A SIMULATION AND GRAPH THEORETICAL ANALYSIS OF CERTAIN PROPERTIES OF SPECTRAL NULL CODEBOOKS

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Abstract: The spectral shaping technique and the design of codes providing nulls at rational sub-multiples of the symbol frequency, as the case with spectral null (SN) codes, have enhanced digital signaling over communication channels as digital mass recorders and metallic cables. The study of the special structure of these codes helps in investigating and analyzing certain of their properties which have been proved and emphasized from a mathematical perspective using graph theory. The cardinality of spectral null codebooks reflects the rate of spectral null codes and therefore the amount of transmitted information data. The rate of these codes can also play a role in their error correction capability. The paper presents in different ways the special structure of spectral null codebooks and analyze better their properties. A possible link between these codes and other error correcting codes as the case of Low Density Parity Check (LDPC) is presented and discussed in this paper.

Key words: Spectral shaping, spectral null codes, error correcting codes.

1. INTRODUCTION

The design of a code having power spectral density (PSD) zero at its DC-component, called DC-free codes [1, 2], becomes a necessity for AC coupling of the signal to the medium. DC-balanced codes have found widespread applications in digital transmission and recording systems [3]– [5]. DC balance can be achieved by using an appropriate transmission code or by balancing each transmitted symbol. Any drift in the transmitted signal from the center baseline level, due to an uncontrolled running digital sum (RDS) or the effects of an AC coupling, will create a DC component, which is known as baseline wander [6], or create an intersymbol interference, which is caused by the AC coupling at various points in the communication channel [4]. In some applications low-frequency channel noise, such as a fingerprint on an optical disk [7] or impulse noise due to dial pulses in a subscriber loop plant, can be filtered out by sending the encoded data through a high pass filter. To minimize the effect of this filtering on the symbol shape of the coded sequence, the encoded data stream must have very little or no DC or low-frequency component. Also magnetic recording systems often require that the channel sequences have a spectral null at zero frequency. This technique is called the spectral shaping technique or the design of nulls at certain specific frequencies in a spectrum.

Spectral null codes are codes with simultaneous nulls at the rational submultiples of the symbol frequency and have great importance in certain applications like in the case of transmission systems employing pilot tones for synchronization and that of track-following servos in digital recording [8, 9].

The paper is organized as follows. In Section 2 we present two different design techniques of spectral null codebooks. Section 3 emphasizes better the relationship in the calculation of the cardinality of the codebook and its corresponding spectral null equation. Section 4 derives and presents proofs of certain properties of spectral null codes. A link and approach between spectral null codebooks and LDPC codes is presented in Section 5. We conclude with an analysis of these properties in Section 6.

2. SPECTRAL NULL CODES DESIGN

In this section we present two different techniques for designing spectral null codes based on the calculation of the power spectral density function and the binary representation of permutation sequences.

2.1 Using Gorog Construction

Gorog [10] was first to simplify and formulate the way of calculating the values of the frequencies for spectral null codes. To calculate the value of the frequencies at the corresponding nulls at the rational submultiples of the symbol frequency f_c for block codes, he considered the vector $y = (y_1, \ldots, y_M)$, $y_i \in \{-1, +1\}$, to be an element of a set S, which is called the codebook of codewords with elements in $\{-1, +1\}$. For the sake of simplification and good presentation, we represent -1 with a 0. Applying the Fourier transform to those codewords we get [10]:

$$Y = \sum_{i=1}^{M} y_i e^{-jiw}, -\pi \leq w \leq \pi. \quad (1)$$

The power spectral density function denoted by $H(w)$ of the concatenated sequence when transmitted serially [11] is defined as:

$$H(w) = \frac{1}{C_S M} \sum_{i=0}^{M-1} \left|Y^i(w)\right|^2, \quad (2)$$

where $Y^i(w)$ is the Fourier transform of the i-th element of S and C_S is the cardinality of S. Having nulls at certain frequencies is the same as having the power spectral density function $H(w)$ equal to zero at those frequencies [7].

A sequence of length M having a null at the frequency $f = \omega/2\pi = 1/N$, with N an integer, means that it is sufficient to satisfy $|Y(2\pi/N)| = 0$. For purposes of simplification we choose the codeword length M as an integer multiple of N, where $f = r/N$ represents the spectral nulls at rational submultiple r/N. The parameter N could be chosen either prime or not prime and divides M [7], i.e.

$$M = Nz. \tag{3}$$

We denote the vector amplitudes by the summation:

$$A_i = \sum_{r=0}^{z-1} y_{i+rN}, \quad i = 1,2,3,\ldots,N. \tag{4}$$

In the case where N is a prime number [12], we have to satisfy [13],

$$A_1 = A_2 = \cdots = A_N, \tag{5}$$

where A_i is the same as in (4). As an example, if $N = 3$ and $M = 6$, the following relationship must hold,

$$\begin{array}{ccccc} A_1 & = & A_2 & = & A_3, \\ y_1 + y_4 & = & y_2 + y_5 & = & y_3 + y_6. \end{array}$$

Definition 1 A spectral null binary block code of length M is any subset $C_b(M,N) \subseteq \{0,1\}^M$ of all binary M-tuples of length M and have spectral nulls at the rational submultiples of the symbol frequency $1/N$.

