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Modeling HF Radio Wave Propagation in 3D Non-Uniform Ionosphere With Smooth Perturbation Method

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ABSTRACT An asymptotic solution of ray equations is used for numerical simulation of the electromagnetic wave propagation in a smoothly non-uniform Earth-ionosphere duct. Being an extension of the adiabatic invariant approach, our method allows one to easily calculate and visualize all descriptive and energetic characteristics of global HF radio waves, including transition between multi-hop and ricochet propagation modes, antipodal focusing, geometric divergence and ohmic absorption. A quasi-parabolic model of F2 layer with the parameters taken from the international model IRI-2012 is used in numerical examples. From the mathematical point of view, it is an example of two-scale asymptotic expansion in a small parameter $\nu \sim H/L \sim L/\pi R$ where H is a characteristic height of the ionospheric layer, L is the scale of its horizontal non-uniformity, R is the Earth radius.

INDEX TERMS Ionosphere, radio wave propagation, ray tracing, asymptotic solution, propagation time, signal attenuation, antipodal focusing.

I. INTRODUCTION

ALTHOUGH high-frequency (HF) radio waves nowadays are gradually losing their importance as a universal communication medium, we cannot underestimate such vital applications as broadcasting over sparsely populated areas, backup emergency messaging, and global ionosphere probing. In this paper we discuss the most theoretically interesting and computationally challenging problem of long-range HF radio wave propagation. Its computational difficulty is determined by a wide hierarchy of parameters ranging from the wavelength $\lambda$of about 15-30 m and the ionospheric layer height $H \sim 150-300$km to the horizontal nonuniformity scale $L \sim 1500-3000$ km and propagation path length up to $\pi R \sim 20000$ km.

A commonly accepted method of ionospheric radio wave simulation is presented by geometric optics (GO) [1]–[2] following from the wave equations for $\lambda/H \sim 10^{-4} \ll 1$. For one-hop radio links, the well-proven approximation – a spherically-layered ionosphere is used, with the electron density height distribution derived from the data of vertical HF radio sounding [3]–[4]. Being a high-frequency asymptotic approximation to Maxwell's equations, geometric optics remains applicable at global distances because the wavelength of decameter radio waves is by three to four orders of magnitude less, compared with the characteristic scales of the ionospheric plasma non-uniformity. Difficulty lies in the accurate calculation of the ray trajectories coming to the observation point, and summing up their contributions, taking into account ray divergence, absorption and EM field enhancement near caustics [5]. A qualitative analysis of long-range HF radio wave propagation can be achieved by applying the adiabatic invariant method [6] based on the similarity between a ray trajectory in a smoothly non-uniform ionospheric wave duct and the oscillatory path of a material particle moving in a slowly varying potential well [7]. With such an approach, it becomes possible not to draw the ray path in detail but derive its slowly varying parameters (reflection heights, angles of arrival, hop lengths, etc.) from approximate conservation laws.

A formal mathematical foundation and further development of the adiabatic invariant concept presents the smooth perturbation approach to the integration of the ray equations developed in our works [8]–[11]. This semi-analytical method, being a kind of the two-scale asymptotic expansion [12], reduces ray tracing time by an order of magnitude and allows one to analyze the global pattern of propagation paths, ray trajectories and signal characteristics with minimum computational burden. Most of the original results, as well as a review paper [13], have been published in Russian and are not familiar to the international radio-physicist community. In what follows, the basic idea of the method is displayed. Omitting the derivation details [8]–[11] we present here the main formulas, illustrated by some examples of asymptotic long-range ray tracing in a commonly adopted International Reference Ionosphere-2012 model [14].

An example of applying this theory to the experimental data obtained at the Ukrainian Antarctic Station Akademik Vernadskii and the idea of using antipodal radio links for monitoring the global ionospheric HF wave duct have been put forward in a joined article [15]. Analytical basis for antipodal sounding and classification of the "ionospheric lens aberrations" was given in our works [16]–[17]. Recently, more papers appeared in the literature, expressing renewed interest to the long-range HF propagation and its practical applications [18]–[20].

II. ASYMPTOTIC ANALYSIS

An idealized global ionospheric layer has the maximum heights varying within the limits $H \sim 150 - 300$ km, and typical horizontal scales $L \sim 1500 - 3000$ km. A two-scale asymptotic expansion in a small parameter $\nu \sim H/L$ allows one to construct an approximate analytical solution of the ray equations and to derive explicit formulas for descriptive and energetic characteristics of the ionospheric HF radio signals. In order to apply the smooth perturbation method [11] to the analysis of long-range radio wave propagation, one has to take into account the effects of the Earth sphericity and to choose an appropriate model of global electron density distribution $N(\vec{r})$. Let $H \sim 300$km be a characteristic height of the ionospheric layer, r – distance from the Earth center, R – its radius, ϑ, φ – geographic latitude and longitude. By neglecting magneto ionic splitting, we write the averaged global distribution of the plasma dielectric permittivity $\varepsilon(\vec{r}) = 1 - 81 \times 10^{-6} N(\vec{r})/f^2 \equiv 1 - f_0^2/f^2$ in modified spherical coordinates $(\zeta, \vartheta, \varphi)$, where $\zeta = RH^{-1}\log(r/R)$. In these variables, all the partial derivatives of the function $\varepsilon(\zeta, \vartheta, \varphi)$ have the order of magnitude about unity, and the ray equations [3]–[4] take the following form:

