^k = \{v_0^k, v_1^k, \ldots, v_{m_k-1}^k, v_{m_k}^k\}$, which denotes a directed path of length $m_k > 1$ from node $i$ to node $j$ such that $v_0^k = i$, $v_{m_k}^k = j$, and $(v_{l-1}^k, v_l^k) \in \hat{\mathscr{E}}$ for all $l \in \mathbb{N}_{m_k}$. Then, the multiplicative weight of path $\pi_{i,j}^k$, denoted by $W(\pi_{i,j}^k)$, is defined as

$$W(\pi_{i,j}^k) = \prod_{l=1}^{m_k} p_{v_l^k v_{l-1}^k}.$$

Since each element of $\mathbf{P}$ represents the probability of the existence of its corresponding edge in $\hat{\mathscr{G}}$ and all edges are characterized by a set of mutually independent binary random variables, the multiplicative weight of a path can be interpreted as its operational probability.

*Definition 3:* Consider $\pi_{i,j}^s$ and $\pi_{i,j}^t$ as two distinct simple paths from node $i$ to node $j$ in $\hat{\mathscr{G}}$, described by the node sets $\pi_{i,j}^s = \{v_0^s, v_1^s, \ldots, v_{m_s-1}^s, v_{m_s}^s\}$ and $\pi_{i,j}^t = \{v_0^t, v_1^t, \ldots, v_{m_t-1}^t, v_{m_t}^t\}$, respectively, with $v_0^s = v_0^t = i$ and $v_{m_s}^s = v_{m_t}^t = j$ for $s, t \in \mathbb{N}_{|\Pi_{i,j}|}$. Let $m_s$ and $m_t$ denote the lengths of two directed paths $\pi_{i,j}^s$ and $\pi_{i,j}^t$, respectively, such that $m_s > 1$ and $m_t > 1$. Then, $\pi_{i,j}^s$ and $\pi_{i,j}^t$ are said to be vertex-disjoint paths if $(\pi_{i,j}^s \backslash \{v_0^s, v_{m_s}^s\}) \cap (\pi_{i,j}^t \backslash \{v_0^t, v_{m_t}^t\}) = \emptyset$.

The notion of the local WVC measure for any ordered pair of distinct nodes $i, j \in \hat{\mathscr{V}}$ in the expected communication graph $\hat{\mathscr{G}}$ is now introduced. This measure, denoted by $\hat{\kappa}_{i,j}(\hat{\mathscr{G}})$, is defined as the maximum of the summation of the multiplicative weights of the vertex-disjoint paths from node $i$ to node $j$ in $\hat{\mathscr{G}}$. In other words, $\hat{\kappa}_{i,j}(\hat{\mathscr{G}})$ represents the maximum of the summation of the operational probability of all paths connecting node $i$ to node $j$ in $\hat{\mathscr{G}}$, which do not share any internal node. Consider $\mathcal{P}(\Pi_{i,j})$ as the power set of $\Pi_{i,j}$, and let $\hat{\mathcal{P}}(\Pi_{i,j}) \subseteq \mathcal{P}(\Pi_{i,j})$ contain all nonempty subsets of $\Pi_{i,j}$ composed of a set of mutually vertex-disjoint paths from $i$ to $j$ in $\hat{\mathscr{G}}$. Then

$$\hat{\kappa}_{i,j}(\hat{\mathscr{G}}) = \begin{cases} \sum_{k=1}^{|\hat{\Pi}_{i,j}|} W\left(\hat{\pi}_{i,j}^k\right), & \text{if } (i,j) \notin \hat{\mathscr{E}} \\ \max\left((|\hat{\mathscr{V}}|-1)p_{ji},\, p_{ji} + \sum_{k=1}^{|\hat{\Pi}_{i,j}|} W\left(\hat{\pi}_{i,j}^k\right)\right), & \text{if } (i,j) \in \hat{\mathscr{E}} \end{cases} \tag{1}$$

where the optimal path set $\hat{\Pi}_{i,j}$ is the solution of the following combinatorial optimization problem:

$$\hat{\Pi}_{i,j} = \underset{\Pi \in \hat{\mathcal{P}}(\Pi_{i,j})}{\operatorname{argmax}} \sum_{k=1}^{|\Pi|} W(\pi^k) \tag{2}$$

and $\Pi = \{\pi^k \,|\, k \in \mathbb{N}_{|\Pi|}\}$ denotes a path set belonging to $\hat{\mathcal{P}}(\Pi_{i,j})$. The WVC degree of $\hat{\mathscr{G}}$, denoted by $\hat{\kappa}(\hat{\mathscr{G}})$, is defined in the following as the global connectivity of the expected communication graph $\hat{\mathscr{G}}$:

$$\hat{\kappa}(\hat{\mathscr{G}}) = \min_{i,j \in \hat{\mathscr{V}}, i \neq j} \hat{\kappa}_{i,j}(\hat{\mathscr{G}}). \tag{3}$$

Thus, the WVC measure $\hat{\kappa}(\hat{\mathscr{G}})$ can be considered as an extension of the VC degree $\kappa(\hat{\mathscr{G}})$, where the relationship between the two notions is addressed in the following proposition.

*Proposition 1:* Let $\hat{\mathscr{G}}$ represent the expected communication graph of a random network with the probability matrix $\mathbf{P}$. It then follows that $\hat{\kappa}(\hat{\mathscr{G}}) \leq \kappa(\hat{\mathscr{G}})$.

*Proof:* To prove this proposition, it suffices to show that $\hat{\kappa}_{i,j}(\hat{\mathscr{G}}) \leq \kappa_{i,j}(\hat{\mathscr{G}})$ for any ordered pair of distinct nodes $i, j \in \hat{\mathscr{V}}$. To this end, two different cases are considered.