For codewords of length M consisting of N interleaved subwords of length z, the cardinality of the codebook $C_b(M,N)$ for the case of N considered as a prime number is presented by the following formula [14],

$$|C_b(M,N)| = \sum_{i=0}^{M/N} \left(\begin{array}{c} M/N \\ i \end{array} \right)^N, \tag{6}$$

where $\left(\begin{array}{c} M/N \\ i \end{array} \right)$ denotes the combinatorial coefficient $\frac{(M/N)!}{i!(M/N-i)!}$.

Example 1 The spectral null codebook for $N = 2$ and $z = 2$ is:

$$C_b(4,2) = \left\{ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right\}.$$

The cardinality of this codebook $C_b(4,2)$ is clearly equal to 6, which could be verified from (6). The spectrum is shown in Fig. 1, where we can see the null appearing at the frequency $1/2$ since $N = 2$.

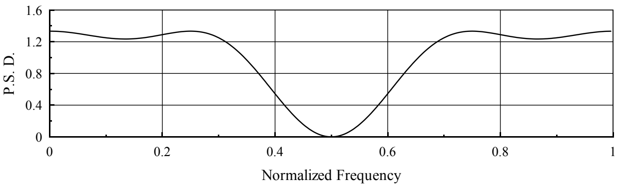

Figure 1: Power spectral density of codebook $N = 2$; $M = 4$.

In the case where N is not prime we have to suppose that $N = cd$, where c and d are integer factors of N. The equation, which leads to nulls, is

$$\begin{array}{l} A_u = A_{u+vc}, \\ u = 0,1,2,\ldots,c-1, \\ v = 1,2,\ldots,d-1, \\ N = cd, \end{array} \tag{7}$$

where A_u is the same as in (4). The complete spectral null codebook for a given N is the union of the solutions to (7) for each possible pair of factors. For example, if $N = 12$, it can be written as the following products: 2×6, 6×2, 3×4 and 4×3 [15].

Example 2 If we take $N = 4$ and $M = 8$, we have the following relationships:

$$\begin{array}{ll} A_1 = A_3, & y_1 + y_4 = y_3 + y_5, \\ A_2 = A_4, & y_2 + y_6 = y_4 + y_8. \end{array} \tag{8}$$

We expect that the null will appear at the frequencies $1/4$ and $3/4$ of the normalized frequency since $N = 4$. The spectrum is shown in Fig. 2.

The corresponding spectral null codebook is:

$$C_b(8,4) = \left\{ \begin{array}{l} 00000000, 00000101, 00001010, 00001111, \\ 00010100, 00011110, 00101000, 00101101, \\ 00111100, 01000001, 01001011, 01010000, \\ 01010101, 01011010, 01011111, 01101001, \\ 01111000, 01111101, 10000010, 10000111, \\ 10010110, 10100000, 10100101, 10101010, \\ 10101111, 10110100, 10111110, 11000011, \\ 11010010, 11010111, 11100001, 11101011, \\ 11110000, 11110101, 11111010, 11111111 \end{array} \right\}.$$

2.2 Using Permutation Sequences

We consider permutation sequences written in the passive form, such as $12\ldots M$, where each of the symbols are written as a binary sequence of length M, with the

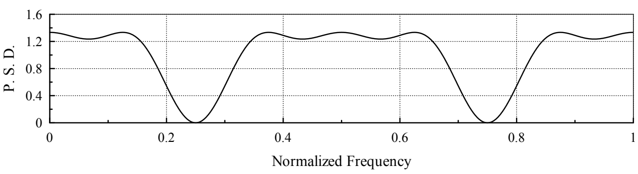

Figure 2: Power spectral density of codebook $N = 4$; $M = 8$.

symbol value indicating where a 1 is to appear and zeros everywhere else (similar to pulse position modulation). For example, if we take $M = 3$, we have

$$\begin{array}{ccccc} 1 & \rightarrow & 1 & 0 & 0, \\ 2 & \rightarrow & 0 & 1 & 0, \\ 3 & \rightarrow & 0 & 0 & 1. \end{array} \tag{9}$$

The permutation sequences for $M = 3$ are thus changed to the binary form as follows:

$$\begin{Bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{Bmatrix} \rightarrow \begin{Bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{Bmatrix}$$

Therefore, each of the $M!$ permutation sequences can be converted to binary sequences of length M^2.

An alternative representation is that of (0,1)-matrices, where only a single 1 is allowed in every column and every row. For example, the permutation sequence 231 will be

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{10}$$

The binary sequence representation of the permutation sequence 231 is then constructed by concatenating the columns to form 010001100. The matrix in (10) has only one single 1 in each row and each column.

We denote by $P_\omega(M^2)$ the binary permutation code that contains all the binary sequences of length M^2 as a result of the conversion of the permutation sequences of length M to binary sequences. The value of ω represents the weight of the binary sequences in each row and each column. For the case of $\omega = 1$, as in the matrix presented in (10), the cardinality of the code $P_1(M^2)$ is $|P_1(M^2)| = M!$.

For the case of $\omega = 2$, the (0,1)-matrix can be constructed from two $\omega = 1$ (0,1)-matrices by XOR-ing them, as shown below

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \tag{11}$$

or equivalently $100010001 \oplus 001100010 = 101110011$ for the binary sequences.