$$
\begin{aligned}
\nu^2 \frac{d^2\zeta}{d\tau^2} &= \frac{1}{2}\frac{\partial}{\partial\zeta}\tilde{\varepsilon}^2(\zeta, \vartheta, \varphi), \\
\frac{d^2\vartheta}{d\tau^2} + \frac{1}{2}\sin 2\vartheta \cdot \left(\frac{d\varphi}{d\tau}\right)^2 &= \frac{1}{2}\frac{\partial}{\partial\vartheta}\tilde{\varepsilon}^2(\zeta, \vartheta, \varphi), \\
\frac{d}{d\tau}\left(\cos^2\vartheta \frac{d\varphi}{d\tau}\right) &= \frac{1}{2}\frac{\partial}{\partial\phi}\tilde{\varepsilon}^2(\zeta, \vartheta, \varphi). \qquad (1)
\end{aligned}
$$

Here, $\nu = H/R \sim 1/20$ is a small parameter characterizing smooth horizontal variation of the ionosphere, $\tilde{\varepsilon} = (r/R)^2\varepsilon(\zeta, \vartheta, \varphi)$ is the modified dielectric permittivity, $\tau = R\int \varepsilon^{-1/2} r^{-2} ds$ is a dimensionless parameter along the ray path. We look for an asymptotic solution of the equation set (1) in the form of a two-scale expansion [12] in the small parameter ν:

$$
\begin{aligned}
\zeta(\tau) &= \zeta_0(t, \tau) + \nu\zeta_1(t, \tau) + \ldots, \\
\vartheta(\tau) &= \Theta_0(\tau) + \nu^2\Theta_2(t, \tau) + \ldots, \\
\varphi(\tau) &= \Phi_0(\tau) + \nu^2\Phi_2(t, \tau) + \ldots \qquad (2)
\end{aligned}
$$

where $t = \frac{1}{\nu}\int_0^\tau \omega(\tau)d\tau$. The zero-order approximation $\zeta_0(t, \tau)$, as a function of t, is governed by an explicitly integrable ordinary differential equation and can be found by inversion of its integral

$$
\begin{aligned}
t - t_0 &= \pm\omega(\tau)\int \frac{d\zeta_0}{\sqrt{\tilde{\varepsilon}(\zeta_0, \Theta_0, \Phi_0) - q(\tau)}} \\
&\approx \pm\frac{\omega(\tau)}{\nu}\int \frac{dr}{r\sqrt{\tilde{\varepsilon}(z, \Theta_0, \Phi_0) - q(\tau)}}. \qquad (3)
\end{aligned}
$$

Here, $r = R + z$ is the altitude of the ray trajectory, measured from the Earth center. In accordance with the idea of smooth perturbation method [11], the slowly varying "energy" $q(\tau) = Q(\Theta_0, \Phi_0, I)$ (by analogy with classical mechanics [7]) and the "frequency"

$$
\omega(\tau) = \Omega(\Theta_0, \Phi_0, I) = -\frac{\pi}{2}Q_I(\Theta_0, \Phi_0, I)
$$

of the vertical ray oscillation in the Earth-ionosphere duct can be found from the condition of boundedness of the first-order correction $\zeta_1(t, \tau)$. It results in the adiabatic invariant conservation law [12] taking, in spherical geometry, the following form [11]:

$$
\begin{aligned}
I &= \int_{\zeta^-}^{\zeta^+} \sqrt{\tilde{\varepsilon}(\zeta, \vartheta, \varphi) - Q(\vartheta, \phi, I)}d\zeta \\
&= \frac{1}{\nu}\int_{r^-}^{r^+} \sqrt{\tilde{\varepsilon}(r, \vartheta, \varphi) - Q(\vartheta, \varphi, I)}\frac{dr}{r} = Const \\
\frac{\pi}{\Omega(\vartheta, \varphi, I)} &= \int_{\zeta^-}^{\zeta^+} \frac{d\zeta}{\sqrt{\tilde{\varepsilon}(\zeta, \vartheta, \varphi) - Q(\vartheta, \phi, I)}} \\
&= \frac{1}{\nu}\int_{r^-}^{r^+} \frac{dr}{r\sqrt{\tilde{\varepsilon}(r, \vartheta, \varphi) - Q(\vartheta, \varphi, I)}} \qquad (4)
\end{aligned}
$$

The notation $r^{\pm}$ is applied to the solutions of the algebraic equation $\tilde{\varepsilon}(r) = Q$. Formulas (3-4) describe both modes of the ray propagation: "hops" (successive reflections from the ionosphere and the Earth's surface) and "ricochet" (smoothly oscillating rays creeping beneath the concave ionospheric layer). In the former case, $Q < 1$ and the lower integration limit in Eq. (4) is $r^- = 0$; in the latter case $Q > 1$, the lower turning point is given by $\tilde{\varepsilon} \equiv r^2/R^2 = Q(\vartheta, \varphi)$ and the ray passes over the Earth surface at the height $z^- = R(\sqrt{Q} - 1)$.