1) In the first case, assume that no directed edge exists from $i$ to $j$ in $\hat{\mathscr{G}}$. Then, the local WVC measure $\hat{\kappa}_{i,j}(\hat{\mathscr{G}})$ is determined by the optimal path set $\hat{\Pi}_{i,j}$ defined in (2), which includes a set of mutually vertex-disjoint paths connecting $i$ to $j$ in $\hat{\mathscr{G}}$ such that the sum of the multiplicative weights of its elements is maximum. Since $0 < W(\hat{\pi}_{i,j}^k) \leq 1$ for all $k \in \mathbb{N}_{|\hat{\Pi}_{i,j}|}$ and $\hat{\Pi}_{i,j} \subseteq \Pi_{i,j}$, it can be concluded that

$$\sum_{k=1}^{|\hat{\Pi}_{i,j}|} W\left(\hat{\pi}_{i,j}^k\right) \leq |\hat{\Pi}_{i,j}| \leq N_{i,j}(\hat{\mathscr{G}}) \tag{4}$$

where $N_{i,j}(\hat{\mathscr{G}})$ represents the maximum number of vertex-disjoint paths connecting $i$ to $j$ in $\hat{\mathscr{G}}$. Moreover, the equality occurs in (4) when $p_{ts} = 1$ for all $(s,t) \in \hat{\mathscr{E}}$. In this case, all simple paths from $i$ to $j$ have unit multiplicative weights, and, as a result, the optimal path set $\hat{\Pi}_{i,j}$ contains the maximum number of vertex-disjoint paths from $i$ to $j$ in $\hat{\mathscr{G}}$ according to its definition. In other words, $N_{i,j}(\hat{\mathscr{G}}) = |\hat{\Pi}_{i,j}|$. It then follows from (4) that $\hat{\kappa}_{i,j}(\hat{\mathscr{G}}) \leq \kappa_{i,j}(\hat{\mathscr{G}})$ for any two distinct nodes $i, j \in \hat{\mathscr{V}}$, provided $(i,j) \notin \hat{\mathscr{E}}$.

2) In the second case, let $(i,j) \in \hat{\mathscr{E}}$. Then, according to (1), the WVC measure $\hat{\kappa}_{i,j}(\hat{\mathscr{G}})$ is either $(|\hat{\mathscr{V}}|-1)p_{ji}$ or $p_{ji} + \sum_{k=1}^{|\hat{\Pi}_{i,j}|} W(\hat{\pi}_{i,j}^k)$, whichever is larger. Given that there is at most $|\hat{\mathscr{V}}| - 2$ vertex-disjoint paths with length greater than one between any ordered pair of distinct nodes in $\hat{\mathscr{G}}$ and on noting that $0 < p_{ji} \leq 1$ and $0 < W(\hat{\pi}_{i,j}^k) \leq 1$ for all $k \in \mathbb{N}_{|\hat{\Pi}_{i,j}|}$ and that $\hat{\Pi}_{i,j} \subseteq \Pi_{i,j}$, it can be deduced that

$$p_{ji} + \sum_{k=1}^{|\hat{\Pi}_{i,j}|} W(\hat{\pi}_{i,j}^k) \leq |\hat{\mathscr{V}}| - 1 \tag{5}$$

for any $(i,j) \in \hat{\mathscr{E}}$. Therefore, it follows from (5) that:

$$\max\left((|\hat{\mathscr{V}}|-1)p_{ji},\, p_{ji} + \sum_{k=1}^{|\hat{\Pi}_{i,j}|} W\left(\hat{\pi}_{i,j}^k\right)\right) \leq |\hat{\mathscr{V}}| - 1 \tag{6}$$

for any ordered pair of distinct nodes $i, j \in \hat{\mathscr{V}}$ such that $(i,j) \in \hat{\mathscr{E}}$. Note that the equality in (6) holds when $p_{ji} = 1$. It then follows from (6) that $\hat{\kappa}_{i,j}(\hat{\mathscr{G}}) \leq \kappa_{i,j}(\hat{\mathscr{G}})$ for the case when $(i,j) \in \hat{\mathscr{E}}$. This completes the proof. ■

*Remark 1:* Note that for two different expected communication graphs $\hat{\mathscr{G}}_1$ and $\hat{\mathscr{G}}_2$ with $\kappa(\hat{\mathscr{G}}_1) > \kappa(\hat{\mathscr{G}}_2)$, it is possible that $\hat{\kappa}(\hat{\mathscr{G}}_1) < \hat{\kappa}(\hat{\mathscr{G}}_2)$, depending on the probability matrices $\mathbf{P}_1$ and $\mathbf{P}_2$. Since the random nature of the communication links is captured by the WVC measure, it is more suitable to use this notion to assess the connectivity of a network whose information exchange is described by a random digraph.

## IV. Procedure to Find the WVC Metric

In this section, an algorithm is proposed to compute the WVC degree of a strongly connected expected communication graph. As the first step, the iterative deepening depth-first search (IDDFS) algorithm given in [25] is used to find all simple paths from a source node to a destination node in $\hat{\mathscr{G}}$. In the second step, the problem of obtaining the optimal path set $\hat{\Pi}_{i,j}$ for any ordered pair of distinct nodes $i, j \in \hat{\mathscr{V}}$ given in (2) is formulated as a maximum weight clique (MWC) problem in the context of combinatorial optimization. To this end, two definitions are provided in the sequel.

*Definition 4:* Consider a weighted undirected graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathrm{H})$ with node set $\mathscr{V}$, edge set $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and weight vector $\mathrm{H} \in \mathbb{R}_{>0}^{|\mathscr{V}|}$, whose $i$th component $h(i)$ is the weight assigned to node $i$. The weight of a node set $\mathcal{S} \subseteq \mathscr{V}$, denoted by $\mathrm{H}(\mathcal{S})$, is defined as $\sum_{i=1}^{|\mathcal{S}|} h(s^i)$. A clique $\mathcal{C}$ in $\mathscr{G}$ is a subset of the node set $\mathscr{V}$ such that the subgraph induced by $\mathcal{C}$ is complete. A MWC in $\mathscr{G}$ is then defined as a clique $\mathcal{C}$ whose weight $\mathrm{H}(\mathcal{C})$ is maximum.