In general, we will use $P_\omega(M^2)$ to denote the code containing all the possible binary sequences that are obtained from (0,1)-matrices with ω 1s in each row and each column.

It is clear that for $P_1(3)$ and $P_1(4)$, we have spectral null codes with nulls at frequency multiples of $1/3$ and $1/4$ respectively, as depicted in Fig. 3 and 4, in addition to it not being DC-free.

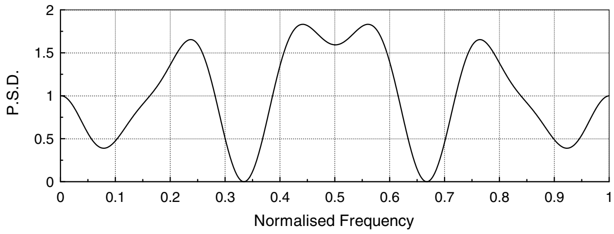

Figure 3: Power spectral density of $P_1(3)$

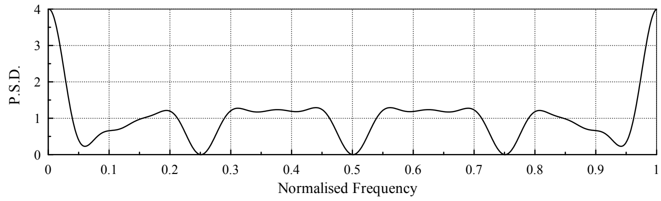

Figure 4: Power spectral density of $P_1(4)$

3. COMPUTATION OF THE SPECTRUM

We present in this section a few examples of designed spectral null codebooks where we compute their cardinalities based on their spectral null equations defined in (5) and (7).

The value of N, can be prime or non prime. In the following section we limit our work only on the case of N prime since the other one case be derived similarly.

In the case of N prime, we substitute (4) into (5), and we get:

$$\begin{aligned} \overbrace{y_1 + \cdots + y_{1+(M-N)}}^{M/N} &= y_2 + \cdots + y_{2+(M-N)} \\ &= \cdots \\ &= y_N + \cdots + y_M \end{aligned} \tag{12}$$

It is clear from (12) that the codeword of length M consists of N groupings of subwords of length $z = M/N$. We can rewrite (4) as follow:

$$A_i = \sum_m y_m, \quad i = 1, 2, \ldots, N, \tag{13}$$

where $m \in \{i, i+N, i+2N, \ldots, i+(M/N-1)N\}$, with $1 \leq i \leq N$.

It is also clear from (12) that the value A_i is the sum of "M/N" binary elements, which could be presented in a limited form as follow:

$$A_i \in \{-M/N, -M/N+2, \ldots, M/N-2, M/N\} \tag{14}$$

A Matlab © program, based on an exhaustive search, was used to calculate all possible binary codewords corresponding to different combinations of A_i as presented in (14). A few results of our Matlab exhaustive search will be presented later in Table 1.

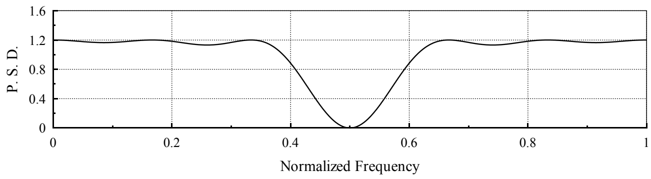

Figure 5: Power spectral density of codebook $N=2, M=6$.

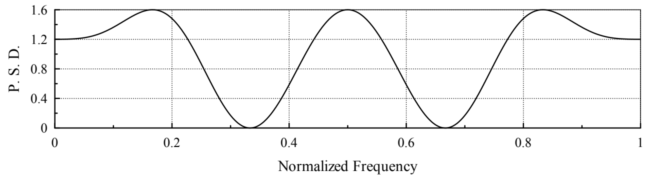

Figure 6: Power spectral density of codebook $N=3$, $M=6$.

To satisfy (12), we need to have the same sum value of the addition of the M/N elements in all different groupings A_i. Thus the number of the binary sequences or binary codewords, which satisfy (12) is the number of codewords in the codebook $C_b(M,N)$ of the spectral null code.

Following are few examples of $C_b(M,N)$ codebooks with their power spectral densities graphs for different values of M and N.

Example 3 For $M=6$ and $N=2$, we have

$$\underbrace{\overbrace{y_1+y_3+y_5}^{3} = y_2+y_4+y_6}_{2}. \tag{15}$$

The cardinality of the codebook $C_b(6,2)$ is the result of a number of combinations that satisfy (15). The value of each grouping A_i could be -3, $+3$, -1 or $+1$ since we are dealing with binary sequences. We can see from (15) that there is one combination of six bits, $A_1=A_2=-3$, when all the elements in the groupings are equal to -1 and another combination, $A_1=A_2=+3$ when all the elements in the groupings are equal to $+1$. There is another combination which yields $A_1=A_2=-1$ and another one which is $A_1=A_2=+1$. The last two combinations are in fact a result of a permutation of the three elements in each grouping, Thus the number of combinations is equal to $3^2=9$. Finally the total number of combinations is $1+1+3^2+3^2=20$, which is in fact equal to the cardinality of the codebook $C_b(6,2)$.