The averaged horizontal projections of the ray $\Theta_o(\tau), \Phi_0(\tau)$, functions of "slow" variable τ, are governed by the following non-linear differential equations:

$$
\begin{aligned}
\Theta_0'' + \frac{1}{2}\Phi_0'^2 \sin 2\Theta_0 &= \frac{1}{2}\frac{\partial Q}{\partial\vartheta}(\Theta_0, \Phi_0, I), \\
\Phi_0''\cos^2\Theta_0 - \Theta_0'\Phi_0'\sin 2\Theta_0 &= \frac{1}{2}\frac{\partial Q}{\partial\varphi}(\Theta_0, \Phi_0, I), \qquad (5)
\end{aligned}
$$

and the initial conditions for a ray launched from the Earth surface at (ϑ_0, φ_0), with the elevation angle β_0 and azimuth ψ_0, are

$$Q(\vartheta_0, \varphi_0, I) \approx \cos^2\beta_0,\ \Theta_0(O) = \vartheta_0,\ \Phi_0(O) = \varphi_0;$$
$$\Theta_0'(O) \approx \cos\psi_0 \cos\beta_0,\ \Phi_0'(O) \approx \frac{\sin\psi_0 \cos\beta_0}{\cos\vartheta_0}. \quad (6)$$

A simple example of the electron density profile $N(z)$, allowing analytical inversion of the integral (3), presents the so-called quasi-parabolic layer [3]–[4]:

$$N(r, \vartheta, \varphi) = N_m\left[1 - \left(\frac{r_m - r}{r_m - r_1}\right)^2 \left(\frac{r_1}{r}\right)^2\right] \quad (7)$$

If we rewrite the input parameters as $r_m = R + Z_m$, $r_1 = r_m - Y_m$, the above equations will express the vertical component of the ray trajectory $z(\tau) \approx Z_0(t, \tau)$ in terms of the forecasted ionosphere characteristics: critical frequency $f_c(\vartheta, \phi) = 9 \times 10^{-3}\sqrt{N_m}$, maximum height $Z_m(\vartheta, \phi)$ and half-thickness $Y_m(\vartheta, \phi)$ of the F_2 layer. For the modified dielectric permittivity $\tilde{\varepsilon}(\vec{r})$, we obtain the following formulas

$$\tilde{\varepsilon} = \begin{cases} \frac{r^2}{R^2}, & r < r_1 \\ \frac{1}{R^2}\left[\left(\varepsilon_m + \frac{b^2}{r_m^2}\right)r^2 - 2\frac{b^2}{r_m}r + b^2\right], & r > r_1 \end{cases} \quad (8)$$

where $\varepsilon_m = 1 - \rho^2$, $b = \frac{\rho r_1 r_m}{r_m - r_1}$, $\rho = \frac{f_c}{f}$.

The function $\tilde{\varepsilon}(r)$ reaches its minimum, equal to $Q_l = \varepsilon_m r_m^2 b^2 R^{-2}(\varepsilon_m r_m^2 + b^2)^{-1}$, at a height $r_l = b^2 r_m(\varepsilon_m r_m^2 + b^2)^{-1}$. The existence condition of the Earth-ionosphere wave duct or an elevated waveguide, for radio waves of the frequency f, follows from the requirement $r_l > r_1$ and looks as $\rho^2 > \frac{r_m - r_1}{r_m} = \frac{Y_m}{R + Z_m}$.

In the quasi-parabolic model (7), with frozen parameters r_m, r_1, f_k, one from the two aforementioned kinds of ray trajectories takes place, depending on the value of Q parameter. For $Q < 1$, we have ray hops with the turning points inside the ionospheric layer:

$$r^+ = r_m\left[b^2 - R\sqrt{(Q - Q_l)\left(\varepsilon_m r_m^2 + b^2\right)}\right]\left(\varepsilon_m r_m^2 + b^2\right)^{-1}$$

and a sharp reflection from the Earth's surface. The case $Q > 1$ corresponds to ricochet-type trajectories oscillating between r^+ and the lowest point $r^- = RQ^{1/2}$ lying beneath the ionospheric layer. Our asymptotic solution describes both types of propagation modes and the transition between them by a smooth variation of the ionospheric parameters [11].

Applying the quasi-parabolic height distribution of the dielectric permittivity (8) to the equation (4), we obtain an explicit formula for the adiabatic invariant

$$I = \frac{1}{\nu}\int_{r^-}^{r^+} \sqrt{\tilde{\varepsilon}(r, \vartheta, \varphi) - Q(\vartheta, \varphi, I)}\frac{dr}{r}$$
$$= \frac{1}{\nu}\left[\sqrt{Q}\left(\arccos\sqrt{Q_1} - \arccos\frac{R\sqrt{Q}}{r_1}\right) - \sqrt{1 - Q_1}\right.$$
$$+ \frac{\sqrt{b^2 - QR^2}}{R}\operatorname{Ar\cosh}\frac{r_m\left(b^2 - QR^2\right) - r_1 b^2}{Rr_1\sqrt{(Q - Q_l)\left(\varepsilon_m r_m^2 + b^2\right)}}$$
$$- \frac{b^2}{R\sqrt{\varepsilon_m r_m^2 + b^2}}\operatorname{Ar\cosh}$$
$$\left.\times \frac{r_m b^2 - r_1\left(\varepsilon_m r_m^2 + b^2\right)}{Rr_1\sqrt{(Q - Q_l)\left(\varepsilon_m r_m^2 + b^2\right)}}\right] \quad (9)$$