*Definition 5:* Let a node be assigned to every path from $i$ to $j$ belonging to the path set $\Pi_{i,j}$ in the expected communication graph $\hat{\mathscr{G}}$. Also, assign an undirected edge between distinct nodes $s$ and $t$ if their corresponding paths, represented by $\pi_{i,j}^s$ and $\pi_{i,j}^t$, are vertex-disjoint. The resulting graph $\mathscr{G}_{i,j}^p = (\mathscr{V}_{i,j}^p, \mathscr{E}_{i,j}^p, \mathrm{H}_{i,j}^p)$ is defined as the weighted undirected path graph associated with the paths belonging to $\Pi_{i,j}$, whose node and edge sets are described as follows:

$$\mathscr{V}_{i,j}^p = \{1, 2, \ldots, |\Pi_{i,j}|\}$$
$$\mathscr{E}_{i,j}^p = \left\{(s,t) \in \mathscr{V}_{i,j}^p \times \mathscr{V}_{i,j}^p \mid \pi_{i,j}^s \text{ and } \pi_{i,j}^t \text{ are vertex-disjoint}\right\}$$

and $h_{i,j}^p(k) = W(\pi_{i,j}^k)$ for all $k \in \mathscr{V}_{i,j}^p$.

*Theorem 1:* The problem of finding the optimal path set $\hat{\Pi}_{i,j}$ in the expected communication graph $\hat{\mathscr{G}}$ given by (2) is equivalent to applying a MWC procedure to the weighted undirected path graph $\mathscr{G}_{i,j}^p$ for any ordered pair of distinct nodes $i, j \in \hat{\mathscr{V}}$.

*Proof:* Consider the weighted undirected path graph $\mathscr{G}_{i,j}^p = (\mathscr{V}_{i,j}^p, \mathscr{E}_{i,j}^p, \mathrm{H}_{i,j}^p)$ associated with the directed paths belonging to $\Pi_{i,j}$, where the weight vector $\mathrm{H}_{i,j}^p$ is formed from the multiplicative weight of every path belonging to $\Pi_{i,j}$. Let the solution to the MWC problem posed on $\mathscr{G}_{i,j}^p$ be denoted by $\hat{\mathcal{S}}$, that is

$$\hat{\mathcal{S}} = \underset{\mathcal{S} \subseteq \mathscr{V}_{i,j}^p}{\operatorname{argmax}} \sum_{k=1}^{|\mathcal{S}|} h_{i,j}^p(s^k) \tag{7}$$

where the node set $\mathcal{S} = \{s^k \mid k \in \mathbb{N}_{|\mathcal{S}|}\}$ corresponds to a clique of size $|\mathcal{S}|$ in $\mathscr{G}_{i,j}^p$. According to (7), $\hat{\mathcal{S}}$ reflects the indices of a set of directed paths from node $i$ to node $j$ in the expected communication graph $\hat{\mathscr{G}}$, which are mutually vertex-disjoint and the sum of their multiplicative weights (as the elements of the weight vector $\mathrm{H}_{i,j}^p$) is maximum. Let $\hat{\Pi}_{i,j}^*$ denote a path set which contains all the paths whose indices are specified by the elements of $\hat{\mathcal{S}}$. It then follows from the definition of $\mathrm{H}_{i,j}^p$ that:

$$\hat{\Pi}_{i,j}^* = \underset{\Pi \in \hat{\mathcal{P}}(\Pi_{i,j})}{\operatorname{argmax}} \sum_{k=1}^{|\Pi|} W(\pi^k)$$

where $\hat{\mathcal{P}}(\Pi_{i,j})$ denotes a subset of $\mathcal{P}(\Pi_{i,j})$ containing all nonempty subsets of $\Pi_{i,j}$ which are composed of a set of vertex-disjoint paths from $i$ to $j$ in $\hat{\mathscr{G}}$ and correspond to all cliques in the undirected path graph $\mathscr{G}_{i,j}^p$. It follows directly from (2) that $\hat{\Pi}_{i,j} = \hat{\Pi}_{i,j}^*$, which means that $\hat{\Pi}_{i,j}$ can be obtained by solving a MWC problem posed on the weighted undirected path graph $\mathscr{G}_{i,j}^p$. This completes the proof. ∎

**Algorithm 1** Procedure to Find the WVC of $\hat{\mathscr{G}}$

Let $\hat{\kappa}(\hat{\mathscr{G}}) = |\hat{\mathscr{V}}| - 1$.
**for all** $i, j \in \hat{\mathscr{V}}$ and $i \neq j$ **do**
  Find $\Pi_{i,j}$ by applying the IDDFS algorithm to $\hat{\mathscr{G}}$.
  Construct the weighted undirected path graph $\mathscr{G}_{i,j}^p$.
  Find $\hat{\Pi}_{i,j}$ by solving the MWC problem posed on $\mathscr{G}_{i,j}^p$.
  **if** $(i,j) \notin \hat{\mathscr{E}}$ **then**
    $\hat{\kappa}_{i,j}(\hat{\mathscr{G}}) = \sum_{k=1}^{|\hat{\Pi}_{i,j}|} W(\hat{\pi}_{i,j}^k)$.
  **else if** $(i,j) \in \hat{\mathscr{E}}$ **then**
    $\hat{\kappa}_{i,j}(\hat{\mathscr{G}}) = \max\left((|\hat{\mathscr{V}}|-1)p_{ji}, p_{ji} + \sum_{k=1}^{|\hat{\Pi}_{i,j}|} W(\hat{\pi}_{i,j}^k)\right)$.
  **end if**
  $\hat{\kappa}(\hat{\mathscr{G}}) = \min(\hat{\kappa}(\hat{\mathscr{G}}), \hat{\kappa}_{i,j}(\hat{\mathscr{G}}))$.
**end for**
**return** $\hat{\kappa}(\hat{\mathscr{G}})$.

A number of algorithms are developed in the literature for the MWC problem. For instance, one can use the procedure given in [26], which is utilized in this brief to find the optimal path set $\hat{\Pi}_{i,j}$ for any ordered pair of distinct nodes $i, j \in \hat{\mathscr{V}}$ according to (2). Then, the local and global WVC measures of $\hat{\mathscr{G}}$ can be obtained by (1) and (3), respectively. A procedure to obtain $\hat{\kappa}(\hat{\mathscr{G}})$ is presented in Algorithm 1.

