OBSERVER DESIGN FOR A BILINEAR MODEL OF A CONTINUOUS COUNTERCURRENT ION EXCHANGE PROCESS.

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Abstract: The paper describes a method and proposes an algorithm for a bilinear observer design as part of a state estimation solution for a Continuous Countercurrent Ion Exchange (CCIX) process used for desalination of water. The aim of the design is to determine immeasurable process state variables using real CCIX measurement data obtained from a different study. The solution requires determination of the observer gain matrix and it is formulated on the basis of the pole placement method. The contribution of the paper is in the developed bilinear dynamic models of the process and the observer and in the extending of the pole placement method for the special matrices of the bilinear observer. An analytical procedure is developed and implemented for calculation of the characteristic equation of the observer as part of calculation of the observer gain matrix.
Matlab and SIMULINK software programs for determining the estimated states are developed. Investigation of the performance of the bilinear observer and the closed loop system under various constant values of the control input and different state initial conditions is performed. The results are presented and discussed.

Key words: linear systems, nonlinear systems, state estimation, observer design, bilinear model, ion exchange, pole placement.

1. INTRODUCTION AND BACKGROUND

A bilinear model of a Continuous Countercurrent Ion Exchange (CCIX) process described in [1–3] has been developed with the overall objective to be applicable to optimal control design of the process [4, 5]. Generally, in developing a model, more especially for the purpose of monitoring and optimal control of a process based on feedback controller design [6], some components of the process state are not known, and or cannot be measured directly from the process due to a lack of a suitable sensor or due to the nature of the plant [6–11]. This necessitates the need for developing a method for determining such states before application of the designed optimal control of the process [12]. [13] suggests that the state estimation problem in bilinear systems is a very important part of designing controllers in these systems. [14] considered the feedback property of the observer. [15] state that a lot of ground was covered in design of observers in determining the importance of observers in feedback control. In designing the observer, generally, two conditions must be met, 1) minimization of the error difference between the model states and the observer estimated states, and 2) keeping error dynamics and its rate at zero or close to zero under changing input or state variables [6, 7]. The observer methods for design can be divided into two main approaches: 1) for linear or 2) nonlinear systems. Unfortunately, a linear state observer is not adequate to reconstruct states of a nonlinear system and this suggests the need for the design of nonlinear observers [6, 13, 16]. Unlike in the linear case [9, 12, 16–20], where the error dynamics are known to be independent of the input signal and the present value of the state; in the nonlinear or bilinear case, the error dynamics are not totally independent of the input signal and the state due to the bilinear dependence between these two variables [6, 13, 20].
With a lot of work in the (1980s) based on physical systems, which are mainly nonlinear or bilinear, the result was a lot of effort directed on linearization of the nonlinear systems [12, 16, 21, 22]. Different observer types are considered in [23]. [12, 21, 24] suggest that the common approach to solving observer design problem for nonlinear systems is to extend the linear *Luenberger observer*, the linear *Kalman filter* design approach or the *pseudo-linearization* techniques to the nonlinear systems. According to [23] these techniques are also valid only for small ranges around the operating point; and if applied in real time applications, they have an extensive computational requirement. A number of authors considered the nonlinear systems to be of *Lipschitz type* in designing their nonlinear observers [16, 23, 25, 26]. [23] gives the full description of the local and global Lipschitz systems. According to [8, 27, 28] research in nonlinear state observer design resulted in the following techniques, by which the above mentioned solutions can be categorized under: *extended linearization*, *feedback linearization*, *variable structure*, *high-gain observer*

design, *extended Kalman filter* and its family, *Lyapunov-based* techniques and state-dependent Riccati equation–based techniques.

According to [13, 29] bilinear systems can be considered as a special class of nonlinear systems, and can also be considered good approximations to the nonlinear systems. Bilinear systems appear naturally as models for physical systems. [13] describes the bilinear systems as a special class of nonlinear systems where the control input appears both additive and multiplicative in the system model. [13, 29, 30] suggest that bilinear systems occur frequently in chemical processes and fault detection processes (dynamics). [21] further suggests that there are a number of reasons for studying bilinear systems, as a class of the nonlinear systems, such as, 1) these systems closely resemble linear systems (which already has a well-developed theory), 2) bilinear systems arise in variety in many physical and practically implementable processes [13, 16]. A lot of work on observer design for bilinear systems (models) appears in the literature [6, 7, 29–34]. In the mid 1970's work on the design of bilinear observers started appearing in the literature and this attracted a lot of interest towards the end of the 1970's to the early 1980's. During this period a lot of significant successes and milestones were reached in estimating unknown state variables for bilinear systems [15, 29, 32–34, 35]. [13, 20] proposed an observer for nonlinear systems where the estimation error decays to zero irrespective of the input. These authors further gave an analysis of observability of nonlinear systems in comparison to that of linear ones.

A number of authors propose the design of asymptotic observers [11, 12, 20]. [11] presented the problem of obtaining an asymptotic estimate of the state of a bilinear system given input and output measurements. [20] proposed an asymptotic observer that is capable of estimating state variables for all initialization of the observer. The initialization of the observer plays an important part in bioprocesses since initial conditions are not measured. Many authors considered stability, observability and controllability and associated necessary conditions for the existence of these properties in their design of observers, including analysis of observability of nonlinear systems [12, 13, 36]. [11, 12] generated necessary and sufficient conditions for the observability of a general system of bilinear and nonlinear equations. [14] discussed reducibility of a bilinear system to a canonical controllability form as a criterion for uniform for observability. [37] generated sufficient conditions for the existence of a stable observer. [12] considered necessary and sufficient conditions for the existence of a special observable form. [8, 30] work showed that the use of Lyapunov framework facilitates and proves asymptotic stabilization of observation errors. [23] gave a procedure for obtaining the observer gain such that there is *quadratic stabilization* of the error dynamics based on condition of existence of a certain Lyapunov function. [10] presented a design of an exponential observer based on Lyapunov method. [8, 30, 34] consider the Lyapunov stabilization procedure and convergence conditions in the observer design. The work of [10, 28, 31] includes solving a Lyapunov equation in determining the state estimator. [20] proved convergence of their design using classical Lyapunov functions. [38] proposed an stable observer design based on Lyapunov stability for bilinear systems. This work considers a state observer with the error that may depend on the system input signal.

The pole placement method is often used to design the state observer matrix [39–42]. It is applicable in the cases of the discrete time implementation of the controller and the state observer of the closed loop system. This approach is applied to various processes in the industry, but till now it has not been applied to the case of the countercurrent ion exchange process. This process, due to the nature of construction of its columns allows measurement only of the output variable and of the input disturbance. All states are not measurable and the proper control of the process requires corresponding state observer equations. There is no described algorithm for design on an observer for the countercurrent ion exchange process till now in the existing literature. This problem of design of the gain matrix of a bilinear observer is considered in the paper.

The aim of the paper is to develop a method and an algorithm of designing an observer for a bilinear system for the case of the Continuous Countercurrent Ion Exchange (CCIX) process – to estimate the unknown states of the system that cannot be directly measured from the CCIX plant. The paper presents an observer design which is an extension of a Luenberger-type observer for bilinear systems. The observer design method developed proposes a solution where the disturbance is assumed constant for a long period of time. The input signal *stays constant throughout the operational sampling period* in some constrained region. This is a more realistic approach since it is a true reflection of the plant's behaviour. The solution of the problem is developed using *pole placement stability* requirement of the characteristic equation of the observer.

The special structure of the state space matrices of the process model makes the derivation of the calculation algorithm of the pole placement method very difficult. The paper proposes an extension of the existing algorithm based on the analytical derivation of the characteristic equation of the bilinear state observes. Further, the behaviour of the state observer and of the process closed loop system dynamics are investigated for various values of the control input and of the initial states of the observer. The observer matrix is calculated with data acquired from an existing counter current Ion Exchange process. The simulation results given here are for a constant input signal over the full plant trajectory. Though assumed constant over the process trajectory, different input signal values have been considered and are presented in the result section of the paper.

The contributions of the paper are as follows:

1) Development of a pole placement method and algorithm for the design of the matrix of a bilinear observer of the ion exchange states in the case of constant values of the input control signal using

the observer characteristic equation stability analysis and the pole placement based approach.

2) Mathematical derivation of the characteristic equation of the observer and the process for the higher dimensional case and special structure of the Ion Exchange process.
3) Development of Matlab/Simulink software for use of the data from the ion exchange process for the design of the observer matrix and the simulation of the system closed loop consisting of a controller, the observer, and the process model.
4) The behavior of the observer and its convergence towards the process model states are investigated for various values of the constant control input and various initial conditions of the estimated states.
5) The behavior of the closed loop system is investigated for various values of the constant control input and various initial conditions of the estimated state.

The significance of the developments described in the paper is the transformation of the ion exchange process in a dynamic state space bilinear model, formulation of the Luenberg type of a state observer, its design through the extension of the pole placement method and the verification of the design by simulation, which is novel in the existing literature. There are no publications which consider the dynamic behavior of the ion exchange process and the connected to it state observers.

The paper is organized as follows: in Section 2, the continuous countercurrent ion exchange (CCIX) process is described. Section 3 covers the model development of a bilinear multistage CCIX process based on component balances in the process column stages, and the formulation of the state estimation problem is discussed. In Section 4, the design of the proposed observer is presented. Section 5 presents the pole placement procedure applied to solve the stability problem for the unknown observer matrix, and in turn find the matrix entries using the observer characteristic equation. In Section 6 software for the observer design and simulation is presented. In Section 7, the observer design data are presented and the application of the data in the design of the observer is illustrated. Further, simulation results are given to demonstrate the effectiveness of the design; and this is followed by the conclusion section.

2. PROCESS DESCRIPTION

According to [43] an ion exchange process is a reversible chemical process where ions with similar polarities are exchanged between solids and an electrolyte solution if the two are contacted. The interaction of the ions happens at different levels, there is ion exchange in the solid phase to the liquid phase and also diffusion of ions within the solid phase [44–46]. The ion exchange process is a very convenient chemical process for wastewater desalination [1, 2, 5]. In water desalination application, ion exchange resins form the solid phase, and water being treated forms the liquid phase. Resins are charged beads of micrometers in diameters that are coated with replacement ions H^+ (hydrogen ions) or OH^- (hydroxide ions) depending on their ionic form (cation or anion). Hydrogen ions are used in the cation resins and hydroxide ions are used in the anion resins. In the cation phase the H^+ ions will exchange with Na (sodium) ions from sodium chloride ($NaCl$) in the water being processed. During the anion phase the Cl^- (chloride) ions exchange with OH^- ions from the acidic output stream from the cation phase (Figure 1). Figure 1 also shows the pilot plant as built in the Chemical Engineering Department of Cape Peninsula University of Technology [1, 2, 4, 5].