Here, $r^- = R\max(1, \sqrt{Q})$, $Q_1 = \min(1, Q)$. The slowly varying parameter $Q(\vartheta, \varphi, I)$, being an analog of the dielectric permittivity in the horizontal ray equations (5), can be found from the transcendent equation (9). After that, for the slowly varying rate of the vertical ray oscillation in the Earth-ionosphere duct $\omega(\tau) = \Omega(\Theta_0, \Phi_0, I)$ we obtain an explicit formula

$$\frac{\pi}{\Omega} = \frac{1}{\nu}\int_{r^-}^{r^+} \frac{dr}{r\sqrt{\tilde{\varepsilon}(r, \vartheta, \varphi) - Q(\vartheta, \varphi, I)}}$$
$$= \frac{1}{\nu}\left[\frac{1}{\sqrt{Q}}\left(\arccos\frac{R\sqrt{Q}}{r_1} - \arccos\sqrt{Q_1}\right)\right.$$
$$\left. + \frac{R}{\sqrt{b^2 - QR^2}}Ar\cosh\frac{r_m\left(b^2 - QR^2\right) - r_1 b^2}{Rr_1\sqrt{(Q - Q_l)\left(\varepsilon_m r_m^2 + b^2\right)}}\right] \quad (10)$$

The inversion of the integral (3) leads to the following expressions for the vertical component of the ray path:

$$r(t, \tau) = \frac{R\sqrt{Q}}{\cos\left[\nu\frac{\sqrt{Q}}{\Omega}\left(\pi - |t - t_0 - \pi(2n - 1)| + \arccos\sqrt{Q_1}\right)\right]}$$

for

$$\pi - \frac{\Omega}{\nu\sqrt{Q}}\left(\arccos\frac{R\sqrt{Q}}{r_1} - \arccos\sqrt{Q_1}\right) < |t - t_0 - \pi(2n - 1)| < \pi \quad (11)$$

i.e., for $r < r_1$, (continued in Eq. (12) shown at the bottom of the next page).

The hop length or the period of the vertical ray oscillation, corresponding to the increment of the fast variable t by 2π, can be calculated from the equations (9-10): $D = 2\pi H\frac{\sqrt{Q}}{\Omega}$.

III. APPLICATION TO LONG-RANGE HF PROPAGATION

An example of a long-range ray trajectory calculated in adiabatic approximation via formulas (9-12), with the F2-layer parameters taken from the international ionosphere model IRI-2012 [14], is shown in Fig. 1. With such a low initial elevation angle, the major part of the ray trajectory corresponds to the ricochet type, without touching the Earth' surface in its lowest points. Such a propagation mode, eliminating substantial energy loss by each ground reflection (about 2 dB per hop [15]), assures low signal attenuation at a long propagation path.

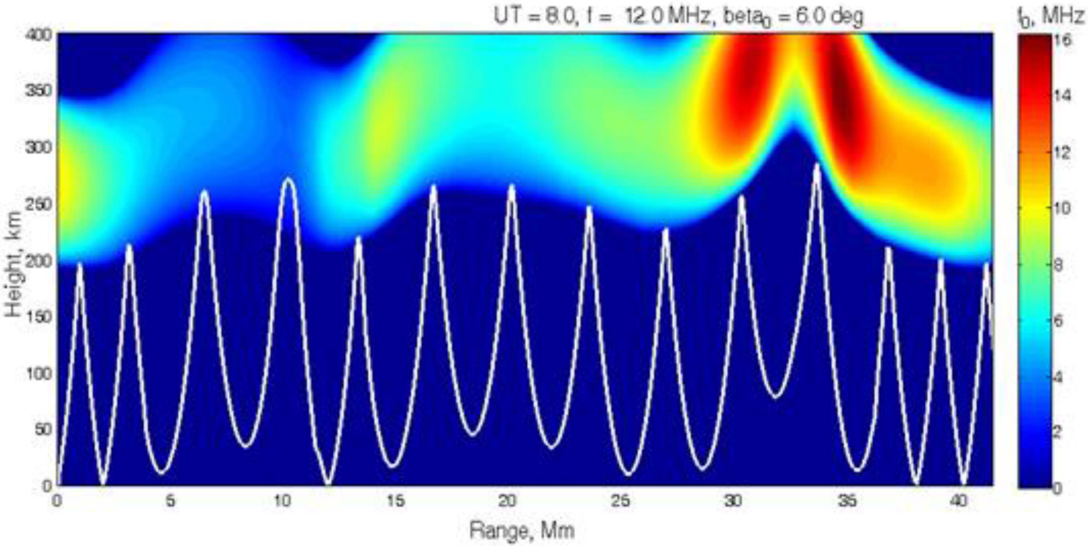

FIGURE 1. Vertical projection of a round-the-world ray, calculated via (9-12).