The spectral shaping codebook for $N=2$ and $z=3$ is:

$$C_b(6,2)=\left\{\begin{matrix} 000000,000011,000110,001001,001100,\\ 001111,010010,011000,011011,011110,\\ 100001,100100,100111,101101,110000,\\ 110011,110110,111001,111100,111111 \end{matrix}\right\}.$$

We can see that the total number of codewords in the codebook $C_b(6,2)$ found by our computer search is the same found by our combinatorial analysis. The spectrum is shown in Fig. 5. Since $N=2$, we expect that the null will appear at the frequency $1/2$ of the normalized frequency. This is confirmed in Fig. 5.

Example 4 For $M=6$ and $N=3$, we have

$$\underbrace{\overbrace{y_1+y_4}^{2} = y_2+y_5 = y_3+y_6}_{3}. \tag{16}$$

Using a similar approach for the codebook $C_b(6,3)$, we note from (16) that the value of each grouping A_i could be -2, $+2$ or 0 since the elements in each grouping are binary bits. We can see from (15) that there is one combination of six bits, $A_1=A_2=A_3=-2$, when all the elements in the groupings are equal to -1 and another combination such that $A_1=A_2=A_3=+2$, when all the elements in the groupings are equal to $+1$. There is another combination which yields $A_1=A_2=A_3=0$. The last combination is in fact a result of a permutation of the two elements in each groupings, so the total number of combinations is $2+2+2=6$. Taking into consideration the permutation of the three groupings A_1, A_2 and A_3, which still satisfy the relationship $A_1=A_2=A_3=0$, we find that the number of combinations is 2. Finally, the total number of combinations is $1+1+2+2+2+2=10$, which is the cardinality of the codebook $C_b(6,3)$. The spectral shaping codebook for $N=3$ and $z=2$ is:

$$C_b(6,3)=\left\{\begin{matrix} 000000,000111,001110,010101,011100,\\ 100011,101010,110001,111000,111111 \end{matrix}\right\}.$$

The total number of codewords in the codebook $C_b(6,2)$ found by our computer search is the same found by our combinatorial analysis. The spectrum is shown in Fig. 6. Since $N=3$, we expect that the nulls will appear at the frequencies $1/3$, $2/3$ of the normalized frequency. This is confirmed in Fig. 6.

Table 1 summarizes few results of the values of cardinalities and their corresponding values of N and z. It is important to mention that the cardinality plays a role in leading to have an idea about the code rate which might be helpful in the improvement of the error correction capability of the code. The cardinality also can be increased by satisfying the spectral null equation and having more codewords in the spectral null codebook.

4. PROPERTIES OF SPECTRAL NULL CODES

4.1 Complementary Symmetry of Codewords

From a simple observation from the design of spectral null codes, we can see that their codebooks are usually half-complement symmetrically. We discuss this property for N prime only. The case of N not prime is similar.

Proposition 1 *For any spectral null codebook* $C_b=\{\forall y_i \in \{-1,+1\}/A_1=\cdots=A_N\}$, *there exists a subset* C'_b, *where* $\overline{C'_b}$ *is a subset of* C_b.

Table 1: Cardinalities for Codeword length $M = Nz$ and spectral null at $f = 1/N$ with N prime

M	N	z	Cardinality	Spectral Null Frequencies
4	2	2	6	1/2
6	2	3	20	1/2
8	2	4	70	1/2
10	2	5	252	1/2
12	2	6	924	1/2
14	2	7	3432	1/2
16	2	8	12870	1/2
18	2	9	48620	1/2
20	2	10	184756	1/2
6	3	2	10	1/3, 2/3
9	3	3	56	1/3, 2/3
12	3	4	346	1/3, 2/3
15	3	5	2252	1/3, 2/3
18	3	6	15184	1/3, 2/3
10	5	2	34	1/5, 2/5, 3/5, 4/5
15	5	3	488	1/5, 2/5, 3/5, 4/5
20	5	4	9826	1/5, 2/5, 3/5, 4/5
25	5	5	206252	1/5, 2/5, 3/5, 4/5

PROOF For all $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_M}) \in \mathbf{C'_b}$ we have:

$$\begin{aligned} A_1 = \cdots = A_N &\Leftrightarrow y_1 + \cdots + y_{1+zN} = \\ &\qquad \cdots = y_n + \cdots + y_{n+zN} \\ &\Leftrightarrow \overline{y_1 + \cdots + y_{1+zN}} = \\ &\qquad \cdots = \overline{y_n + \cdots + y_{n+zN}} \\ &\Leftrightarrow \overline{y}_1 + \cdots + \overline{y}_{1+zN} = \\ &\qquad \cdots = \overline{y}_n + \cdots + \overline{y}_{n+zN} \\ &\Leftrightarrow \overline{A}_1 = \cdots = \overline{A}_N, \end{aligned}$$

therefore for all $\mathbf{y} \in \mathbf{C'_b}$ we have all $\overline{\mathbf{y}} \in \overline{C'_b}$ and thus $\overline{C'_b}$ is a subset of C_b.

4.2 Repetition of Codewords

As defined previously, N represents the number of groupings and z represents the number of elements in each grouping. Satisfying (5), in the case of N prime as example, means having the same value of the sum in each grouping. The value of z can be reduced or increased by either eliminating or adding certain number of elements equally in each grouping. The power spectral density is not effected by the variations of the value of z, since the nulls are always a multiple of $1/N$, where N stays the same. In this section we show that for any value of N we have codebooks, that are included in other codebooks with longer codewords.