The basic ion exchange countercurrent configuration consists of four columns, two for each phase of the process. The system has a cation load column and a cation regeneration column for the cation phase, and an anion load column and an anion regeneration column for the anion phase. The columns operate in a multistage fashion with primary and secondary cycles. During the cation load primary cycle, $NaCl$ is extracted from feed water, resulting in an acidic output stream; and during the primary cycle of the anion load, this acidic stream from the cation load column is split to produce product water. The secondary columns are for regeneration of partially exhausted resins back to their refreshed form using either an acid solution or a base solution for each phase respectively. This is one of the greatest advantages of the ion exchange process; resins are reusable for a number of years, depending on their type [1, 3, 4, 43 ,44, 46]. The considered ion exchange process is a *countercurrent process with the solid and liquid phases flowing in* opposite directions. The exchange is said to be countercurrent in that the moving resin bed and liquid move in opposite directions. The resins are in a packed bed form and they move from column to column in a cyclic form.

This cyclic operation has three distinct periods for each of the two main cycles mentioned above in each phase, the primary and secondary cycles. Each cycle has three periodic flows for moving liquid streams and moving resin beds, 1) an *up-flow period*, 2) *the settle time* and 3) the *pulldown period*. During the primary cycle of the cation phase, liquid to be treated is pumped up the load column through a packed bed of resin to fluidize the resin beds held by each stage of the column and this is referred to as an *up-flow period*. After a certain predetermined period, resin beds are allowed to settle by stopping the up-flow stream. This is known as the settle time. And then finally, the resin bed at the bottom stage is pulled out to prepare it for regeneration; this period is referred to as *pulldown period*. Before pulldown can be initiated there must be enough resins to fill up a stage in the top holding vessels (Figure 1). During the secondary cycles of the regeneration columns, resin beds of specific amounts are intermittently moved from the top stage of the regeneration column down to the bottom stage in a controlled fashion [1, 2, 4, 46].

3. MODEL DEVELOPMENT AND PROBLEM FORMULATION

In the considered countercurrent ion exchange process the change in concentration of the feed flow at the first stage of the column is considered a disturbance to the system, and it needs to be modelled accordingly. The process is run under a global aim of the optimization. The optimization strategy of the process is to ensure that for a given desalination level, maximum purified water output is obtained with the minimum usage of the regenerant chemicals [1, 2, 3, 44, 46].

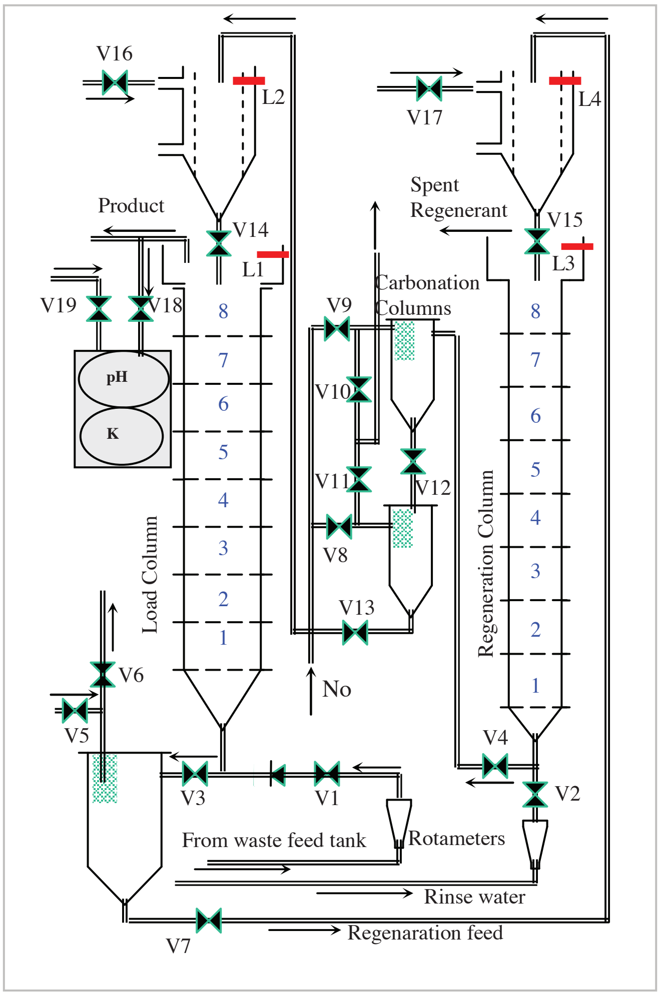

Figure 1: Continuous countercurrent ion exchange (CCIX) process as built at Chemical Engineering Department of Cape Peninsula University of Technology

Technically, this aim can be formulated as an optimization problem using concentration of salt in the product water as a process output and the resin flow rate into the column as the control action for the process on the basis of the steady state balance [4, 5]:

$$u(t) = F_R(t) = \frac{hd}{T(t)} \tag{1}$$

Where

$u(t) =$ the control input to the process,

$F_R(t) =$ the moving resin bed flow rate in the cation load column, considered as a control input,

$T(t) =$ the upflow period for the cation load column,

$h =$ the resin holdups and

$d =$ the liquid-resin fractional balance constant defining the fraction of the resin hold up which is moved from one stage to another and $d = 2/3$

The observed column behavior requires the developed mathematical process model to predict liquid and resin composition in each stage for every process cycle. The model design is developed on the basis of the steady state balance between the principal operating parameters of the ion exchange, the liquid flow rate $F_L(t)$ and the resin flow rate $F_R(t)$. Up-flow cycle time $T(t)$ determines the control action for the plant [1, 2, 4, 5], as given by equation (1). There are some models developed in the literature for the same process [44–46], however, these models are very complex and could not be used for the purposes of optimization and real-time control based on the intended overall control strategy. These models are more suitable for chemical engineering or analytical chemistry type analysis. The process model in the paper is developed for the purpose of control, and calculations in the model derivation are based on equilibrium and kinetic data, resin and liquid flow rates.

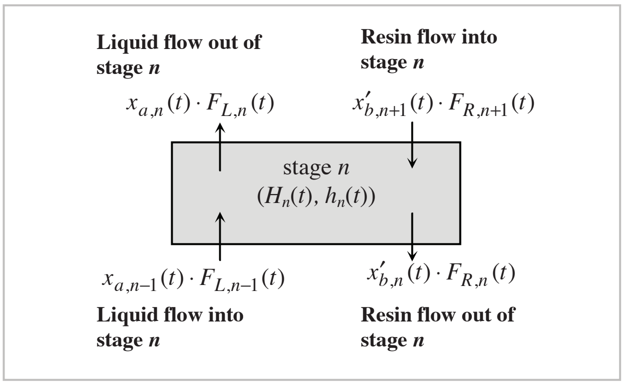

Figure 2: Countercurrent flows of liquid and resin in a single stage of a multistage column of a CCIX process

These calculations are performed from the bottom to the top column stages, as the changes in the lower stage(s) states affect the following upper stage(s) states. Model equations are developed on the basis of the input and output rates of the mass balances for every process stage, Figure 2, based on the following assumptions [1, 2, 44]: (1) there is equal volume and amount of liquid and resin holdups just before the resin transfer from one stage to the other, (2) there is perfectly mixed fluidized phase in each stage and no back mixing occurs, (3) the process is in a steady state (electro-neutrality is maintained), and (4) a linear equilibrium exists between the liquid and resin phases. In Figure 2, $x_{a,n}(t)$ is the mole fraction of sodium (Na) in the liquid phase $F_{L,n}(t)$, $x'_{b,n}(t)$ is the mole fraction of Na in the resin phase $F_{R,n}(t)$, $H_n(t)$ is the liquid hold up, and $h_n(t)$ is the resin hold up for the n-th stage [1, 2, 4, 5, 44]. The ion exchange model is obtained based on the i^{th} component mass balance on stage n with N representing the total number of stages in the process columns, Figure 1 [4, 5]. The component mass law of material balance based on rate of accumulation and rate of materials formation can be expressed as:

$$\frac{d(H_n(t)x_{a,n}(t))}{dt}+\frac{d(h_n(t)x'_{b,n}(t))}{dt}= \\ =\left[F_{L,n-1}(t)\cdot x_{a,n-1}(t)+F_{R,n+1}(t)\cdot x'_{b,n+1}(t)\right]- \\ -\left[F_{L,n}(t)\cdot x_{a,n}(t)+F_{R,n}(t)\cdot x'_{b,n}(t)\right],\ n=\overline{1,N} \quad (2)$$

Where

$F_{L,n}(t)=$ the molar flow rate of the liquid,

$F_{R,n}(t)=$ the molar flow rate of the resin,

$F_{L,n-1}(t)\cdot x_{a,n-1}(t)=$ the rate of material input with liquid coming from stage $n-1$,

$F_{R,n+1}(t)\cdot x'_{b,n+1}(t)=$ the rate of material input with resin coming from stage $n+1$,

$F_{L,n}(t)\cdot x_{a,n}(t)=$ the rate of material output with liquid leaving plate n for plate $n+1$,

$F_{R,n}(t)\cdot x'_{b,n}(t)=$ the rate of material output with resin leaving plate n for $n-1$,

$\frac{d(H_n(t)x_n(t))}{dt}=$ the rate of accumulation of sodium specie in the liquid phase on plate n, and

$\frac{d(h_n(t)x'_{b,n}(t))}{dt}=$ the rate of accumulation of sodium specie in the resin phase on plate n.

Based on assumptions made, the liquid and resin holdups and flow rates are considered constants, i.e., $H_n=H$, $h_n=h$, $F_{R,n}(t)=F_R$, and $F_{L,n}(t)=F_L$.

The equilibrium between the liquid and the resin fractions is assumed to be linear to maintain electroneutrality, and the relationship between the exchanging cations is given by [1, 2]

$$x'_{b,n}(t)=a_n x_{a,n}(t)+b_n \quad (3)$$

Where

$a_n, b_n=$ the coefficients of the rate of the ion exchange reaction.