In an averaged model of the 3D-nonuniform ionosphere, parameters Q and Ω are slowly varying functions of the geographical coordinates (ϑ, φ). In order to find, to the first approximation, their variation along the ray path: $q(\tau) = Q(\Theta_0, \Phi_0, I)$ and $\omega(\tau) = \Omega(\Theta_0, \Phi_0, I)$, one has to solve the averaged equations of the horizontal ray projections (5). By neglecting the transversal gradients of the electron density, the equations (5) can be easily integrated:

$$\Theta_o(\tau, I) \equiv 0, \quad \tau = \int_0^{\Phi_0(\tau,I)} \frac{d\varphi}{\sqrt{Q(\varphi, I)}}.$$

Then, putting $x(\tau) \approx R\,\Phi_0(\tau, I)$ and substituting into (10-11) the obtained expression for the fast variable of the two-scale asymptotic expansion

$$t = \frac{1}{\nu}\int_0^{\tau} \Omega[\Phi_0(\tau, I), I]d\tau,$$

we get an approximate analytic description of the ray trajectory, taking into account the smooth change of the ionospheric parameters along the propagation path. In a general case, the calculation scheme looks like this [11], [16]. The initial value $Q(\vartheta_0, \varphi_0, I) \approx \cos^2\beta_0$ defines the adiabatic invariant for a fan of ray trajectories launched from the point (ϑ_0, φ_0) with a definite elevation angle β_0 and different azimuths ψ_0:

$$I = \frac{1}{\nu}\int_{r^-}^{r^+} \sqrt{\tilde{\varepsilon}(r, \vartheta, \varphi) - \cos^2\beta_0}\,\frac{dr}{r} \tag{13}$$

Having found the slowly varying function $Q(\vartheta, \varphi, I)$ from Eq. (9), we can solve the "horizontal" equations (5) for $\Theta_0(\tau, I, \psi_0)$, $\Phi_0(\tau, I, \psi_0)$ by numerical methods, with a large calculation step. Moreover, in most cases (except the lowest frequencies) it turns out that the function $Q(\vartheta, \varphi, I)$ does not differ essentially from its global average: $Q(\vartheta, \varphi, I) = Q_0(I) + Q_1(\vartheta, \varphi, I)$, $|Q_1| << Q_0$, and we can use the perturbation theory for the lateral ray deviations. The solution takes the simplest form in a special system of spherical coordinates (Θ, Φ) whose equator $\Theta = 0$ coincides with the unperturbed trajectory. The first-order corrections have the following form:

$$\Theta_1(\tau) = \frac{1}{2Q_0}\int_0^{\Phi} \sin(\Phi - \chi)\frac{\partial Q_1}{\partial \Theta}(0, \chi)d\chi \equiv \Theta(\Phi, I, \psi_0),$$
$$\Phi_1(\tau) = \frac{1}{2Q_0}\int_0^{\Phi} Q_1(0, \chi)d\chi. \tag{14}$$

Here, $\Phi = \Phi_0(\tau) = \tau\sqrt{Q_0}$ is the angular length of the non-perturbed trajectory $\Theta_0(\tau) \equiv 0$. With such choice of the variables, the horizontal ray equations do not have singularities due to the spherical coordinates' degeneration at the poles. Coming back to the original coordinates (ϑ, φ) by the formulas of spherical trigonometry we obtain a global representation of the family of rays spreading out from the initial point (ϑ_0, φ_0) with different azimuthal angles ψ_0.

$$r(t, \tau) = \frac{\left(b^2 - QR^2\right) r_m}{R\sqrt{(Q - Q_I)\left(\varepsilon_m r_m^2 + b^2\right)}\cosh\left[\nu\frac{\sqrt{(b^2 - QR^2)}}{R\Omega}(t - t_0 - (2n-1)\pi)\right] + b^2}$$

$$\text{for } |t - t_0 - \pi(2n-1)| < \pi - \frac{\Omega}{\nu\sqrt{Q}}\left(\arccos\frac{R\sqrt{Q}}{r_1} - \arccos\sqrt{Q_1}\right), \ \text{i.e. for } r > r_1 \tag{12}$$

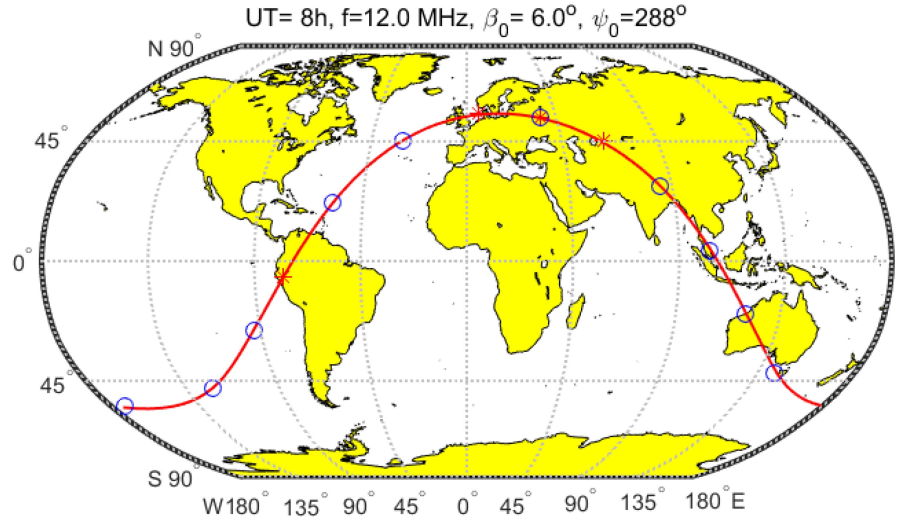

FIGURE 2. A round-the-world trajectory with marked ray minima (circles) and three landing points (stars). Transmitter: 54°N, 37°E; initial elevation angle $\beta_0 = 6°$, azimuth $\psi_0 = 288°$.