## V. Approximate Weighted Vertex Connectivity Metric

The number of all possible paths between any pair of distinct nodes increases exponentially with the network size. On the other hand, the MWC problem is NP-hard; thus, it is desired to seek an approximation of the WVC metric, which can be obtained using a polynomial-time algorithm. To this end, the most reliable path in $\hat{\mathscr{G}}$ is defined next.

*Definition 6:* Given an expected communication graph $\hat{\mathscr{G}}$ with probability matrix $\mathbf{P}$, let $\Pi_{i,j}$ denote the set of all simple paths from node $i$ to node $j$ with length greater than one in $\hat{\mathscr{G}}$. The most reliable path directed from $i$ to $j$ in $\hat{\mathscr{G}}$, denoted by $\overline{\pi}_{i,j}$, is defined as a path in $\Pi_{i,j}$ with the largest multiplicative weight. The path $\overline{\pi}_{i,j}$ is, in fact, a path from $i$ to $j$ with the highest probability of existence whose length is greater than one.

*Definition 7:* Let $\overline{\Pi}_{i,j}$ represent a set of sequential most reliable paths from node $i$ to node $j$ with length greater than one in $\hat{\mathscr{G}}$ such that $\overline{\Pi}_{i,j} = \{\overline{\pi}_{i,j}^k \mid k \in \mathbb{N}_{|\overline{\Pi}_{i,j}|}\}$. The $k$th most reliable path from node $i$ to node $j$, denoted by $\overline{\pi}_{i,j}^k$, is defined as the most reliable path from $i$ to $j$ after removing the internal nodes (and their corresponding adjacent edges) of the $k-1$ previously found most reliable paths $\overline{\pi}_{i,j}^l$, $l \in \mathbb{N}_{k-1}$, from $\hat{\mathscr{G}}$ sequentially (by definition $\overline{\pi}_{i,j}^1 = \overline{\pi}_{i,j}$).

Note that $|\overline{\Pi}_{i,j}|$ is the total number of the sequential vertex-disjoint most reliable paths from $i$ to $j$ with length greater than one such that after deletion of their internal nodes (along with the corresponding adjacent edges) from $\hat{\mathscr{G}}$ in a sequential manner, no path will remain from $i$ to $j$. The notion of local *approximate WVC* (AWVC) of $\hat{\mathscr{G}}$ for any ordered pair of distinct nodes $i, j \in \hat{\mathscr{V}}$, denoted by $\overline{\kappa}_{i,j}(\hat{\mathscr{G}})$, is defined as follows:

$$\overline{\kappa}_{i,j}(\hat{\mathscr{G}}) = \begin{cases} \sum_{k=1}^{|\overline{\Pi}_{i,j}|} W\left(\overline{\pi}_{i,j}^{k}\right), & \text{if } (i,j) \notin \hat{\mathscr{E}} \\ \max\left((|\hat{\mathscr{V}}|-1)p_{ji},\, p_{ji} + \sum_{k=1}^{|\overline{\Pi}_{i,j}|} W\left(\overline{\pi}_{i,j}^{k}\right)\right), & \text{if } (i,j) \in \hat{\mathscr{E}} \end{cases} \tag{8}$$

where $W(\overline{\pi}_{i,j}^{k})$ represents the multiplicative weight of the $k$th most reliable path from node $i$ to node $j$ in $\hat{\mathscr{G}}$ for all $k \in \mathbb{N}_{|\overline{\Pi}_{i,j}|}$. Subsequently, $\overline{\kappa}(\hat{\mathscr{G}})$ is defined as the global AWVC of $\hat{\mathscr{G}}$, which is related to the local AWVC measure as follows:

$$\overline{\kappa}(\hat{\mathscr{G}}) = \min_{i,j \in \hat{\mathscr{V}}, i \neq j} \overline{\kappa}_{i,j}(\hat{\mathscr{G}}).$$

The relation between the WVC metric $\hat{\kappa}(\hat{\mathscr{G}})$ and its approximation $\overline{\kappa}(\hat{\mathscr{G}})$ is described in the following proposition.

*Proposition 2:* Let $\hat{\mathscr{G}}$ represent the expected communication graph of a random network with the probability matrix $\mathbf{P}$. Then, the AWVC of $\hat{\mathscr{G}}$ provides a lower bound on the WVC degree of $\hat{\mathscr{G}}$, i.e., $\overline{\kappa}(\hat{\mathscr{G}}) \leq \hat{\kappa}(\hat{\mathscr{G}})$.

*Proof:* According to (1), the local WVC metric is obtained based on the solution of the combinatorial optimization problem (2) which gives the optimal path set $\hat{\Pi}_{i,j}$ whose elements represent a set of vertex-disjoint paths, and have the largest sum of the multiplicative path weights. Therefore, any other set of vertex-disjoint paths such as the set of sequential vertex-disjoint most reliable paths from node $i$ to node $j$ in $\hat{\mathscr{G}}$ with length greater than one, denoted by $\overline{\Pi}_{i,j}$, satisfies the following inequality:

$$\sum_{k=1}^{|\overline{\Pi}_{i,j}|} W\left(\overline{\pi}_{i,j}^{k}\right) \leq \sum_{k=1}^{|\hat{\Pi}_{i,j}|} W\left(\hat{\pi}_{i,j}^{k}\right) \tag{9}$$

when $(i, j) \notin \hat{\mathscr{E}}$. It then follows from (9) that $\overline{\kappa}_{i,j}(\hat{\mathscr{G}}) \leq \hat{\kappa}_{i,j}(\hat{\mathscr{G}})$ for every ordered pair of distinct nonadjacent nodes $i, j \in \hat{\mathscr{V}}$. The same result holds when there exists an edge from $i$ to $j$, i.e., $(i, j) \in \hat{\mathscr{E}}$. This implies that the AWVC measure provides a lower bound on the WVC degree. ■

The next proposition identifies the elements of the path set $\overline{\Pi}_{s,t}$, introduced in Definition 7, using the standard shortest path algorithms.