From previous sections it is clear that the variables of any codeword $\mathbf{y}$ element of the set C_b, satisfies the spectral null equation of the corresponding codebook C_b. Similarly with sub-sets, if any codebook $C'_b \subset C_b$, the codewords of the codebook C'_b satisfy the spectral null equation of the codebook C_b. We can prove this in a detailed way in the following proposition.

Proposition 2 *For two different spectral null codebooks C_b and C_b^α, with the same value of N and different values of z, where $z^\alpha = z + \alpha$, $\alpha \geq 1$, we have $\mathbf{y} \in \mathbf{C_b} \Rightarrow \mathbf{y} \in \mathbf{C_b^\alpha}$.*

As we know

$$A_i = \sum_{\lambda=0}^{z-1} y_{i+\lambda N}, \quad i = 1, 2, \ldots, N,$$

we consider $M = Nz$ the length of the codewords of the codebook C_b and $M^\alpha = Nz^\alpha$ the length of the codewords of the codebook C_b^α.

PROOF In this case we have:

$$A_1 = A_2 = \cdots = A_N$$

In the case where $M^\alpha = Nz^\alpha$, with $z^\alpha = z + \alpha$, $\alpha \geq 1$, which means we have more elements in each grouping, the codeword length can be written as follows:

$$\begin{aligned} M^\alpha &= Nz^\alpha \\ M^\alpha &= N(z+\alpha) \\ &= Nz + N\alpha \\ &= M + N\alpha. \end{aligned} \tag{17}$$

For all $\mathbf{y}^\alpha \in \mathbf{C_b^\alpha}$ and all $\mathbf{y} \in \mathbf{C_b}$, we have length $(\mathbf{y}^\alpha)$ = length $(\mathbf{y})$ + $N\alpha$ as shown below,

$$\begin{aligned} &\forall \mathbf{y}^\alpha \in \mathbf{C_b^\alpha} \Rightarrow \mathbf{y}^\alpha = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_M}, \mathbf{y_{M+1}}, \ldots, \mathbf{y_{M+N}}, \\ &\qquad \ldots, y_{M+\alpha N}), \\ &\forall \mathbf{y} \in \mathbf{C_b} \Rightarrow \mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_M}). \end{aligned}$$

It is clear from (17), that any spectral null codebook with codewords of length M^α is different to any other spectral null codebook with codewords of length M, only with an extra number of bits which is equal to αN. As is known for any codebook with longer codewords, we have higher cardinality. This will let us predict that the spectral null codebook for the codewords of length M can be found in the codebook with codewords of length M^α. The addition or the reduction of the number of elements within a grouping could be achieved whether we use zeros or ones.

Taking into consideration (4), for $\forall \mathbf{y} \in \mathbf{C_b}$, (5) could be written as:

$$\begin{aligned} y_1 + y_{1+N} + \cdots + y_{1+(z-1)N} &= y_2 + y_{2+N} + \\ &\quad \cdots + y_{2+(z-1)N} \\ &= \quad \vdots \\ &= y_N + y_{2N} + \\ &\quad \cdots + y_{zN} \end{aligned} \tag{18}$$

We can extend (18), by adding αN elements from the codeword $\mathbf{y}$, which can be 0 or 1. We can then show the idea in (19) by using the canceled variables, such that

$$\begin{aligned} y_1+\cdots+\cancel{y_{1+(z-1)N+N}}+ & \\ \cdots+\cancel{y_{1+(z-1)N+\alpha N}} &= y_2+\cdots+y_{2+(z-1)N}+ \\ & \quad \cancel{y_{2+(z-1)N+N}}+\cdots+ \\ & \quad \cancel{y_{2+(z-1)N+\alpha N}} \\ &= \quad \vdots \\ &= y_N+\cdots+y_{N+(z-1)N}+ \\ & \quad \cancel{y_{N+(z-1)N+N}}+\cdots+ \\ & \quad \cancel{y_{N+(z-1)N+\alpha N}} \end{aligned} \tag{19}$$

The addition of y_i, of the same value as shown before regarding the elements in each grouping, to all the equations will not change the sum of the equations. We have then the following relation,

$$\begin{cases} \forall \mathbf{y} \in \mathbf{C_b} \\ A_1 = A_2 = \cdots = A_N \end{cases} \Rightarrow \begin{cases} \forall \mathbf{y}^\alpha \in \mathbf{C_b^\alpha} \\ A_1^\alpha = A_2^\alpha = \cdots = A_N^\alpha \end{cases} \tag{20}$$

The equations in (20) show that all the elements of the codebook C_b are also elements of the codebook C_b^α. We denote by A_i^α, the same value of the grouping A_i but for the values of z^α. The equations in (20) can be proven from the opposite direction, which means from the elements of A_i^α to the elements of A_i and this just by deducting elements.

Example 5 The following example shows the codebook C_b is within the codebook C_b^α.