After substituting the conditions for the linear equilibrium (3), the component mass law of material balance, based on the rate of accumulation and the rate of materials formation, the model (2) can be expressed as:

$$H\frac{dx_{a,n}(t)}{dt}+a_n h\frac{d_{xa,n}(t)}{dt}=F_L\left[x_{a,n-1}(t)-x_{a,n}(t)\right]+ \\ +F_R\left[(a_{n+1}x_{a,n+a}(t)+b_{n+1})-a_n x_{a,n}(t)+b_n\right], \\ n=\overline{1,N}. \quad (4)$$

The model (4) can be further simplified to,

$$\frac{dx_{a,n}(t)}{dt}=\left[\frac{F_L}{(H+a_n h)}\right]x_{a,n-1}(t)-\left[\frac{F_L}{(H+a_n h)}\right]x_{a,n}(t)- \\ -\left[\frac{a_n}{(H+a_n h)}\right]x_{a,n}(t)F_R+\left[\frac{a_{n+1}}{(H+a_n h)}\right]x_{a,n+1}(t)F_R+ \\ +\left[\frac{b_{n+1}-b_n}{(H+a_n h)}\right]F_R,\quad n=\overline{1,N} \quad (5)$$

After selecting $x_{a,n-1}(t)=x_0(t)=x_f(t)$, for $n=1$ as the column input flow concentration, $y(t)=x_{a,N}(t)$ as the output of the system, $x_{a,n}(t)=x(t)=\left[x_1(t),x_2(t),..,x_n(t),..,x_N(t)\right]^T$ as the state space vector, and $F_R(t)=u(t)$ as the control input, the state space model of the ion exchange process (5) can be written as

$$\frac{dx(t)}{dt}=Ax(t)+B_1x(t)u(t)+Bu(t)+Wx_f(t),\quad x(0)=x_0 \\ y(t)=Cx(t) \quad (6)$$

Where

$$A=\begin{bmatrix} \frac{-F_L}{H+a_1h} & 0 & 0 & ... & 0 \\ \frac{F_L}{H+a_2h} & \frac{-F_L}{H+a_2h} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & \frac{F_L}{H+a_{N-1}h} & \frac{-F_L}{H+a_{N-1}h} & 0 \\ 0 & 0 & ... & \frac{F_L}{H+a_Nh} & \frac{-F_L}{H+a_Nh} \end{bmatrix}$$

$$B_1=\begin{bmatrix} \frac{-a_1}{H+a_1h} & \frac{a_2}{H+a_1h} & 0 & ... & 0 \\ 0 & \frac{-a_2}{H+a_2h} & \frac{a_3}{H+a_2h} & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & \frac{-a_{N-1}}{H+a_{N-1}h} & \frac{a_N}{H+a_{N-1}h} & 0 \\ 0 & 0 & ... & 0 & \frac{-a_N}{H+a_Nh} \end{bmatrix}$$

$$B=\begin{bmatrix}\dfrac{(b_2-b_1)}{H+a_1h}\\ \dfrac{(b_3-b_2)}{H+a_2h}\\ \vdots\\ \dfrac{(b_N-b_{N-1})}{H+a_nh}\\ \dfrac{b_N}{H+a_Nh}\end{bmatrix}, W=\begin{bmatrix}\dfrac{F_L}{H+a_1h}\\ 0\\ \ldots\\ 0\\ 0\end{bmatrix}, C=\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix}^T$$

The model matrices are:

$A\in R^{N\times N}=$ the matrix of the state,
$B_1\in R^{N\times N}=$ the matrix of bilinear term of the state and input control signal
$B\in R^{N\times 1}=$ the matrix of the input signal
$W\in R^{N\times 1}=$ the matrix of the disturbance,
$C\in R^{1\times N}=$ the matrix of the output signal.

The model variables are

$u(t)\in R^{1\times 1}=$ the control signal,
$x(t)\in R^{N\times 1}=$ the state vector and
$x_f(t)\in R^{1\times 1}=x_{a,0}\in R^{1\times 1}=$ the disturbance to the system,
$y(t)\in R^{1\times 1}=$ the output of the system.

The disturbance is considered as the change of liquid concentration that enters the first stage (the most bottom stage) of the cation column in the continuous countercurrent ion exchange process. Once the model has been developed and before using it for real time optimal control, one needs to estimate the unknown parameters. In the same breath, in this presentation, the liquid concentration of the CCIX plant cannot be measured in all the stages of the column except for the first and the last stages, one needs to estimate liquid concentration of all other stages using state estimation techniques. The aim is to develop a new algorithm for solving state estimation problem given that the system is nonlinear in terms of variables but linear in terms of parameters. The proposed solution is presented in the next section.

4. DESIGN OF THE OBSERVER

Consider the model of a bilinear system given by (6) above; the objective is to design a full-order observer to be able to identify all the unknown states. The proposed observer is constructed in the analogy of a Luenberger type of observer for linear time–invariant systems. The observer equation for the model (6) is expressed so as to correspond to the structure of the bilinear system, as follows:

$$\begin{aligned}\frac{d\hat{x}(t)}{dt}&=L_1\hat{x}(t)+L_2\hat{x}(t)u(t)+L_3u(t)+L_4x_f(t)+Ly(t)\\ \hat{y}(t)&=C\hat{x}(t)\end{aligned}\tag{7}$$

Where

$\hat{x}(t)\in R^{N\times 1}=$ the estimated state vector,
$\hat{y}(t)\in R^{1\times 1}=$ the estimated system output based on estimated states,
$L_1\in R^{N\times N}=$ the observer state matrix,
$L_2\in R^{N\times N}=$ the bilinear term matrix of the observer,
$L_3\in R^{N\times 1}=$ the observer control input matrix, and
$L\in R^{N\times 1}=$ the output (the observation) matrix.

The main aim of the observer is to minimize the error between the process states $x(t)$ and the estimated states $\hat{x}(t)$ and to be convergent, i.e., asymptotically stable. In order to achieve this aim, the design of the observer is based on fulfillment of two conditions: 1) for the *value of the error* and 2) for *the rate of change of the error* between the real and estimated states. The first condition states that $\|e(t)\|\to 0$ for $t\to\infty$ and this is presented by the error equation,

$$e(t)=x(t)-\hat{x}(t)\to 0\tag{8}$$

This means that the estimated state vector is defined by $\hat{x}(t)=x(t)-e(t)$. Based on *the condition for the error rate dynamics* of the observed process and *the observer states*, the second condition is for the *minimum error rate value* and is expressed by

$$\frac{de(t)}{dt}=\left[\frac{dx(t)}{dt}-\frac{d\hat{x}(t)}{dt}\right]\cong 0\tag{9}$$

The error rate dynamic equation can therefore be expressed as

$$\begin{aligned}\frac{de(t)}{dt}&=\left[Ax(t)+B_1x(t)u(t)+Bu(t)+Wx_f(t)\right]-\\ &-\left[L_1\hat{x}(t)+L_2\hat{x}(t)u(t)+L_3u(t)+L_4x_f(t)+Ly(t)\right]\end{aligned}\tag{10}$$

From the output process equation, the error rate can be expressed using the output and state variables, $y(t)=Cx(t)$. This equation is used to incorporate the measured data $y(t)$ into the equation (11) as a requirement for one of the inputs to the observer before producing the estimates. The error rate equation then becomes:

$$\begin{aligned}\frac{de(t)}{dt}&=\left[Ax(t)+B_1x(t)u(t)+Bu(t)+Wx_f(t)\right]-\\ &-\left[L_1\hat{x}(t)+L_2\hat{x}(t)u(t)+L_3u(t)+L_4x_f(t)+L(Cx(t))\right]\end{aligned}\tag{11}$$

The process of observer design involves determining the matrices L_1, L_2, L_3, and L in such a way that the two minimization error requirements for the error difference are met: the 1) *minimization of the error* between the *process state* and the *estimated state* $e(t) = x(t) - \hat{x}(t) \to 0$; and 2) *keeping error rate at zero or close to zero*, $\frac{de(t)}{dt} = 0$. From this requirement it is important to note that $x(t)$ and $u(t)$ cannot be zero in this case for the condition to hold. The mathematical derivation of the matrices of the observer is done following the steps below:

1) The error equation (12) is rewritten in the form:

$$\frac{de(t)}{dt} = (A - LC)x(t) - L_1\hat{x}(t) + \\ + \left[B_1 x(t) - L_2\hat{x}(t)\right]u(t) + (B - L_3)u(t) + (W - L_4)x_f(t) \tag{12}$$

2) Condition 1 requirements are considered to be fulfilled: if $\to$ $e(t) = 0$ or $x(t) = \hat{x}(t)$. Then, the equation (12) can be rewritten in the following manner

$$\frac{de(t)}{dt} = (A - LC - L_1)x(t) + (B_1 - L_2)x(t)u(t) + \\ + (B - L_3)u(t) + (W - L_4)x_f(t) \tag{13}$$

In order for $\frac{de(t)}{dt}$ to approach zero, and since $x(t) \neq 0$, $u(t) \neq 0$, $x_f(t) \neq 0$, $t \to \infty$ the matrices in front of these variables have to be equal to zero. The following expressions can be generated from this condition

$$\begin{aligned} &A - LC - L_1 = 0 \text{ and } \therefore L_1 = A - LC \\ &B_1 - L_2 = 0 \text{ and } \therefore L_2 = B_1 \\ &B - L_3 = 0 \text{ and } \therefore L_3 = B \\ &W - L_4 = 0 \text{ and } \therefore L_4 = W \end{aligned} \tag{14}$$

According to equation (14) the observer matrices L_2, L_3 and L_4 are determined.

3) Condition 2 requirements are such that:

The observer matrix L_1 is not determined at this moment. This can be done through fulfillment of the second requirement mentioned above after back substitution of L_1, L_3 and L_4 obtained in equation (14). The error rate equation then becomes

$$\begin{aligned} \frac{de(t)}{dt} &= \left[A - LC\right](x(t) - \hat{x}(t)) + B_1(x(t) - \hat{x}(t))u(t) \\ &= (A - LC)e(t) + B_1 e(t)u(t) \\ &= \left[A - LC + B_1 u(t)\right]e(t) \to 0 \end{aligned} \tag{15}$$

The design of the observer is to find a method for calculation of the observer matrix L in such a way that the rate of the error $e(t)$ between the process model state and the observer state is minimized and $\|e(t)\| \to 0$ when $t \to \infty$ under some assumptions about the control input $u(t)$. The above requirement fulfillment means that the system to be observed is detectable, which means if it cannot be observed, it is still asymptotically stable and the observer to be designed should converge to the real system. The control signal can be considered to be unconstrained or known and fixed. The second case is considered in the paper. In this case, the methods for observer design of linear time invariant linear systems can be applied.

Using equation (15) it becomes possible to determine the L matrix using the second requirement. For (15) to converge to zero, the requirement is that the term $\left[(A - LC) + B_1 u(t)\right]$ must be a stable matrix. From this requirement, the entries for the observer matrix L can be determined. If the value of the control input is *fixed and constrained* , the values of L can be determined from the matrix $[\overline{A} - LC]$, where $\overline{A} = [A + B_1 u(t)]$ and the observed behavior is as of a linear system. [13] suggested that if the proper choice of the gain matrix L is made, then the error will go to zero with arbitrary exponential decay. From these interpretations, the observer design problem can be summarized as a problem of choosing or selecting the observer gain matrix L such that the error rate dynamics goes to zero. The pole placement method is used to calculate the observer matrix L.

5. POLE PLACEMENT METHOD FOR DESIGN OF THE OBSERVER MATRIX

5.1 Procedure for design of the observer gain matrix L

The procedure for the design of the matrix L is built on the basis of the pole placement method. The solution for the procedure is given for the case where the *input is assumed constant for the entire process time trajectory*, or for the case *when the matrix L is calculated at every time in the sampling period* in which the control is considered constant.