*Proposition 3:* Consider a pair of distinct nodes $s, t \in \hat{\mathscr{V}}$ in the expected communication graph $\hat{\mathscr{G}}$. The problem of finding the most reliable path from $s$ to $t$ in $\hat{\mathscr{G}}$ with the weight matrix $\mathbf{P} = [p_{ij}]$ is equivalent to finding the shortest path connecting $s$ to $t$ in $\hat{\mathscr{G}}$ with the modified weight matrix $\overline{\mathbf{P}} = [\overline{p}_{ij}]$, where $\overline{p}_{ij} = -\log(p_{ij})$ for all $i, j \in \hat{\mathscr{V}}$, $i \neq j$.

*Proof:* Let $\overline{\pi}_{s,t}$ be the desired most reliable path from $s$ to $t$ in $\hat{\mathscr{G}}$ described by the node set $\overline{\pi}_{s,t} = \{\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_{\overline{m}-1}, \overline{v}_{\overline{m}}\}$, where $\overline{v}_0 = s$, $\overline{v}_{\overline{m}} = t$, and $(\overline{v}_{k-1}, \overline{v}_k) \in \hat{\mathscr{E}}$ for all $k \in \mathbb{N}_{\overline{m}}$. Then, the multiplicative weight of $\overline{\pi}_{s,t}$ is given by

$$W(\overline{\pi}_{s,t}) = \prod_{k=1}^{\overline{m}} p_{\overline{v}_k \overline{v}_{k-1}}.$$

Map the elements of $\mathbf{P}$ to the new modified weight matrix $\overline{\mathbf{P}}$ using the relation $\overline{p}_{ij} = -\log(p_{ij})$, for any ordered pair of distinct nodes $(i, j) \in \hat{\mathscr{V}}$. By applying a standard shortest path algorithm (e.g., the Bellman–Ford or Dijkstra algorithm) to $\hat{\mathscr{G}}$ with the modified weight matrix $\overline{\mathbf{P}}$, $\check{\pi}_{s,t}$ is obtained as the shortest path from $s$ to $t$ in $\hat{\mathscr{G}}$, which is the solution to the following minimization problem:

$$\check{\pi}_{s,t} = \underset{\pi \in \Pi_{s,t}}{\operatorname{argmin}} \sum_{k=1}^{m} \overline{p}_{v_k v_{k-1}} = \underset{\pi \in \Pi_{s,t}}{\operatorname{argmin}} \sum_{k=1}^{m} -\log(p_{v_k v_{k-1}}). \tag{10}$$

Using the properties of the logarithm, (10) can be simplified to

$$\check{\pi}_{s,t} = \underset{\pi \in \Pi_{s,t}}{\operatorname{argmax}} \log\left(\prod_{k=1}^{m} p_{v_k v_{k-1}}\right). \tag{11}$$

Since $\log(\cdot)$ is a concave function, it has no effect on the solution of the maximization problem in (11). Thus, (11) can be rewritten as

$$\check{\pi}_{s,t} = \underset{\pi \in \Pi_{s,t}}{\operatorname{argmax}} \prod_{k=1}^{m} p_{v_k v_{k-1}}. \tag{12}$$

It follows from (12) and the definition of $\overline{\pi}_{s,t}$ that $\overline{\pi}_{s,t} = \check{\pi}_{s,t}$. Therefore, the most reliable path from $s$ to $t$ with the weight matrix $\mathbf{P}$ is the shortest path from $s$ to $t$ in digraph $\hat{\mathscr{G}}$ with the modified weight matrix $\overline{\mathbf{P}}$. This completes the proof. ■

We now seek to develop a polynomial-time algorithm to obtain the global AWVC measure $\overline{\kappa}(\hat{\mathscr{G}})$ as a lower bound on the computationally expensive WVC metric $\hat{\kappa}(\hat{\mathscr{G}})$. To this end, the local AWVC measure $\overline{\kappa}_{i,j}(\hat{\mathscr{G}})$ is computed for all ordered pairs of distinct nodes $i, j \in \hat{\mathscr{V}}$ using a standard shortest path method such as Dijkstra's algorithm repeatedly by considering the modified weight matrix $\overline{\mathbf{P}}$ according to Proposition 3. The weighted digraph $\hat{\mathscr{G}}_{i,j} = (\hat{\mathscr{V}}_{i,j}, \hat{\mathscr{E}}_{i,j}, \overline{\mathbf{P}})$ is initially formed by setting $\hat{\mathscr{V}}_{i,j} = \hat{\mathscr{V}}$ and $\hat{\mathscr{E}}_{i,j} = \hat{\mathscr{E}}$. Then, the most reliable path $\overline{\pi}_{i,j}$ from node $i$ to node $j$ in $\hat{\mathscr{G}}_{i,j}$ with length greater than one is identified and $\overline{\pi}_{i,j}$ is added to the path set $\overline{\Pi}_{i,j}$, where $\overline{\pi}_{i,j} = \{\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_{\overline{m}-1}, \overline{v}_{\overline{m}}\}$, $\overline{v}_0 = i$, $\overline{v}_{\overline{m}} = j$, and $(\overline{v}_{k-1}, \overline{v}_k) \in \hat{\mathscr{E}}_{i,j}$ for all $k \in \mathbb{N}_{\overline{m}}$. Then, all internal nodes of $\overline{\pi}_{i,j}$ along with the edges adjacent to them are removed from $\hat{\mathscr{G}}_{i,j}$ such that $\hat{\mathscr{V}}_{i,j} = \hat{\mathscr{V}}_{i,j} \setminus \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_{\overline{m}-1}\}$ and $\hat{\mathscr{G}}_{i,j}$ is updated as a weighted digraph induced by the modified node set $\hat{\mathscr{V}}_{i,j}$. The next step starts by applying Dijkstra's algorithm again to the modified $\hat{\mathscr{G}}_{i,j}$ and repeating the same process until no directed path from $i$ to $j$ with length greater than one exists in $\hat{\mathscr{G}}_{i,j}$. The local AWVC degree $\overline{\kappa}_{i,j}(\hat{\mathscr{G}})$ is then determined according to (8). Finally, by comparing the computed local