Consider $N=2$ and $z=2$ for $C_b = C_b(4,2)$ and $z^1 = 3$ for $C_b^1 = C_b(6,2)$. This means that in this example we have $\alpha = 1$, so $M^1 = M+2$ as shown in the following codebook:

C_b (21)

N	bits					
0	0	0	0	0	0	C_b^1
0	0	0	0	1	1	
0	0	0	1	1	0	
0	0	1	0	0	1	
0	0	1	1	0	0	
0	0	1	1	1	1	
0	1	0	0	1	0	$C_b^{'}$
0	1	1	0	0	0	
0	1	1	0	1	1	
0	1	1	1	1	0	
1	0	0	0	0	1	
1	0	0	1	0	0	
1	0	0	1	1	1	
1	0	1	1	0	1	
1	1	0	0	0	0	C_b^1
1	1	0	0	1	1	
1	1	0	1	1	0	
1	1	1	0	0	1	
1	1	1	1	0	0	
1	1	1	1	1	1	

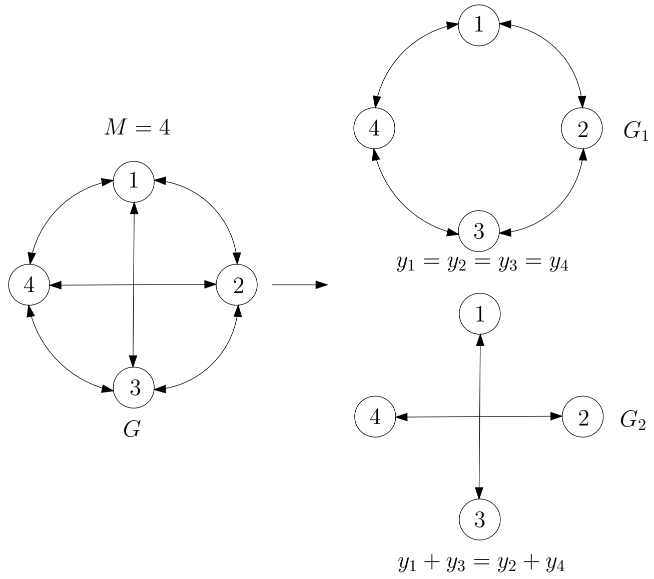

Figure 7: Equation representation for Graph $M = 4$

This shows clearly the difference between the codewords of length 6 for C_b^1 and 4 for C_b as it has been explained previously. It also shows that $C_b \subset C_b^1$ as it was defined previously and thus the codewords from C_b appear as elements of the codebook C_b^1.

4.3 Concept of Graph Theory

In this section we present and emphasize certain properties of spectral null codebooks from graph theoretical perspective. The concept of subsets and subgraphs [16]–[17] are studied. We link between the indices of the variables in a spectral null equation and the permutation sequences formed from these indices.

As an example if we take the case of $M = 4$ with $N = 2$, we have the spectral null equation:

$$A_1 = A_2 \rightarrow y_1 + y_3 = y_2 + y_4 \tag{22}$$

The corresponding permutation sequences to the variables in (22) is (1)(3)(2)(4). These permutation symbols can be presented graphically by just being lying on a circle, which it is called a state. The state design follows the order of appearance of the indices in (22). The symbols are connected in respect of the addition property of their corresponding variables in (22) as depicted in Fig. 7.

The elimination of states from any graph corresponding to the index-permutation symbols is in fact the same as the elimination of the corresponding variables from the spectral null equation (5). The elimination of the variables is performed in such a way that the spectral null equation is always satisfied. This leads to the basic idea of eliminating an equivalent number of variables equal to N as a total number from different groupings in the spectral null equation. This is true when we eliminate only one variable from each grouping. In the case when we eliminate t

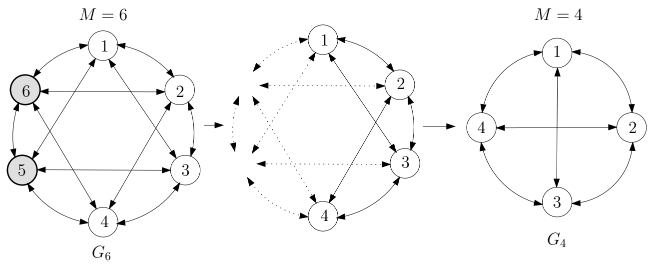

Figure 8: Subgraph design from $M = 6$ to $M = 4$ with $N = 2$

variables with $1 < t < z$ from each grouping, we have a total number of eliminated variables of $t \times N$.

Example 6 We construct a spectral null code for the case of $M = 6$, with $N = 2$ and $z = 3$, which is represented by the codebook $C_b(6,2)$ in (21) and which is designed from the spectral null equation (23). The corresponding graph is G_6 in Fig. 8.

From the spectral null equation (23), we can eliminate the variables y_5 and y_6 using the addition property and this will lead to the equation (24), which is the spectral null equation for the case of $M = 4$ with $N = 2$.

$$\overbrace{\overbrace{y_1 + y_3 + y_5}^{z=3} = \overbrace{y_2 + y_4 + y_6}^{z=3}}^{N=2} \tag{23}$$

The obtained codebook is denoted by $C_b(4,2)$. Fig. 8 depicts the elimination of the states from a graph theory perspective. The elimination of the states "5" and "6" results in the elimination of the links between them and the other states.

$$\overbrace{\overbrace{y_1 + y_3}^{z=2} = \overbrace{y_2 + y_4}^{z=2}}^{N=2} \tag{24}$$

It is clear that in the codebook presented in (21), we have $C_b(4,2) \subset C_b(6,2)$, in terms of the existence of the elements from the codebook $C_b(4,2)$ in the codebook $C_b(6,2)$, which is the same as for the subgraphs where we have $G_4 \subset G_6$.