The solution is derived from *the stability requirement of the error rate dynamics*, equation (15). This requirement translates to the *pole placement procedure for system stability*, i.e., the real parts of the poles of the characteristic equation of the observer must be on the negative side of the Cartesian plane, but not far from the imaginary line in order for the observer to have fast dynamics, faster than the process ones. This is another reason why the solution is considered to be based on the *pole placement method*. This method can be used for calculation of the Matrix L coefficients at every sampling period, and in real time if the values of the input signal are changing at every sampling period. Two characteristic equations of the observer are needed for the solution, one is an equation (16) for the determinant of the *observer error rate equation* and the other is the desired one, determined by the equation (17) for the determinant of the *desired observer error rate equation*.

$$\det_{obs}(s) = \left| sI - \left[A - LC + B_1 u(t)\right] \right| \tag{16}$$

$$\begin{aligned} &\det_{des}(s) = (s+p_1)(s+p_2)...(s+p_N), \quad or \\ &\det_{des}(s) = (s+p)^N = 0 \end{aligned} \tag{17}$$

Where

$s =$ the Laplace variable,
$L = [l_1 \quad l_2 \quad ... \quad l_N]^T \in R^{N\times 1}$,
$p_1, p_2, ..., p_N =$ the desired poles for every stage of the process and
$p =$ the desired pole that has the same value for all stages of the process.

The coefficients of the observer gain matrix for the given value of the control input are determined by equalization of the two determinants, as follows:

$$\det_{obs}(s) = \det_{des}(s) \tag{18}$$

5.2 *Algorithm of the pole placement method for determination of the L matrix*

The algorithm for the solving the pole placement problem is given by:

1) Give a trajectory of $u(t), t = \overline{0, t_f}$,
2) Select the input signal $u_t = u(t) -$ for a given time $t, t = \overline{0, t_f}$
3) Form the determinant of the observer $\det_{obs}$, with unknown observer gain matrix L, $L(t) = [l_1(t), l_2(t), ..., l_N(t)]^T$,
4) Form the desired determinant $\det_{des}$,
5) Compare the two determinants and calculate the gain matrix elements, $l_i(t)$, $i = \overline{1, N}$

The solution to step 5) can be obtained numerically or analytically, e.g., using *fsolve* functions in Matlab software program provides a numerical solution. The paper proposes an analytical technique of back substitution between the two characteristic equations. The derivation is explained in section 5.3.

5.3 *Derivation of the determinant det_{obs} equations*

The observer characteristic equation formed from the equation (16) is derived for the considered case of the ion exchange process as follows:

$$\det_{obs}(s) = $$

$$\det\left\{ \begin{bmatrix} s & 0 & ... & 0 & 0 \\ 0 & s & 0 & ... & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & ... & ... & s \end{bmatrix} - \begin{array}{l} -\begin{bmatrix} -a_{11} & 0 & ... & ... & ... & 0 \\ a_{21} & -a_{22} & 0 & ... & ... & 0 \\ 0 & a_{32} & -a_{33} & 0 & ... & 0 \\ 0 & ... & ... & \ddots & 0 & 0 \\ 0 & ... & ... & ... & a_{65} & -a_{66} \end{bmatrix} - \\ -[l_1 \quad l_2 \quad ... \quad l_6]^T \times [0 \quad 0 \quad ... \quad 1] + \\ +u(t)\times \begin{bmatrix} -b_{11} & b_{12} & 0 & ... & 0 & 0 \\ 0 & -b_{22} & b_{23} & 0 & ... & 0 \\ 0 & 0 & \ddots & 0 & ... & 0 \\ 0 & ... & ... & \ddots & -b_{55} & b_{56} \\ 0 & ... & ... & ... & 0 & -b_{66} \end{bmatrix} \end{array} \right\} \tag{19}$$

Where the elements of the observer matrix L are unknown.
In evaluating (19), the resulting simplified expression of the $\det obs(s)$ is a $R^{6\times 6}$ matrix given by (20),

$$\det_{obs}(s) = \det\{[A_{obs}]\} \in R^{6\times 6} \tag{20}$$

Where

$$A_{obs} = \begin{bmatrix} s+a_{11}-g_{11} & g_{12} & 0 & 0 & 0 & l_1 \\ -a_{21} & s+a_{22}-g_{22} & g_{23} & 0 & 0 & l_2 \\ 0 & -a_{32} & s+a_{33}-g_{33} & g_{34} & 0 & l_3 \\ 0 & 0 & -a_{43} & \ddots & \ddots & l_4 \\ 0 & 0 & 0 & \ddots & \ddots & (l_5+g_{56}) \\ 0 & 0 & 0 & 0 & -a_{65} & (s+a_{66}-g_{66}+l_6) \end{bmatrix}$$

Where

$g_{ii} = u(t)b_{ii} =$ the product of control input and b coefficients of the bilinear term, $i = \overline{1,6}$
$g_{ij} = u(t)b_{ij} =$ the product of control input and coefficients of the bilinear term for $i = \overline{1,5}$ and $j = \overline{2,6}$.

The mathematical derivation of the characteristic equation from the determinant $|A_{obs}|$ is very complex because of the higher dimension of the matrix A_{obs}. In order to simplify the calculations the resulting determinant is presented as a sum of sub-determinants which further aids in simplifying the calculation. The sub-determinants are derived from the first row A_{obs}

$$\det_{obs}(s) = \det_1(s) + \det_2(s) + \det_6(s) \tag{21}$$

Where

$$\det_1 = (s + a_{11} - g_{11}) \times$$
$$\times \begin{vmatrix} s + a_{22} - g_{22} & g_{23} & 0 & 0 & l_2 \\ -a_{32} & s + a_{33} - g_{33} & g_{34} & 0 & l_3 \\ 0 & -a_{43} & \ddots & \ddots & l_4 \\ 0 & 0 & \ddots & \ddots & (g_{56} + l_5) \\ 0 & 0 & 0 & -a_{65} & (s + a_{66} - g_{66} + l_6) \end{vmatrix} \tag{22}$$

$$\det_2(s) = -g_{12} \times$$
$$\times \begin{vmatrix} -a_{21} & g_{23} & 0 & 0 & l_2 \\ 0 & s + a_{33} - g_{33} & g_{34} & 0 & l_3 \\ 0 & -a_{43} & s + a_{44} - g_{44} & g_{45} & l_4 \\ 0 & 0 & \ddots & \ddots & (g_{56} + l_5) \\ 0 & 0 & 0 & -a_{65} & (s + a_{66} - g_{66} + l_6) \end{vmatrix} \tag{23}$$

$$\det_6(s) = -l_1 \times$$
$$\times \begin{vmatrix} -a_{21} & s + a_{22} - g_{22} & g_{23} & 0 & 0 \\ 0 & -a_{32} & s + a_{33} - g_{33} & g_{34} & 0 \\ 0 & 0 & -a_{43} & s + a_{44} - g_{44} & g_{45} \\ 0 & 0 & 0 & -a_{54} & s + a_{55} - g_{55} \\ 0 & 0 & 0 & 0 & -a_{65} \end{vmatrix} \tag{24}$$

Equations (20) – (24) are determined for every sampling period if the control vector is not constant. Using the same approach the sub-determinants $\det_1(s)$, $\det_2(s)$, and $\det_6(s)$ are presented by their sub-determinants, (25) – (27)

$$\det_1(s) = (s + a_{11} - g_{11}) \times$$
$$\times \left[(s + a_{22} - g_{22}) \det_{11}(s) - g_{23} \det_{12}(s) + l_2 \det_{16}(s)\right] \tag{25}$$

$$\det_2(s) =$$
$$g_{12}\left[a_{21} \det_{21}(s) - g_{23} \det_{22}(s) + l_2 \det_{26}(s)\right] \tag{26}$$

$$\det_6(s) = -l_1 \times$$
$$\times \left[-a_{21} \det_{61}(s) - (s + a_{22} - g_{22}) \det_{62}(s) + g_{23} \det_{63}(s)\right] \tag{27}$$

The procedure above is followed for all new sub-determinants until their mathematical expressions are derived and the characteristic equation of the observer is obtained, written according to the power of the Laplace variable s. The desired characteristic equation is calculated for a real negative desired pole $p_i = -5$, $i = \overline{1, N}$. Determination of its equation is via a Matlab software *system function*, *simplify*( ) and *expand*( ) functions,

$$\left.\begin{array}{l} syms \quad s \\ ce = simplify\,((s + p)^\wedge N) \\ \theta = expand\,(ce) \end{array}\right\} \tag{28}$$

Where ce stands for characteristic equation.

After evaluating both characteristic equations, the resulting simplified equations are equated to determine values of $l_i, i = \overline{1, N}$, as follows:

$$\det_{des}(s) =$$
$$s^6 + 30s^5 + 375s^4 + 2500s^3 + 9375s^2 + 1875s + 15625 =$$
$$\det_{obs}(l_i); \quad i = \overline{1,6} \tag{29}$$

The algorithm for calculation of the observer gain matrix is based on the derivations in point 5. It is shown in Figure 3 and Figure 4.

6. SOFTWARE FOR DESIGN AND SIMULATION

The solution of the observer gain matrix is obtained by the development of a Matlab/Simulink software program. The software program is written according to the proposed algorithm in Figure 3 using the Simulink part as described in Figure 4. The algorithm and the developed software have two parts:

1) Matrix calculation. The first part is used to calculate the model parameters based on the experimental data; then the observer gain matrix L is calculated based on the *observer and the observer desired characteristic equations*.
2) Verification of the observer gain matrix. Once the observer parameters are determined, the second part of the program is to simulate the system error based on various specified constant control input and initial conditions for the states of the observer for the full trajectory of the system until *the error and error rate* requirements are met. If these requirements are not met, then new desired poles are generated and the calculation of the observer gain matrix is calculated again, and so on. Simulink environment, Figure 4 is used for this part. The results from the simulations in this part of the program provide means to evaluate, for which area of constant values of the control input the proposed algorithm will be applicable. The same is done for the initial conditions of the estimated states.