Notwithstanding the seeming complexity of the above formulas, numerical solution of the transcendental equation (10) and subsequent calculation of the ray trajectories by the explicit expressions (11), (12), as shown at the bottom of the previous page, reduces the computing time by an order of magnitude, compared with traditional ray tracing codes. This performance gain is especially obvious in the problems of long-range HF radio propagation. Even more important is the fact that the obtained analytic expressions allow one to easily calculate all geometric and energetic characteristics of the radio signal (angles of arrival, propagation time, amplitude, etc.).

For example, the condition that a ray returns to the Earth's surface has the form

$$t \equiv \frac{1}{\nu}\int_0^\tau \Omega[\Phi_0(\tau, I, \psi_0), I]d\tau = 2\pi n \tag{15}$$

where n is the hop number. In order to resolve Eq. (15) one does not require ray tracing – just to mark several "landing points" on a round-the world great circle – see Fig. 2.

By resolving equations (16) and (6) for I and ψ_0 we obtain a number of initial angles (β_0, ψ_0) providing the arrival of the ray at the given point (ϑ_1, φ_1). The vertical angle of arrival is given by the explicit formula $\cos\beta_1 = \sqrt{Q(\vartheta_1, \varphi_1, I)}$. An example of the horizontal ray projection of a round-the-world ray trajectory, with the intermediate points of arrival, is shown in Fig. 2.

In order to find the signal amplitude, we have to calculate the geometric divergence of the family of rays starting from the initial point with different elevation angles and azimuths. Within geometrical optics approximation, it holds

$$A = F(\beta_0, \psi_0)\sqrt{\frac{r^2\cos\beta_0}{RJ}} \tag{16}$$

Here, $F(\beta_0, \psi_0)$ is the scaled radiation pattern, and

$$J(\tau, \beta_0, \psi_0) = \left|\frac{D(x_1, x_2, x_3\}}{D(\tau, \beta_0, \psi_0)}\right| \tag{17}$$

is the Jacobian of the transition from Cartesian $x_1 = r\cos\vartheta\cos\varphi$, $x_2 = r\cos\vartheta\sin\varphi$, $x_3 = r\sin\vartheta$, to the ray coordinates τ, β_0, ψ_0.

The approximate separation of horizontal and vertical variables facilitates the analysis of the ray trajectories.

In particular, the ray divergence factor (17), to the first-order approximation, splits into two multipliers describing the wave focusing in the corresponding planes: $J(\tau, \beta_0, \psi_0) = J_{vert} \cdot J_{hor}$. For the vertical divergence component of the ray fan, spreading from the source (ϑ_0, φ_0) with the fixed elevation angle β_0 and coming to the Earth surface at (ϑ_1, φ_1) point, the two-scale asymptotic method yields the following expression

$$\begin{aligned} J_{vert} &= \frac{1}{\nu r}\left|r_{\beta_0}\Phi_\tau - r_\tau\Phi_{\beta_0}\right|_{r=R} \\ &= \frac{\partial I}{\partial\beta_0}\frac{\sqrt{Q(\vartheta_1, \varphi_1, I)[1 - Q(\vartheta_1, \varphi_1, I)]}}{\nu\Omega(\vartheta_1, \varphi_1, I)} \end{aligned} \tag{18}$$

In order to study the divergence of the horizontal ray projections, it is convenient to use the approximate solution (14). As the ray deviations from the unperturbed great circle are relatively small, the horizontal divergence $J_{hor} = \cos\vartheta|r_{\psi_0}\varphi_\Phi - r_\Phi\varphi_{\psi_0}|$, as a rule, is close to the standard factor of spherical focusing $\sin\Phi$. An exception present the vicinities of the launching point and its antipode. Despite the smallness of the corrections Θ_1, Φ_1, expressed by the formulas (14), they lead to substantial effects in the vicinity of the transmitter antipode $(-\vartheta_0, \varphi_0 + \pi)$ and the initial point (ϑ_0, φ_0) itself, after the round-the-world propagation. If the ionosphere were spherically symmetric, at these points one would observe a field maximum caused by coherent summation of the signals coming from all directions.

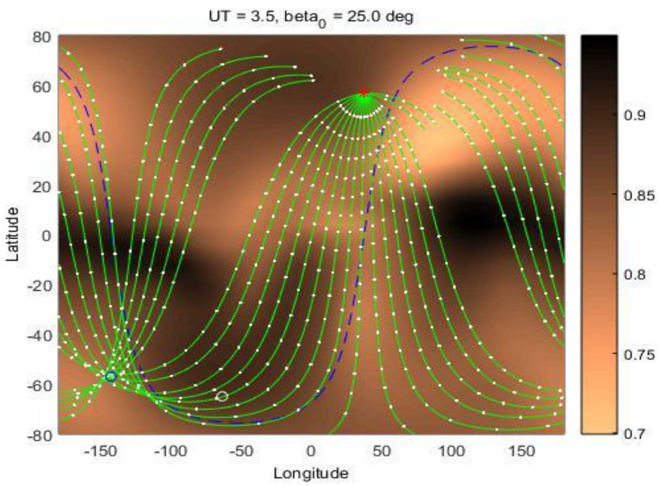

FIGURE 3. **Horizontal projections of rays emitted from transmitter point (star) with a fixed elevation angle $\beta_0 = 25^{\circ}$. Antipode is marked with a dark circle, white circle – UAS "Akademik Vernadsky"; dashed line indicates solar terminator. Color background depicts global $Q(\vartheta,\varphi,I)$ distribution.**

Horizontal gradients destroy the diffraction focus, leading to the formation of a more complicated focal spot with characteristic dimensions of the order of 1-2 thousand kilometers. Its structure can be analyzed by means of formulas (14).