4.4 *Frequency Spectra of Spectral Null Codes*

From the designed spectral null codes C_b, we can observe that each codebook has balanced codewords within it. These balanced codewords form DC-free subsets of the designed spectral null codes denoted by C_b^B. Another property that can be observed from the designed spectral null codebook is that they have codewords with a sequence where half of it, is a complement of the other half or with another word like a mirror of the other half. We call this class of codes the complementary symmetrical codes, which are subsets of the spectral null codes and denoted by C_b^S.

Definition 2 A balanced code, denoted by C_b^B has all its codewords with an even length where the number of ones and zeros are equal.

Definition 3 A complementary symmetrical code, denoted by C_b^S has all its codewords with an even length in such a way that its first half is the conjugate of its second half.

From Table 2, we can see that for the same length of the codeword and certain specific values of N and z, we always have $C_b^B \subset C_b$ and $C_b^S \subseteq C_b^B$.

Taking into consideration the definitions, we have summarized our results in Table 2 where it can be seen that we have a few important properties to be derived from these results:

1. For any prime value of z, we cannot design a symmetric codebook except for the special case of $z = 2$.
2. For any not prime value of z, we can design a balanced code and we can produce a symmetric codebook with a predictable cardinality equal to $\left|C_b^S\right| = 2^{n/2}$.
3. For the values of z, which are not prime, we can have nulls at the Nyquist frequency for the following conditions:
   (a) if $z = 2$ and N not prime with $N \geq 2$ we can get nulls at the Nyquist frequency,
   (b) if $z \geq 4$ and $\forall N$ we can always have nulls at the Nyquist frequency.
4. In the case of symmetric codes, we can always predict the values of the nulls and their corresponding frequencies as shown in the following equation:

$$f_M = 2(i-1)/M, \quad i = 1, \ldots, M/2.$$

5. SPECTRAL NULL CODES APPROACH: LOW-DENSITY PARITY-CHECK CODES

Ouahada *et al* [18] have shown that for any permutation sequences of length N, the binary representation of these permutation symbols, where the bit 1 represents the symbols at its corresponding position, e.g. $123 \rightarrow 100010001$, is a subset of spectral null codes with $N = z$ and codewords length of $M = N^2$ and cardinality of $N!$. The obtained codebook is a $N! \times N^2$ matrix, denoted by M and the number of 1s in each row is equal to N.

The LDPC matrices, denoted by H, were first introduced by Gallagar [19], who defined them as (n, j, k) matrices with n columns that have j ones in each, and k ones in each row, and zeros elsewhere.

The number of 1s in each row in the obtained codebook is equal to N with a rate of $p_r = N/N^2$ and the number of 1s in each column is equal to $(N-1)!$, which represents a rate of $p_c = (N-1)!/N!$. We can see that $p_r = p_c = 1/N$,

Table 2: Frequency Spectra and Cardinalities of Spectral Null Codes

M	N	z	$\|C_b\|$	Nulls	$\|C_b^B\|$	Nulls	$\|C_b^S\|$	Nulls
4	2	2	6	1/2	4	0, 1/2, 1	$4 = 2^2$	0, 1/2, 1
6	2	3	20	1/2	—	—	—	—
8	2	4	70	1/2	36	0, 1/2, 1	$16 = 2^4$	0, 1/4, 1/2, 3/4, 1
10	2	5	252	1/2	—	—	—	—
12	2	6	924	1/2	400	0, 1/2, 1	$64 = 2^6$	0, 1/6, 1/3,1/2, 2/3, 5/6, 1
14	2	7	3432	1/2	—	—	—	—
8	4	2	36	1/4, 3/4	18	0, 1/4, 3/4, 1	$16 = 2^4$	0, 1/4, 1/2, 3/4, 1
12	4	3	400	1/4, 3/4	164	0, 1/4, 3/4, 1	—	—
16	4	4	4900	1/4, 3/4	1810	0, 1/4, 3/4, 1	$256 = 2^8$	0, 1/8, 1/4, 3/8,5/8, 3/4, 7/8, 1
12	6	2	250	1/6, 5/6	90	0, 1/6, 5/6, 1	$64 = 2^6$	0, 1/6, 1/3,1/2, 2/3, 5/6, 1

which means that the rates are very low at very large values of N.

We can define two numbers that describe a low-density parity-check matrix with a dimension of $n \times m$; w_r for the number of 1s in each row and w_c for the columns. To have a low-density parity-check matrix we need to satisfy two conditions $w_c \ll n$ and $w_r \ll m$.

Proposition 3 *The matrix $H = \mathcal{M}^T$, is a regular LDPC matrix, for $N \geq 4$.*

PROOF The matrix $\mathcal{M}$ is a $N! \times N^2$ matrix. So H is a $N^2 \times N!$ matrix, with $n = N!$, $k = (N-1)!$ and $j = N$, which means that H is regular [20].

It is clear that the Gallagar condition is satisfied, where the number of rows is $N^2 = nj/k$. We can also see that each submatrix of $N \times N!$, has a single 1 in each one of its columns.