**Algorithm 2** Procedure to Find the AWVC of $\hat{\mathscr{G}}$

```
Let κ̄(Ĝ) = |V̂| − 1.
Construct the modified weight matrix P̄ = [p̄_ij].
for all i, j ∈ V̂ and i ≠ j do
    N̄_{i,j}(Ĝ) = 0.
    V̂_{i,j} = V̂ ; Ê_{i,j} = Ê ; Ĝ_{i,j} = Ĝ.
    while there is a path from i to j in Ĝ_{i,j} with length
    greater than one do
        Find π̄_{i,j} by applying Dijkstra's algorithm to Ĝ_{i,j}
        with modified weight matrix P̄ such that
        π̄_{i,j} = {v̄_0, v̄_1, ..., v̄_{m̄−1}, v̄_{m̄}}.
        Find W(π̄_{i,j}) based on the elements of P.
        N̄_{i,j}(Ĝ) = N̄_{i,j}(Ĝ) + W(π̄_{i,j}).
        V̂_{i,j} = V̂_{i,j} \ {v̄_1, v̄_2, ..., v̄_{m̄−1}}.
        Update Ĝ_{i,j} as a graph induced by the modified V̂_{i,j}.
    end while
    if (i, j) ∉ Ê then
        κ̄_{i,j}(Ĝ) = N̄_{i,j}(Ĝ).
    else if (i, j) ∈ Ê then
        κ̄_{i,j}(Ĝ) = max((|V̂| − 1)p_ji, p_ji + N̄_{i,j}(Ĝ)).
    end if
    κ̄(Ĝ) = min(κ̄(Ĝ), κ̄_{i,j}(Ĝ)).
end for
return κ̄(Ĝ).
```

AWVC degrees for all ordered pairs of distinct nodes in $\hat{\mathscr{G}}$, the smallest value, denoted by $\overline{\kappa}(\hat{\mathscr{G}})$, is found. This procedure is elaborated in Algorithm 2.

*Theorem 2:* Algorithm 2 has a time complexity of $O(|\hat{\mathscr{E}}||\hat{\mathscr{V}}|^3+|\hat{\mathscr{V}}|^4\log(|\hat{\mathscr{V}}|))$, where $\hat{\mathscr{V}}$ and $\hat{\mathscr{E}}$ are, respectively, the node set and edge set of the corresponding expected communication graph $\hat{\mathscr{G}}$.

*Proof:* Since the maximum possible number of the sequential vertex-disjoint most reliable paths with length greater than one between any two distinct nodes in $\hat{\mathscr{G}}$ is $|\hat{\mathscr{V}}|-2$, it is required to apply Dijkstra's algorithm $|\hat{\mathscr{V}}|-2$ times, at most, in a sequential manner, to find the path set $\overline{\Pi}_{i,j}$ composed of the sequential most reliable paths from $i$ to $j$, for any ordered pair of distinct nodes $i, j \in \hat{\mathscr{V}}$. Since the fastest implementation of Dijkstra's algorithm has a time complexity of $O(|\hat{\mathscr{E}}|+|\hat{\mathscr{V}}|\log(|\hat{\mathscr{V}}|))$, every local AWVC metric can be obtained in $O(|\hat{\mathscr{E}}||\hat{\mathscr{V}}|+|\hat{\mathscr{V}}|^2\log(|\hat{\mathscr{V}}|))$ time. Also, note that the number of all ordered pairs of distinct nodes in $\hat{\mathscr{G}}$ is of order $O(|\hat{\mathscr{V}}|^2)$. Given the requirement of finding the local AWVC measure for all ordered pairs of distinct nodes $i, j \in \hat{\mathscr{V}}$ in Algorithm 2, the overall time complexity of the algorithm is $O(|\hat{\mathscr{E}}||\hat{\mathscr{V}}|^3+|\hat{\mathscr{V}}|^4\log(|\hat{\mathscr{V}}|))$. This concludes the proof. ■

### A. Computational Example of the WVC and AWVC Metrics

*Example 1:* An illustrative example is given here to demonstrate the required steps for finding the proposed WVC and AWVC measures for a random network composed of six nodes. Consider the expected graph $\hat{\mathscr{G}}$ shown in Fig. 1.

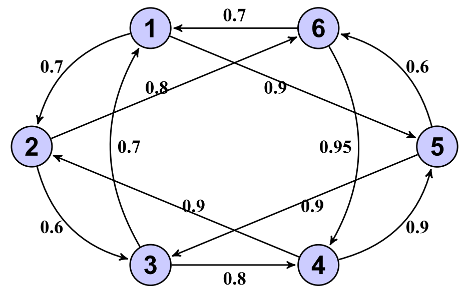


Fig. 1. Expected graph $\hat{\mathscr{G}}$ of Example 1.

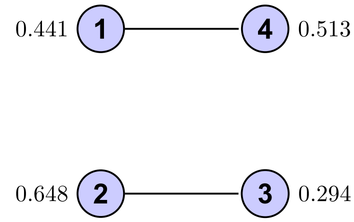


Fig. 2. Weighted undirected path graph $\mathscr{G}^p_{5,2}$ of Example 1.

Note that the existence probability of each link appears as a weight on its corresponding edge in $\hat{\mathscr{G}}$, which yields the following probability matrix $\mathbf{P}$:

$$\mathbf{P}=\begin{bmatrix} 0 & 0 & 0.7 & 0 & 0 & 0.7 \\ 0.7 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0.95 \\ 0.9 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0.6 & 0 \end{bmatrix}.$$

It can be observed from Fig. 1 that at least two vertex-disjoint paths exist between any ordered pair of distinct nonadjacent nodes in $\hat{\mathscr{G}}$, which implies that $\kappa(\hat{\mathscr{G}})=2$. For this example, the minimum local WVC measure corresponds to the directed paths from node 5 to node 2, i.e., $\hat{\kappa}(\hat{\mathscr{G}})=\hat{\kappa}_{5,2}(\hat{\mathscr{G}})$. In order to compute $\hat{\kappa}_{5,2}(\hat{\mathscr{G}})$, it is first required to find the set $\Pi_{5,2}$ containing all distinct paths from node 5 to node 2 with length greater than one in $\hat{\mathscr{G}}$. This results in four different paths $\pi^k_{5,2}$, $k \in \mathbb{N}_4$, as follows:

$$\pi^1_{5,2}=\{5,3,1,2\},\quad \pi^2_{5,2}=\{5,3,4,2\},$$
$$\pi^3_{5,2}=\{5,6,1,2\},\quad \pi^4_{5,2}=\{5,6,4,2\}.$$