Figure 3 presents an example – horizontal projections of the rays emitted from the transmitter point with a fixed elevation angle $\beta_0$and different azimuths ψ_0. The color background depicts the "horizontal permittivity" $Q(\vartheta, \varphi, I)$derived from the IRI-2012 ionosphere model, using the adiabatic invariant (9), for a given value of the initial elevation angle $\beta_0 = 25^{\circ}$(in this example $Q = \cos^2\beta < 1$ – all the rays propagate by hops). The depicted rays span the southern angular sector, calculation of the northern trajectories would require a modified polar ionosphere model.

The most pronounced feature of the horizontal ray pattern in Fig. 3 is an extended antipodal focal spot bounded with a triangle caustic of about 2000 km in diameter. Our smooth perturbation method allows one to describe this phenomenon analytically. The caustic surrounding the antipodal focal spot is determined by the equation

$$\frac{\partial\vartheta}{\partial\Phi}\frac{\partial\varphi}{\partial\psi_0} - \frac{\partial\varphi}{\partial\Phi}\frac{\partial\vartheta}{\partial\psi_0} = 0 \tag{19}$$

which can be easily derived by linearizing formulas of spherical trigonometry with respect to the small difference $\Phi - \pi$. As the function $\Theta(\pi, \psi_0, I)$ is, evidently, 2π-periodic with respect to the initial azimuth ψ_0, the caustic (19) is a closed curve, maybe self-crossing. Due to horizontal non-uniformity of the ionosphere, the focal spot may have a complicated structure. A full classification of antipodal and round-the world ray structures on a non-uniform sphere was given in [16]–[17]. Whereas outside the focal area the wave field is determined by the interference of two rays corresponding to the minimal and maximal values of the optical path, the number of rays increases by two with each crossing the caustic. With the possible caustic self-crossing, its interior may be divided into the parts with 4, 6 and more directions of arrival and different field amplitudes corresponding to the wave divergence and absorption along each ray. Besides, at the caustic itself and its sharp points one has to expect the wave field enhancement determined by the competing effects of geometric focusing and diffraction blurring [5]. By expanding the function $\theta(\pi, \psi_0, I)$ in a Fourier series in ψ_0 parameter, we obtain a useful representation of the antipodal focal pattern as a superposition of elementary aberrations of the global‘ "ionospheric lens" [11], [16]–[17]. A similar classification of round-the-world focal structures is obtained by expanding $\theta(2\pi, \psi_0, I)$ in a Fourier series in ψ_0. The complicated ray structure of the EM wave field inside the antipodal and round-the world focal spot explains multiple azimuths of the received long-range HF signals and their diurnal dynamics [21]–[22].

IV. PROPAGATION TIME, SIGNAL ATTENUATION

In the problems of oblique and backward ionosphere sounding [3]–[4], [9], it is necessary to calculate the propagation time of the radio signal, proportional to the group path $P = \int \frac{ds}{\sqrt{\varepsilon}} = \frac{1}{R}\int r^2 d\tau$. The two-scale expansion method opens an easy way to the calculation of this integral, as well as of similar integral representations for the phase path and ohmic absorption. By denoting $r(\tau) = R\rho(t - t_0, \tau)$, where $\rho(t, \tau)$ is an even periodic function of the fast variable t, we derive an explicit formula, [13]:

$$\begin{aligned} P(\tau) = R\sum_{n-0}^{\infty}\int_0^{\tau} C_n(\tau)\cos\left[\frac{1}{\nu}\int_0^{\tau}\omega(\tau')d\tau' - t_0\right]d\tau \approx R \\ \times\left\{\int_0^{\tau} C_0(\tau)d\tau + \nu\sum_{n-0}^{\infty}\right. \\ \times\left[\int_0^{\tau}\frac{C_n(\tau)}{n\omega(\tau)}\sin\left(\frac{1}{\nu}\int_0^{\tau}\omega(\tau')d\tau' - t_0\right)\right. \\ \left.\left. + \frac{C_n(0)}{n\omega(0)}t_0\right]\right\} \end{aligned} \tag{20}$$