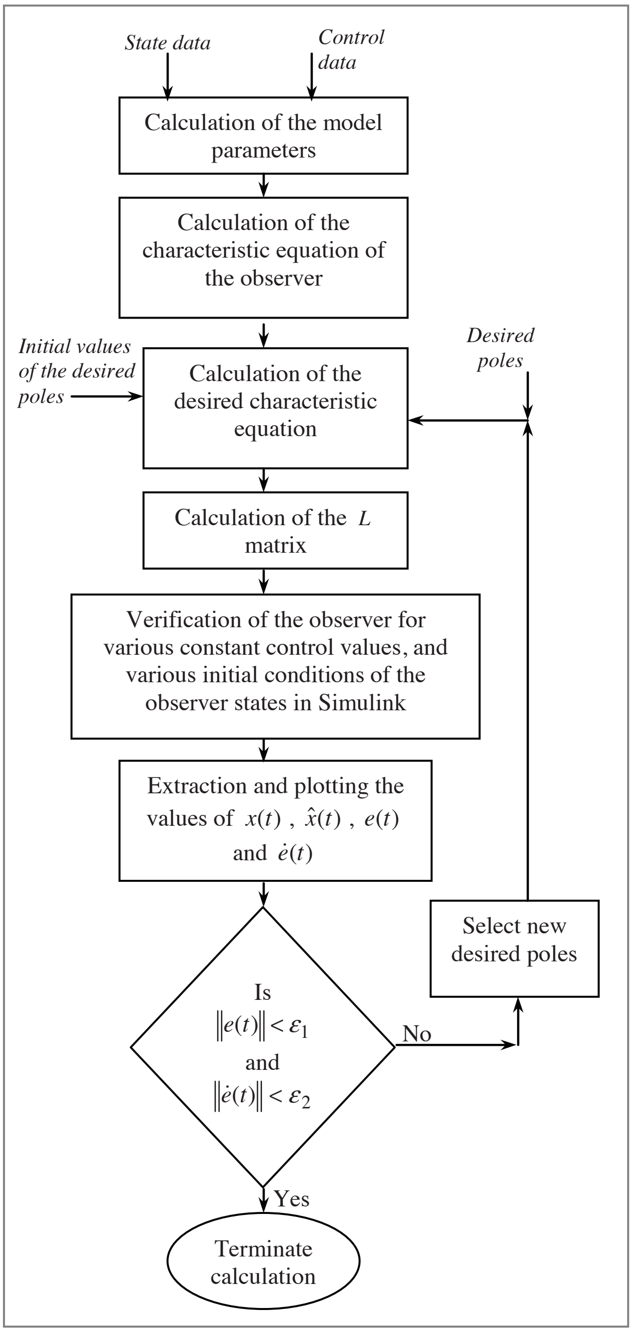

Figure 3: Algorithm for the design process

The proposed algorithm for design can be used for real-time control too, when the control action is changing, but having constant values in the sampling periods for the calculation of the observer gain matrix. In this case the algorithm will be run for every new value of the control signal and the obtained value of L will be directly used for calculation of the estimated state values. These values will be used for calculation of the control action. The structure of the algorithm is similar as in Figure 3, but the difference is that the real-time system replaces the Simulink block, Figure 3, and the calculated observer gain matrix is directly used to produce the real-time state estimates. No verification of the L values is performed in real-time.

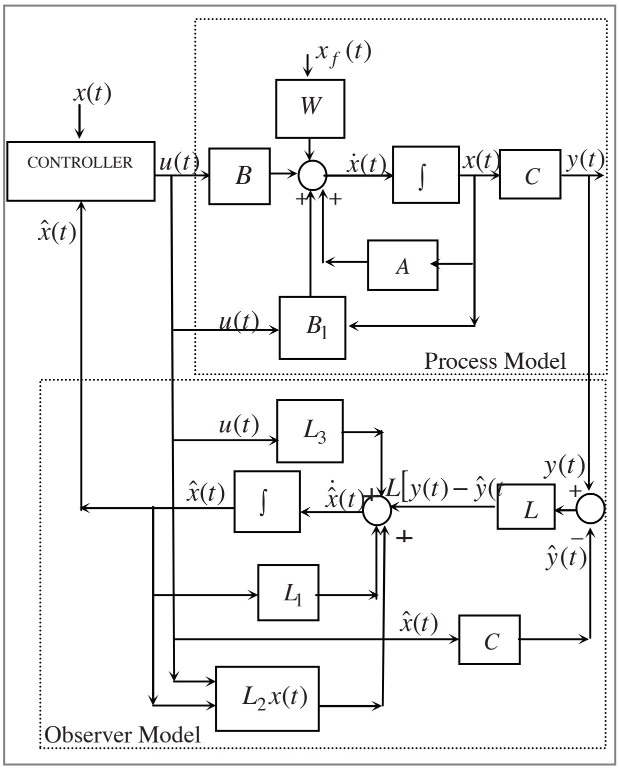

Figure 4: Simulation diagram of the closed loop system

7. SIMULATION RESULTS AND DISCUSSION

The observer gain matrix entries are finally determined from Matlab using data obtained from the previous CCIX project [1, 2, 4, 5], Table I, and the error dynamics are observed through Simulink.

7.1 Description of the data

Concentration measurement data from [1, 2, 3] were used to estimate the state variables of the newly developed bilinear model of the continuous countercurrent ion exchange (CCIX) process. The results are based on normalized data of a six stages CCIX process column. Process parameters were calculated for the input flow rate $F_L = 2000 m^3/h$, the up-flow period of $T = (17/60)h$, resin liquid constant ratio $d = 2/3$, the liquid holdups $42.809l$ and the resin holdups of $32.93l$ [1, 2, 4, 5].
The normalised original data measured from the University of Cape Town (UCT) project [1] is presented in Table I below.

TABLE I. CCIX DATA SHOWING CONCENTRATION IN EACH STAGE OF THE CATION COLUMN AS PER UCT PROJECT

Stage H^+ fractional change in liquid concentration using data obtained from UCT Project [Hendry, 1982b ,Vol. 4]					
Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6
0.221	0.000	0.000	0.000	0.000	0.000
0.577	0.140	0.040	0.000	0.000	0.000

0.730	0.314	0.066	0.004	0.000	0.000
0.847	0.523	0.184	0.035	0.003	0.000
0.920	0.656	0.295	0.082	0.020	0.000
0.936	0.766	0.454	0.168	0.052	0.001
0.968	0.842	0.601	0.277	0.113	0.024
0.974	0.886	0.690	0.361	0.167	0.033
0.981	0.900	0.758	0.440	0.207	0.063
0.989	0.933	0.804	0.522	0.340	0.124
0.988	0.958	0.877	0.698	0.474	0.167
0.997	0.963	0.881	0.784	0.547	0.233
1.000	0.982	0.951	0.899	0.780	0.482
1.000	0.974	0.965	0.931	0.860	0.539
1.000	0.991	0.972	0.966	0.899	0.672
1.000	0.994	0.981	0.966	0.905	0.779
1.000	0.993	1.000	0.991	0.975	0.940
1.000	0.993	0.988	0.991	0.973	0.972

This data set shows the concentration in *fractional change* of sodium ions, H^+ as determined from the equation,

$$FC_n = \frac{I.F.H_n^+ - I.F.H_{initial}^+}{I.F.H_{final}^+ - I.F.^+_{initial}} \qquad (30)$$

Where

$FC_n =$ the fractional change [2]

$I.F.H_n^+ =$ the ionic fraction of H^+ ions of the current measurement from the step change moment [2],

$I.F.H_{initial}^+ =$ the initial ionic fraction of the H^+ ions in stage [2],

$I.F.H_{final}^+ =$ the final ion fraction of the H^+ ions in the stage [2].

7.2 Calculation of model matrices

Matlab software programs were run using the data from Table I above to determine the unknown process parameters. The matrices of the process model were calculated to be:

$$A = \begin{bmatrix} -26.406 & 0 & 0 & 0 & 0 & 0 \\ 28.921 & -28.921 & 0 & 0 & 0 & 0 \\ 0 & 31.966 & -31.966 & 0 & 0 & 0 \\ 0 & 0 & 35.726 & -35.726 & 0 & 0 \\ 0 & 0 & 0 & 37.959 & -37.959 & 0 \\ 0 & 0 & 0 & 0 & 43.382 & -43.382 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.013 & 0.011 & 0 & 0 & 0 & 0 \\ 0 & -0.012 & 0.009 & 0 & 0 & 0 \\ 0 & 0 & -0.010 & 0.006 & 0 & 0 \\ 0 & 0 & 0 & -0.007 & 0.0054 & 0 \\ 0 & 0 & 0 & 0 & -0.006 & 0.002 \\ 0 & 0 & 0 & 0 & 0 & -0.002 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0400 \\ 0.0500 \\ 0.1000 \\ 0.1000 \\ 0.0500 \\ 0.3500 \end{bmatrix}, \quad W = \begin{bmatrix} 26.4065 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

The observer gain matrix evaluated with the constant input signal value and model parameters from the above model matrices is shown in equation (31).

$$L = 1\times 10^5 \begin{bmatrix} 0.782 & 1.975 & 1.235 & 0.024 & 0.020 & 0.0016 \end{bmatrix}^T \qquad (31)$$

7.3 Experiments for validation of the observer gain matrix and closed loop system performance

Multiple simulation runs were performed with different *input signal values*, *initial conditions for the observer* and the same *initial conditions for the process*, as per Table II below. A selected sample of the runs is presented here. The steps followed in the simulation procedure are also presented. The initial values of the estimated states are the same for every stage of the column. The simulation programs are run according to values in Table II until the dynamic error rate reaches zero.

TABLE II. THE VALUES USED FOR VALIDATION OF THE OBSERVER PERFORMANCE

	Experiments performed				
	Run 1	Run 2	Run 3	Run 4	Run 5
Input value $u(k)$	0.0	1.0	5.0	10.0	20.0
Process Init. Cond.	1.0	1.0	1.0	1.0	1.0
Observer Init. Cond.					
Set 1	– 1.0	– 1.0	– 1.0	– 1.0	– 1.0
Set 2	0.0	0.0	0.0	0.0	0.0

Set 3	0.5	0.5	0.5	0.5	0.5
Set 4	1.0	1.0	1.0	1.0	1.0
Set 5	10.0	10.0	10.0	10.0	10.0
Set 6	20.0	20.0	20.0	20.0	20.0

7.4 Graphs and results of the simulation process

The following graphs (Figures 5–22) show the behaviour of the process states (from the model) versus that of the estimated states (observer) based on Simulink simulation runs with a constant input signal of 1.0 and constant initial conditions of the states at 1.0 based on the number of upflow cycles. The different runs are based on the changing observer (estimated) states initial conditions of -1.0, 0, 1/2, 1.0, 5.0 and 10.0; these initial conditions are the same for every stage per each run, i.e., for $\hat{x}_1, \hat{x}_2, ..., \hat{x}_6$. On the graphs the estimated states are represented by $x^{\sim}n$, $n = \overline{1,6}$. The error and error rate display (Figures 5–21) colour coding matches that of the process and the observer.