By constructing the weighted undirected path graph $\mathscr{G}^p_{5,2}=(\mathscr{V}^p_{5,2},\mathscr{E}^p_{5,2},\mathrm{H}^p_{5,2})$, it can be observed that $\mathscr{V}^p_{5,2}=\{1,2,3,4\}$, $\mathscr{E}^p_{5,2}=\{(1,4),(2,3)\}$, and $\mathrm{H}^p_{5,2}=[0.441\ \ 0.648\ \ 0.294\ \ 0.513]$, where $\mathscr{G}^p_{5,2}$ is shown in Fig. 2. Moreover, the path set $\hat{\mathcal{P}}(\Pi_{5,2})$ is given by

$$\hat{\mathcal{P}}(\Pi_{5,2})=\left\{\pi^1_{5,2},\pi^2_{5,2},\pi^3_{5,2},\pi^4_{5,2},\left\{\pi^1_{5,2},\pi^4_{5,2}\right\},\left\{\pi^2_{5,2},\pi^3_{5,2}\right\}\right\}$$

every element of which corresponds to a clique in $\mathscr{G}^p_{5,2}$. By solving the MWC problem for $\mathscr{G}^p_{5,2}$, one arrives at the optimal path set $\hat{\Pi}_{5,2}=\{\pi^1_{5,2},\pi^4_{5,2}\}$, which results in $\hat{\kappa}_{5,2}(\hat{\mathscr{G}})=W(\pi^1_{5,2})+W(\pi^4_{5,2})=0.954$; this means that $\hat{\kappa}(\hat{\mathscr{G}})=0.954$. In order to find the AWVC metric, note that for this example $\overline{\kappa}(\hat{\mathscr{G}})=\overline{\kappa}_{5,2}(\hat{\mathscr{G}})$, i.e., the minimum local AWVC measure is given by $\overline{\kappa}_{5,2}(\hat{\mathscr{G}})$. By applying Dijkstra's algorithm to $\hat{\mathscr{G}}$

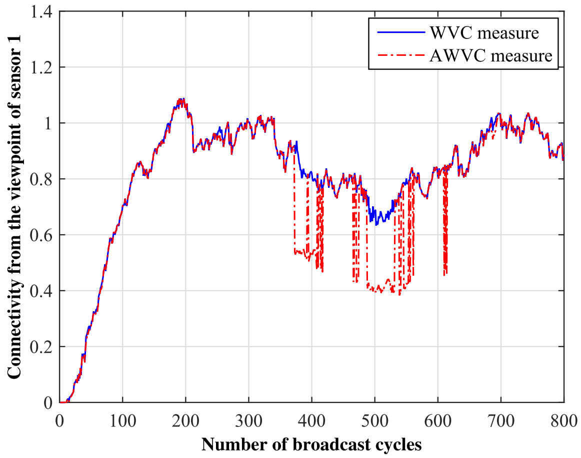


Fig. 3. WVC measure $\hat{\kappa}(\hat{\mathscr{G}})$ and AWVC measure $\overline{\kappa}(\hat{\mathscr{G}})$ from the viewpoint of sensor 1.

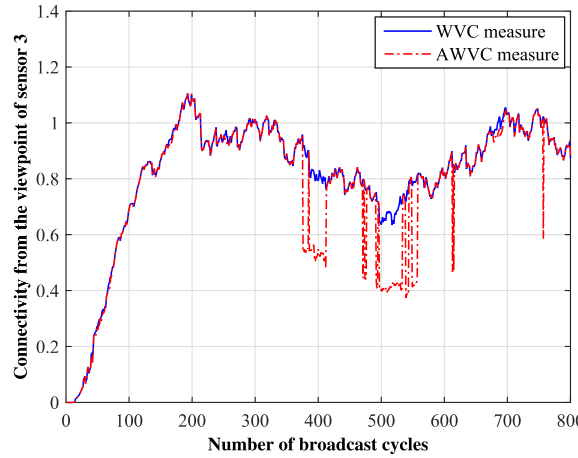


Fig. 4. WVC measure $\hat{\kappa}(\hat{\mathcal{G}})$ and AWVC measure $\overline{\kappa}(\hat{\mathcal{G}})$ from the viewpoint of sensor 3.

with the modified weight matrix $\overline{\mathbf{P}}$, the first most reliable path from node 5 to node 2 is obtained as $\overline{\pi}^1_{5,2} = \pi^2_{5,2}$. After removing the internal nodes of $\overline{\pi}^1_{5,2}$ from $\hat{\mathscr{G}}$ along with the edges adjacent to them and applying Dijkstra's algorithm for the second time, one obtains $\overline{\pi}^2_{5,2} = \pi^3_{5,2}$. Since no path exists from 5 to 2 in $\hat{\mathscr{G}}$ after removing the internal nodes of $\overline{\pi}^2_{5,2}$, it can be concluded that $|\overline{\Pi}_{5,2}| = 2$ and $\overline{\kappa}_{5,2}(\hat{\mathscr{G}}) = W(\pi^2_{5,2}) + W(\pi^3_{5,2}) = 0.942$. Therefore, $\overline{\kappa}(\hat{\mathscr{G}}) = 0.942$, which satisfies the inequality $\overline{\kappa}(\hat{\mathscr{G}}) \leq \hat{\kappa}(\hat{\mathscr{G}})$ as expected.