For example for $N = 4$, with

$$H = \begin{bmatrix} 111111000000000000000000 \\ 000000111111000000000000 \\ 000000000000111111000000 \\ 000000000000000000111111 \\ \hline 000000110000110000110000 \\ 110000000000001100001100 \\ 001100001100000000000011 \\ 000011000011000011000000 \\ \hline 000000001010001010001010 \\ 001010000000100001100001 \\ 100001100001000000010100 \\ 010100010100010100000000 \\ \hline 000000000101000101000101 \\ 000101000000010010010010 \\ 010010010010000000101000 \\ 101000101000101000000000 \end{bmatrix},$$

we can see that

$$\left.\begin{array}{l} n = N! = 24 \\ k = (N-1)! = 6 \\ j = N = 4 \end{array}\right\} \Rightarrow \frac{24 \times 4}{6} = 4^2 = N^2.$$

It is important to notice that for $N = 3$, we have $H = \mathcal{M}$. For example, we obtain for $N = 3$ the following,

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

where H is our low-density parity-check matrix with the dimension of $N! \times N^2$.

The example $N = 3$ is just used to show how our binary representation of permutation codes is a low-density parity-check matrix. In reality we should have N very large.

Our low-density parity-check matrix is regular. As can be seen in the case of $N = 3$, where w_r and w_c are constant.

The regularity is also clear when we form the Tanner graph depicted in Fig. 9, where we have the same number of incoming edges for every v nodes and also for all the c nodes.

For all codewords v, we have

$$v \cdot H^T = 0.$$

Any LDPC code is encoded via generator matrix G. For a given information vector u, the corresponding codeword v is encoded via

$$v = u \cdot G,$$

$H = [H_1 | H_2]$, where H_1 and H_2 have dimensions $(n-k) \times k$ and $(n-k) \times (n-k)$, respectively. H_2 should be non-singular. In the case where H_2 is singular, we have

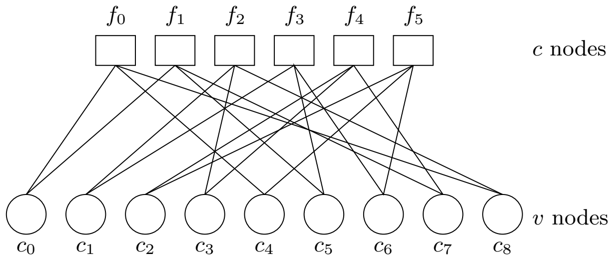

Figure 9: Tanner graph

to eliminate some rows and columns to get a non-singular matrix

$$G = [I|H_1^T H_2^{-T}].$$

As example $N = 3$, we have

$$H = \begin{bmatrix} 100010001 \\ 100001010 \\ 010100001 \\ 010001100 \\ 001100010 \\ 001010100 \end{bmatrix} \Rightarrow \begin{cases} v_1 + v_5 + v_9 = 0 \\ v_1 + v_6 + v_8 = 0 \\ v_2 + v_4 + v_9 = 0 \\ v_2 + v_6 + v_7 = 0 \\ v_3 + v_4 + v_8 = 0 \\ v_3 + v_5 + v_7 = 0 \end{cases}$$

with

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In the case where N is very large we have $N! \gg N^2$ and this will cause some problems to get the previous conditions of H satisfied.

We denote by H^μ, where μ is the number of concatenated LDPC matrices, the generalized form of the construction of our low-density parity check matrix from our binary representation,

$$H^\mu = \overbrace{[H|H|\dots|H]}^{\mu}. \tag{25}$$

Putting H in serial concatenation μ times can increase the weight w_r. We can see that H is always a regular matrix with a dimension equal to $N! \times (\mu N^2)$.

For example if $N = 4$, we have H with $w_c = 6$ and $w_r = 4$. We choose $\mu = 4$, and we get a H^4 with $w_c = 6$ and $w_r = 16$.

It is important to notice that the concatenated construction might causes the dependency in the columns of the matrix H^μ. Thus some columns could be eliminated and the matrix might become a singular matrix. To solve this problem we can permute randomly the columns of each H. We denote by H_{p_φ} the matrix H when we permute its columns φ times. Thus (25) will be presented as follows:

$$H^\mu = \overbrace{[H_{p_\varphi}|H_{p_\varphi}|\dots|H_{p_\varphi}]}^{\mu}, \quad 1 \leq \varphi \leq N^2.$$

It is important to mention that the values of w_r and w_c can be further increased by satisfying the spectral null equation, which leads to the increase of ones in the LDPC matrix. Therefore the code rate will be increased. From Fig. 9 we can also see that the girth of the code is higher than four, which means that we have good error correction codes.

6. CONCLUSION

In this paper, with certain observations of the structure of spectral null codes, we could have derived important properties that can be useful in the field of digital communications. The paper does not present constructions of any type of codes but just analysis of existent properties of spectral null codes.

The relationship between the spectral null equations, the generated nulls and the cardinalities of spectral null codes were investigated. The importance of the cardinality of the codebook and the corresponding rate of the code and also the error correction capability are emphasized and clarified.

The properties and the approaches that we have presented using the binary structure of the codebooks and the graph theory approach could help in similar research in discovering more properties that can be used in important applications telecommunications and data recording to help improve the quality of the transmitted date information.

Certain spectral null codes properties can lead to certain error correcting codes for certain channels as the example in [21, 22] or the improvement in the structures of certain spectral null codebooks for better design of Low Density Parity Check codes.

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