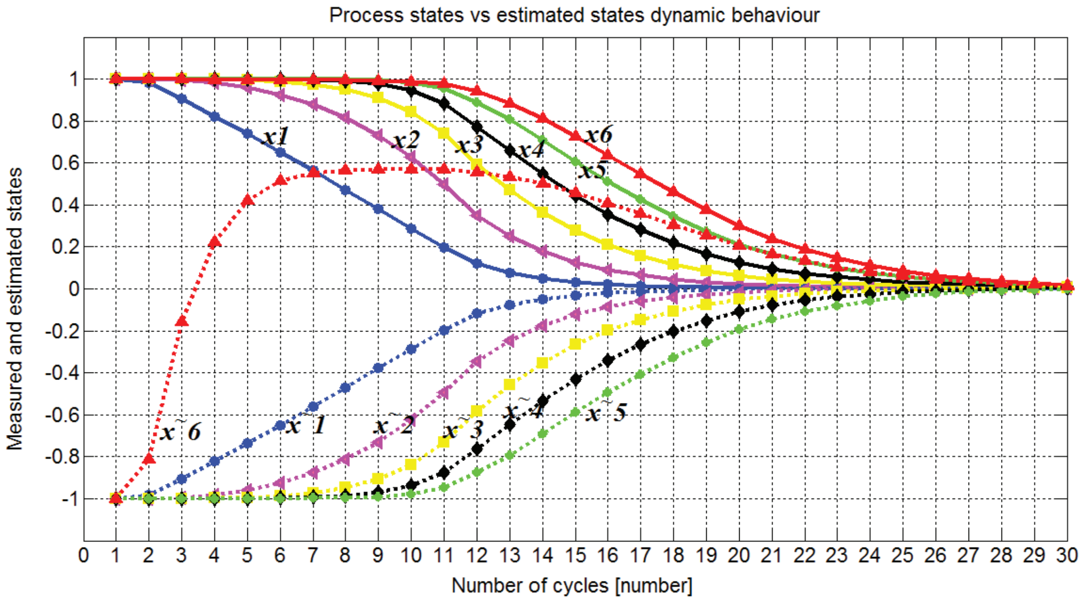

Figure 5:Model states and estimated states for observer initial conditions of –1.0 and model initial conditions of 1.0 using a constant control input of 1.0.

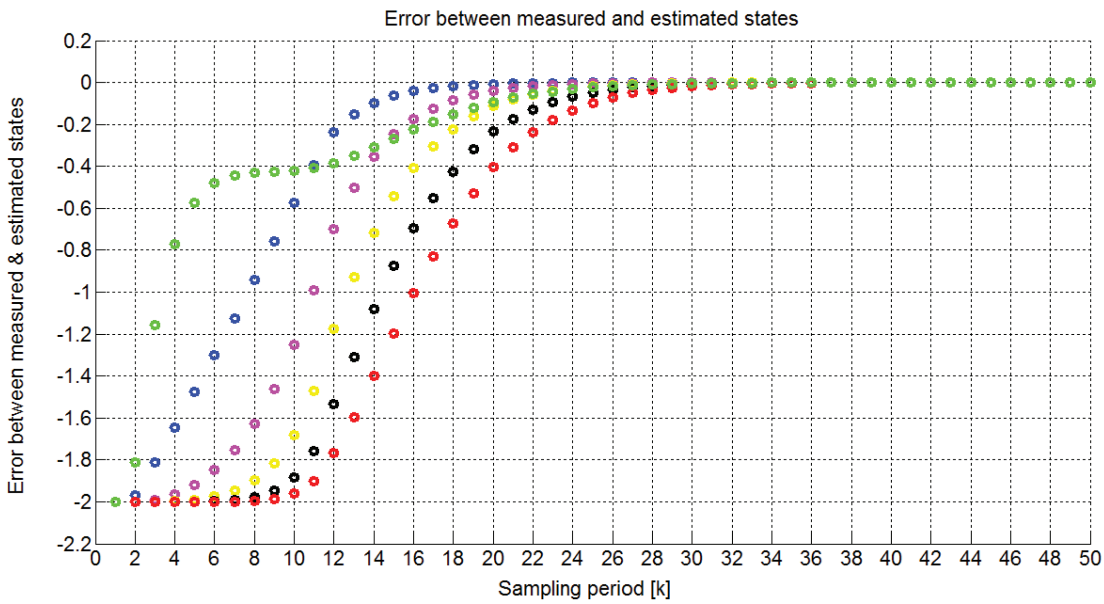

Figure 6: Error between process states and estimated states for observer initial conditions of –1.0 and model initial conditions of 1.0 using a constant control input of 1.0.

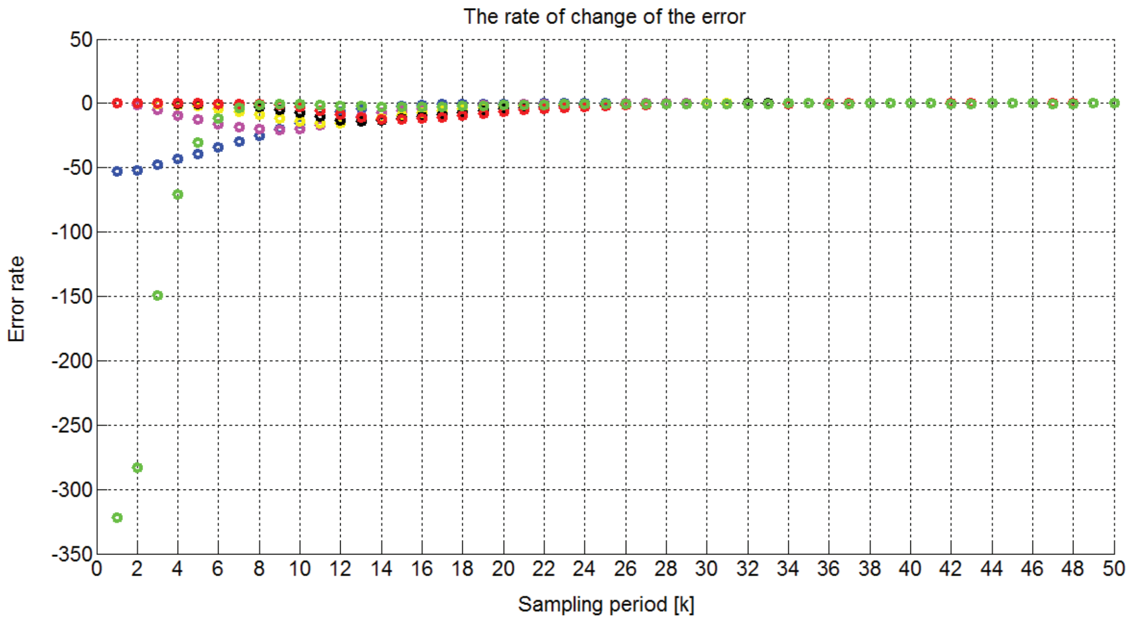

Figure 7: Rate of change of the error for observer initial conditions of –1.0 and model initial conditions of 1.0 using a constant control input of 1.0.

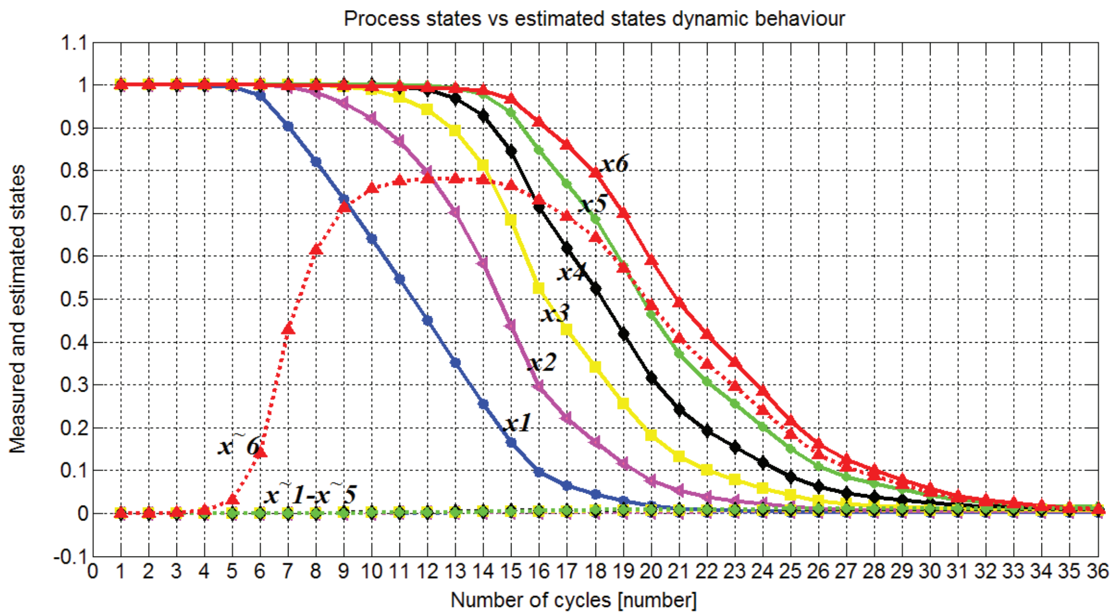

Figure 8: Model states and estimated states for observer initial conditions of 0.0 and model initial conditions of 1.0 using a constant control input of 1.0.

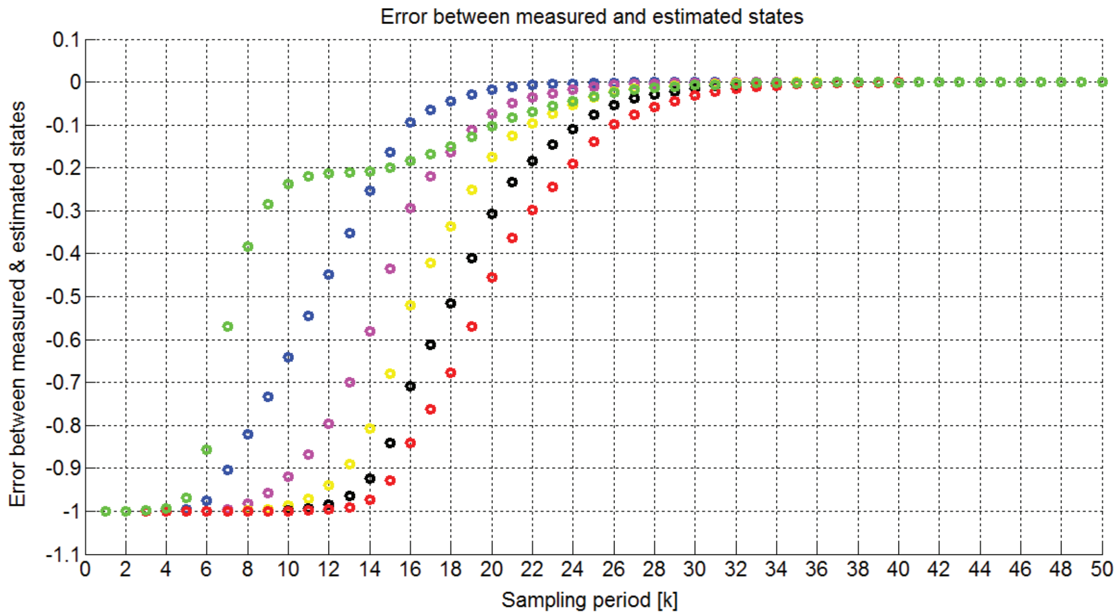

Figure 9: Error between process states and estimated states for observer initial conditions of 0.0 and model initial conditions of 1.0 using a constant control input of 1.0.