Here $C_n(\tau)$ are the Fourier coefficients of the function $\rho^2(t, \tau)$. By summing the Fourier series we obtain the final expression for the group path:

$$\begin{aligned} P(\tau) = \frac{1}{R}\left\{\int_0^{\tau}\overline{r^2(\tau)}d\tau + \frac{\nu}{\omega(0)}\int_0^{\tau}\left[r^2 - \overline{r^2(0)}\right]d\tau\right. \\ \left. + \frac{\nu}{\omega(\tau)}\int_0^{\tau}\left[r^2 - \overline{r^2(\tau)}\right]d\tau\right\} \end{aligned} \tag{21}$$

containing a monotonously growing integral of the squared radius average value

$$\begin{aligned} \overline{r^2(\tau)} \equiv R^2 C_0(\tau) = \frac{R^2}{2\pi}\int_{-\pi}^{\pi}\rho^2(t, \tau)dt \\ = \frac{\omega(\tau)}{\pi\nu}\int_{r^-}^{r^+}\frac{rdr}{\sqrt{\tilde{\varepsilon}(r, \Theta_0, \Phi_0) - Q(\Theta_0, \Phi_0, I)}} \end{aligned} \tag{22}$$

and oscillating corrections of the order of $O(\nu)$.

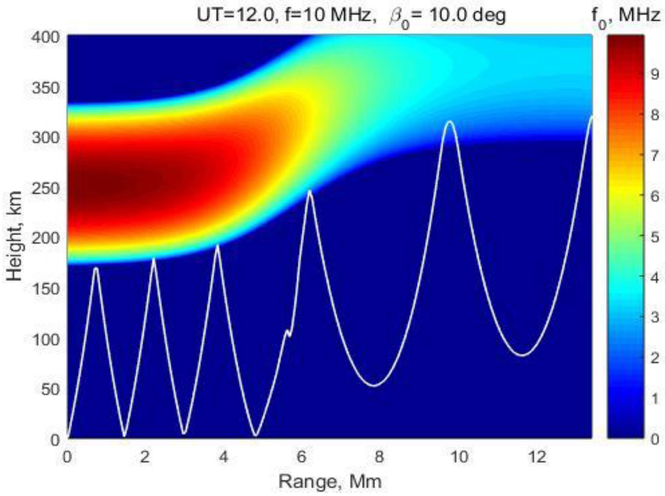

FIGURE 4. Gradual transition from multi-hop propagation to ricochet-type trajectory.

In a similar way, the mean absorption on a long-range path can be found [13]:

$$E \sim A \exp[-\Gamma(\tau)], \ \ \Gamma(\tau) = \frac{e^2}{mcf} \int \frac{\nu_{eff} N}{\sqrt{\varepsilon}} ds$$
$$\simeq \frac{e^2}{mcfR} \int \overline{\nu_{eff} N r^2} d\tau + O(\nu) \tag{23}$$

Here, e and m are the electron charge and mass, $N(r, \vartheta, \varphi)$ and $\nu_{eff}(r, \vartheta, \varphi)$ are the electron density and collision rate in the ionospheric plasma; the mean value is defined as in Eq. (23):

$$\overline{\nu_{eff} N r^2(\tau)} = \frac{\omega(\tau)}{\pi \nu} \int_{r^-}^{r^+} \frac{\nu_{eff} N r dr}{\sqrt{\tilde{\varepsilon}(r, \Theta_0, \Phi_0) - Q(\Theta_0, \Phi_0, I)}} \tag{24}$$

With the parameters N_m, r_m, r_1 gradually changing along the propagation path, a radical change of the propagation mode can take place – the detachment of the ray trajectory from the Earth surface. The naïve use of the asymptotic solution (2)-(5) in this situation leads to a noticeable lost of smoothness, due to a non-analytic behavior of the “hop” branch ($Q < 1$)of the solution (9)-(12) at the transition points $Q(\vartheta, \varphi, I) = 1$ - see Fig. 4. In our works it was shown that a good accuracy can be achieved by analytic continuation of the “ricochet” branch to the Earth surface and matching it with the last ray hop by the law of specular reflection, which leads to a to a small jump of the phase lag t_0 and a minor correction to the adiabatic invariant value I [10], [23]:

$$\Delta t_0 = \frac{\Omega}{\nu\sqrt{Q}} \arccos\sqrt{Q} \approx \frac{\Omega\sqrt{1-Q}}{\nu\sqrt{Q}} = O\left(\nu^{1/4}\right),$$
$$\Delta I = O\left(\nu^{5/4}\right) \tag{25}$$

Furthermore, asymptotic analysis shows that, in order to preserve the solution accuracy at large distances from the transmitter, one has to introduce small corrections $O(\nu)$ in the initial conditions [10]. However, in the problems of ionospheric HF propagation these purifications do not change the global pattern of ray trajectories. A more essential effect is related to the electromagnetic wave refraction in the ionospheric E-layer, which requires adding an extra term to the quasi-parabolic F2 model (7).

V. CONCLUSION

We have presented an analytical approach to the problem of long-range HF electromagnetic wave propagation, based on asymptotic solution of the ray equations by smooth perturbation method (two-scale expansion in a small parameter characterizing horizontal gradients of the ionospheric plasma). The obtained approximate solution, combined with a standard ionosphere model, facilitates global ray tracing and calculation of radio signal physical parameters. It quantitatively describes angular and amplitude variations of long-range HF radio signals propagating in the non-unifofm and non-stationary ionospheric wave duct. An example of global ray tracing on the Moscow–UAS Akademik Vernadsky radio link [22] explains Doppler spectrum of a harmonic HF signal, observed at the Ukrainian Antarctic Station [15], by a propagation path splitting in the vicinity of moving solar terminator.

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