## VI. Simulation Results

Consider a network of six underwater acoustic sensors which broadcast their data periodically as described in [12]. Assume that the existence probabilities of the communication links of the network are given by the time-varying probability matrix in the following:

$$\mathbf{P}(k) = \begin{bmatrix} 0 & 0 & 0.7 & 0 & 0 & p_{16}(k) \\ p_{21}(k) & 0 & 0 & 0.9 & 0 & 0 \\ 0 & p_{32}(k) & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0.95 \\ 0.9 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & p_{65}(k) & 0 \end{bmatrix}$$

where $k \in \mathbb{N}$ denotes the $k$th broadcast cycle, $p_{16}(k) = 0.7 + 0.25\sin(0.01k)$, $p_{21}(k) = 0.7 + 0.2\sin(0.005k)$, $p_{32}(k) = 0.6 + 0.1\sin(0.01k)$, and $p_{65}(k) = 0.6 + 0.1\sin(0.005k)$. The entries of the probability matrix $\mathbf{P}(k)$ along with the topology of the expected communication graph $\hat{\mathscr{G}}$ are estimated by each sensor using the topology estimation algorithms given in [12]. In Figs. 3 and 4, the proposed WVC and AWVC measures of the expected communication graph $\hat{\mathscr{G}}$ versus the number of broadcast cycles are shown using Algorithms 1 and 2 from the viewpoint of sensors 1 and 3, respectively. The topology of $\hat{\mathscr{G}}$ along with the time-varying probability matrix $\mathbf{P}(k)$ is first estimated by each sensor, and then, the global WVC and AWVC metrics are obtained for the estimated probability matrix $\mathbf{P}(k)$ in Figs. 3 and 4. Note that at some time instants, the estimated AWVC measure is equal to the WVC measure and at some other times it is not. However, the AWVC metric never exceeds the WVC measure, as expected.

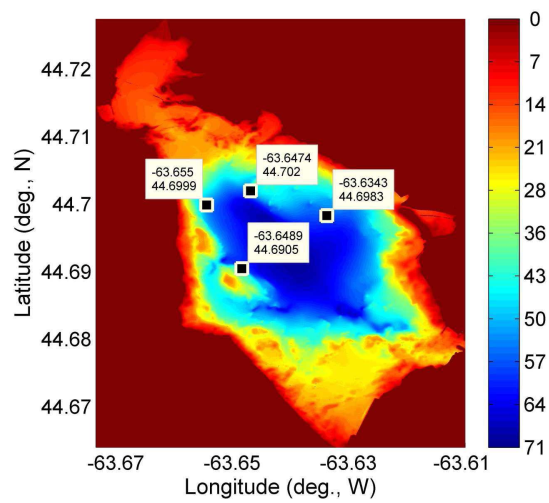


Fig. 5. Location of deployed nodes in Bedford Basin.

TABLE I

Node Locations and the Basin Depth at Mooring Locations

| | [Latitude, Longitude] | Depth (m) | Location |
|---|---|---|---|
| Node 1 | [44.6905 -63.6489] | 50 | Southwest |
| Node 2 | [44.6999 -63.6550] | 52 | Northwest |
| Node 3 | [44.7020 -63.6474] | 52 | Northeast |
| Node 4 | [44.6983 -63.6343] | 57 | Southeast |

There are sudden changes in the estimated AWVC measure as shown in Figs. 3 and 4.

## VII. Experimental Results

The experimental results were obtained during a sea trial conducted in Bedford Basin (N.S., Canada) in July and August of 2014 over a 20-day period. Four nodes were deployed in the Bedford Basin at various locations shown in Fig. 5, where the coordinates of the node locations along with the sea-floor depth at the mooring location of each node is given in Table I. Moreover, the approximate distance separating each node from the others, ranging from 640 m to 1.64 km, is shown in Table II.

Each node was equipped with a radio transceiver, a Global Positioning System, a battery pack, an acoustic modem, two acoustic releases, a mooring, ropes, and cables. The acoustic modem of each node was set at a depth of 5 m below

TABLE II
NODE-TO-NODE DIRECT SEPARATION DISTANCES

| Node 1 to 2 | Node 1 to 3 | Node 1 to 4 |
|---|---|---|
| 1.15km | 1.29km | 1.45km |
| Node 2 to 3 | Node 2 to 4 | Node 3 to 4 |
| 0.64km | 1.64km | 1.11km |

surface to ensure that it would reside in the mixed layer, where acoustic communication is usually difficult to establish. During the trial, various parameters of the acoustic modems, including power level, bit rate, and message length, were altered to investigate their effects on the operational probability of the underwater acoustic communication channels.

All experiments were operated remotely and automatically from Defense Research and Development Canada's (DRDC's) main building through an antenna and a repeater installed on the platform located at DRDC's calibration barge position. The probing signals were modulated messages communicated frequently.

The existence probabilities of all underwater acoustic communication channels for a particular combination of the power level, bit rate, and message length for all nodes were then obtained resulting in the probability matrix **P** as follows:

$$\mathbf{P} = \begin{bmatrix} 0 & 0.36 & 0.35 & 0.47 \\ 0.26 & 0 & 0.58 & 0.11 \\ 0.31 & 0.85 & 0 & 0.31 \\ 0.08 & 0 & 0.09 & 0 \end{bmatrix}. \tag{13}$$

The WVC metric is then obtained as $\hat{\kappa}(\hat{\mathscr{G}}) = 0.105$. It can be verified that in this example the AWVC metric $\overline{\kappa}(\hat{\mathscr{G}})$ is also equal to 0.105. This shows that the network is not well-connected. More experiments with different configurations are planned in order to determine a configuration with a sufficiently high connectivity.

## VIII. CONCLUSION

Connectivity assessment of random sensor networks is investigated in this brief. The WVC is proposed as a novel measure to evaluate the connectivity of a weighted digraph representing the expected communication graph of a sensor network. The elements of the weight matrix denote the operational probabilities of their corresponding communication links in the random network. This measure is, in fact, an extension of the VC metric introduced in the literature. It describes the combined effects of the path reliability and network robustness to node failure on the connectivity of the expected communication graph, which is directly related to the performance of the cooperative algorithms in a random sensor network. An algorithm is presented to compute this metric, which comprises a sequence of algorithms for the IDDFS and MWC problems. A computationally efficient algorithm is subsequently proposed to provide an approximation of the above-mentioned metric, which can be evaluated by applying a series of polynomial-time shortest path procedures.

The results were assessed using real data from an experimental underwater acoustic sensor network, which show the effectiveness of the proposed measures and the corresponding algorithms in practice.

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