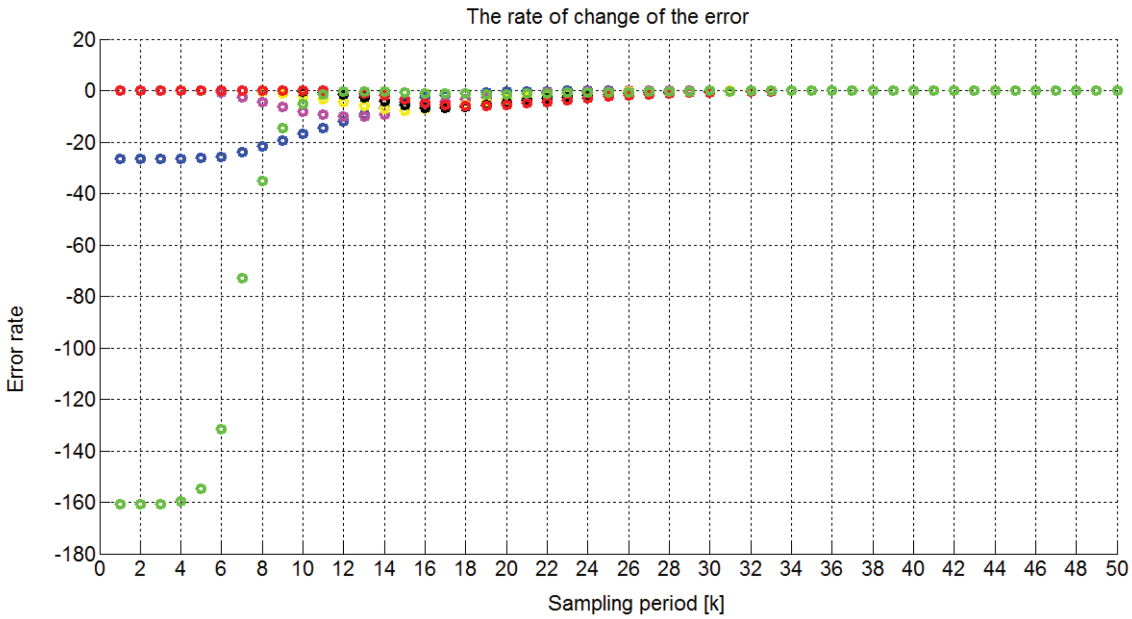

Figure 10: Rate of change of the error for observer initial conditions of 0.0 and model initial conditions of 1.0 using a constant control input of 1.0.

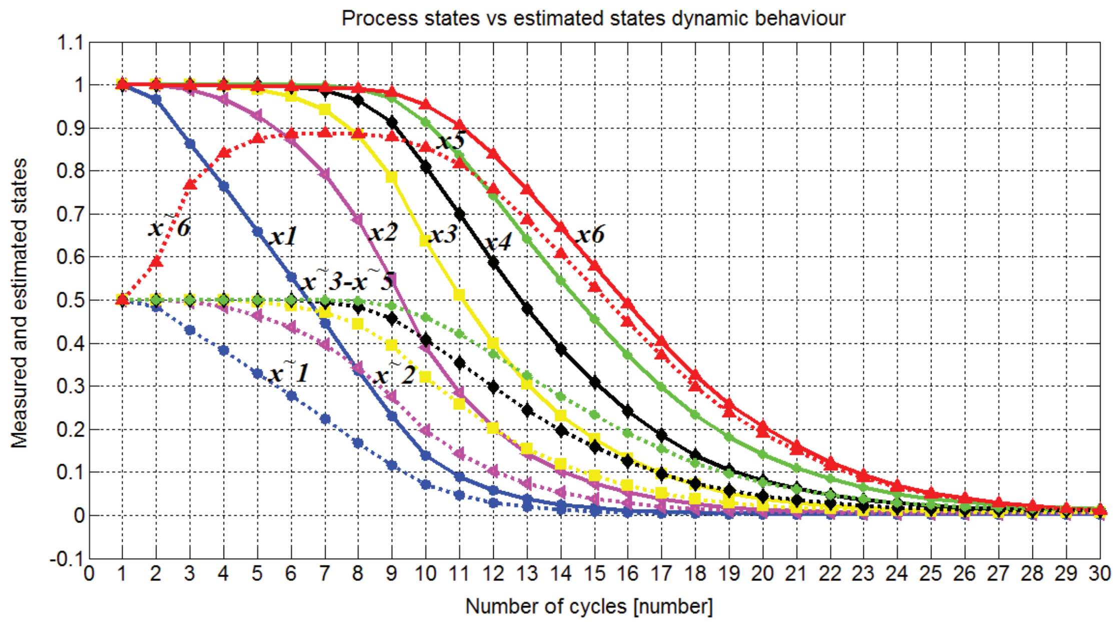

Figure 11: Model states and estimated states for observer initial conditions of 0.5 and model initial conditions of 1.0 using a constant control input of 1.0.

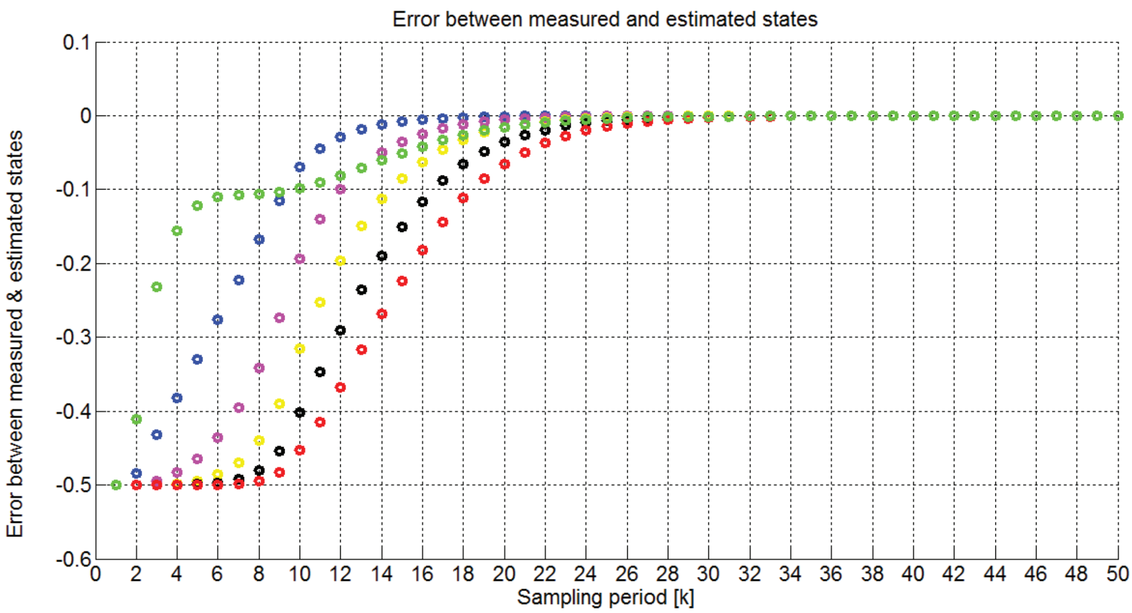

Figure 12: Error between process states and estimated states for observer initial conditions of 0.5 and model initial conditions of 1.0 using a constant control input of 1.0.

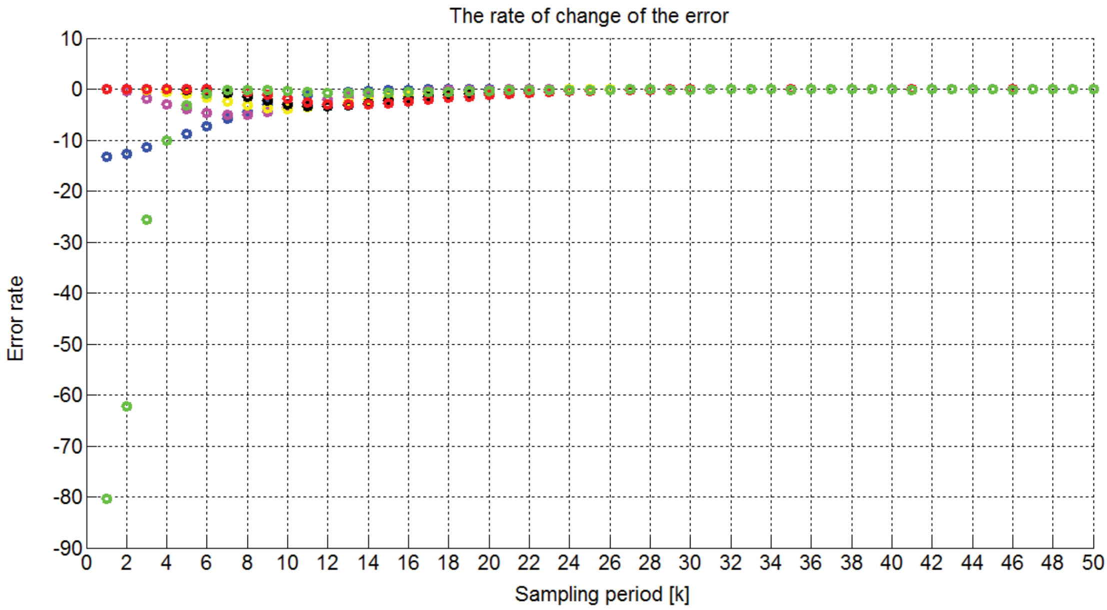

Figure 13: Rate of change of the error for observer initial conditions of 0.5 and model initial conditions of 1.0 using a constant control input of 1.0.

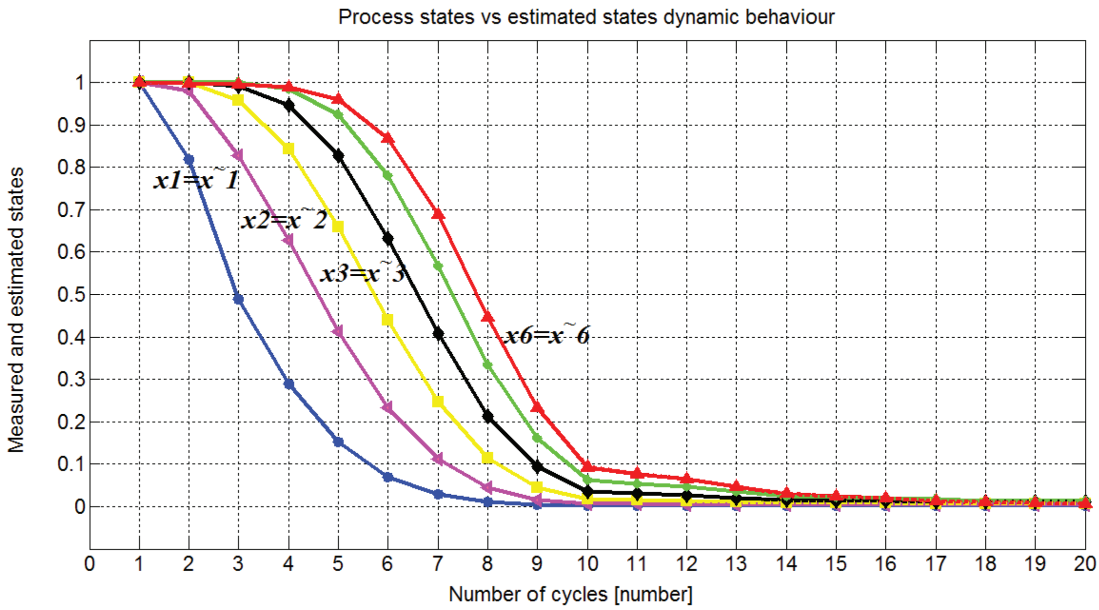

Figure 14: Model states and estimated states for initial conditions of 1.0 for both the observer and model using a constant input signal of 1.0.

In Figure 14, in the same fashion for $x1 = x^{\sim}1$, $x2 = x^{\sim}2$, etc., $x4 = x^{\sim}4$ and $x5 = x^{\sim}5$.

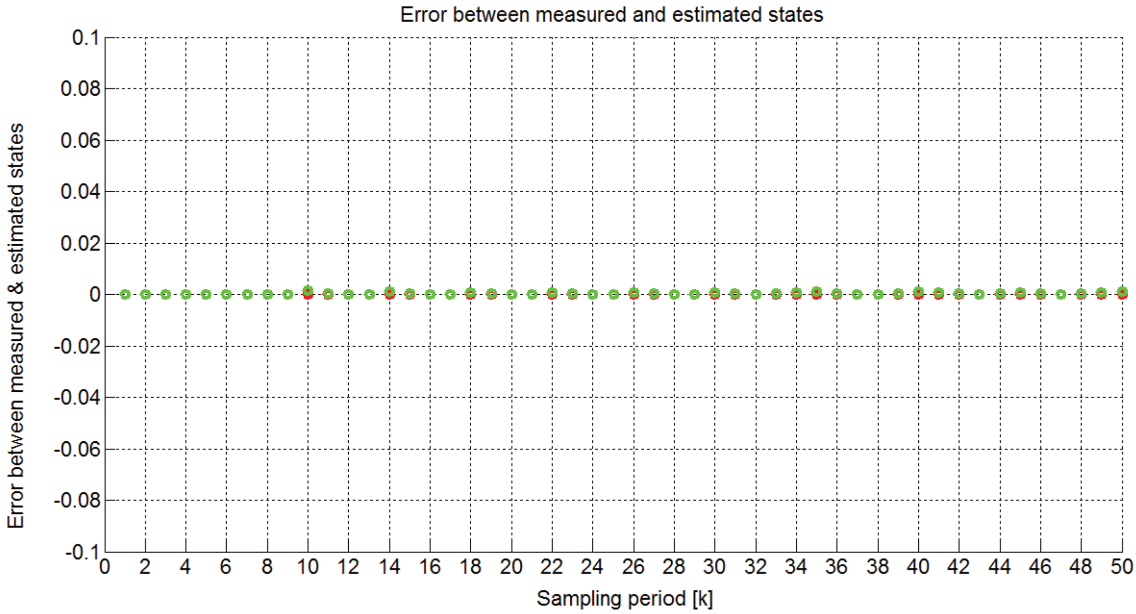

Figure 15: Error between process states and estimated states for observer initial conditions of 1.0 and model initial conditions of 1.0 using a constant control input of 1.0.

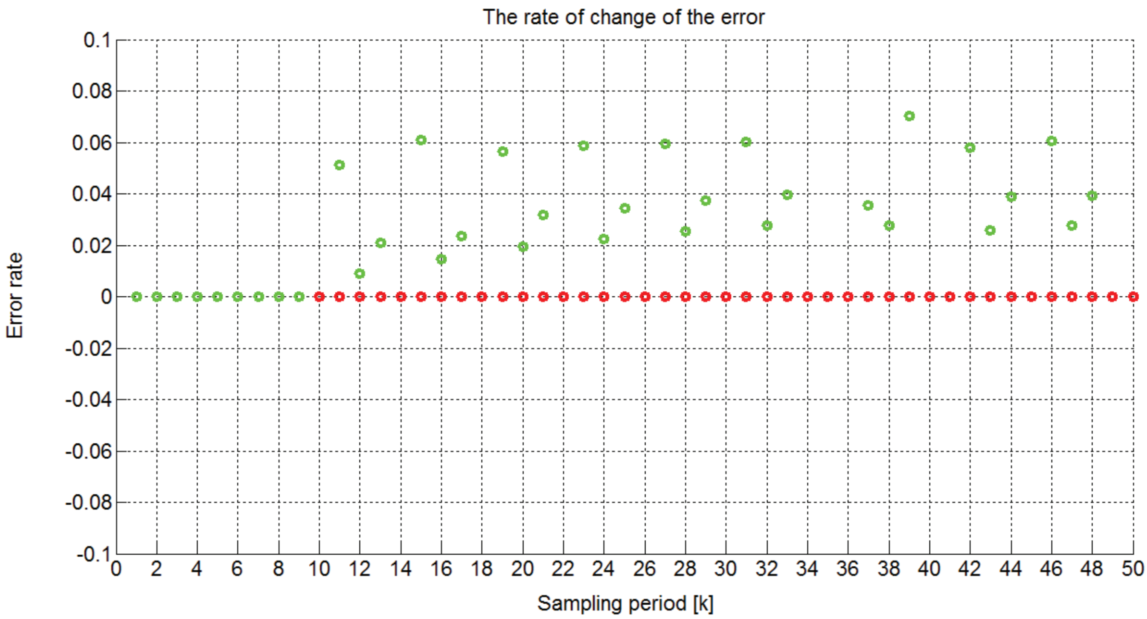

Figure 16: Rate of change of the error for observer initial conditions of 1.0 and model initial conditions of 1.0 using a constant control input of 1.0.

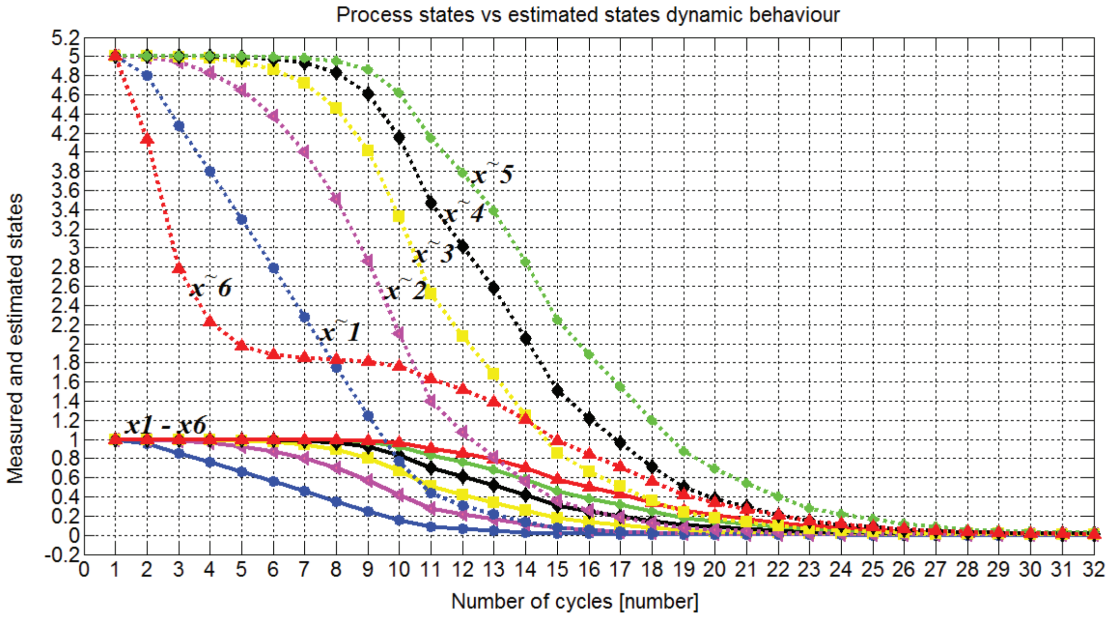

Figure 17: Model states and estimated states for observer initial conditions of 5.0 and model initial conditions of 1.0 using a constant input signal of 1.0.

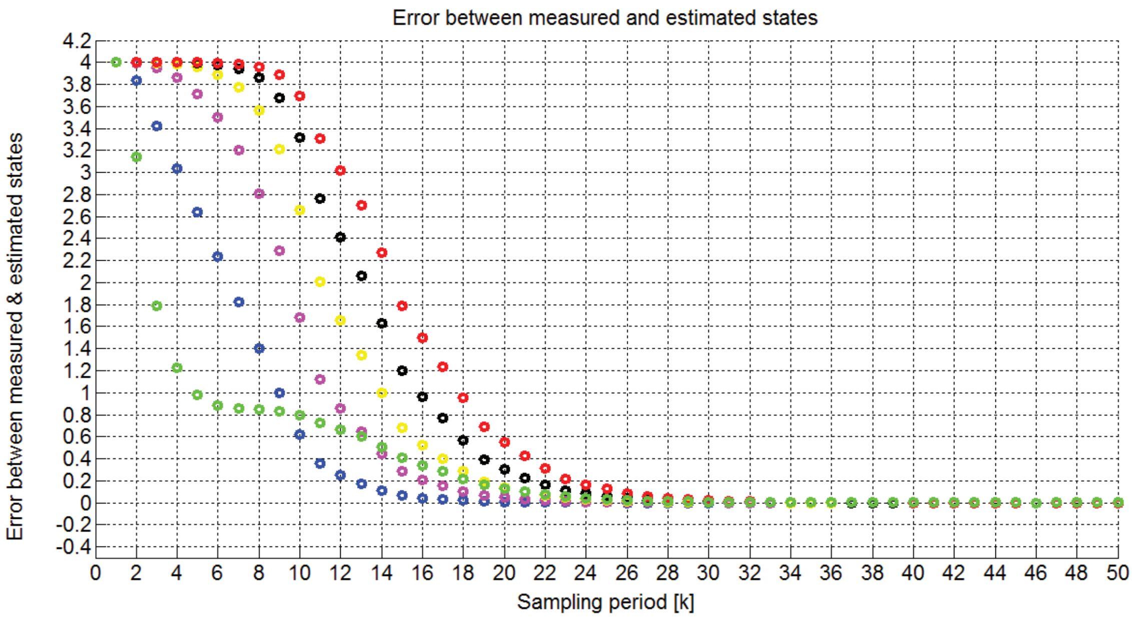

Figure 18: Error between process states and estimated states for observer initial conditions of 5.0 and model initial conditions of 1.0 using a constant control input of 1.0.

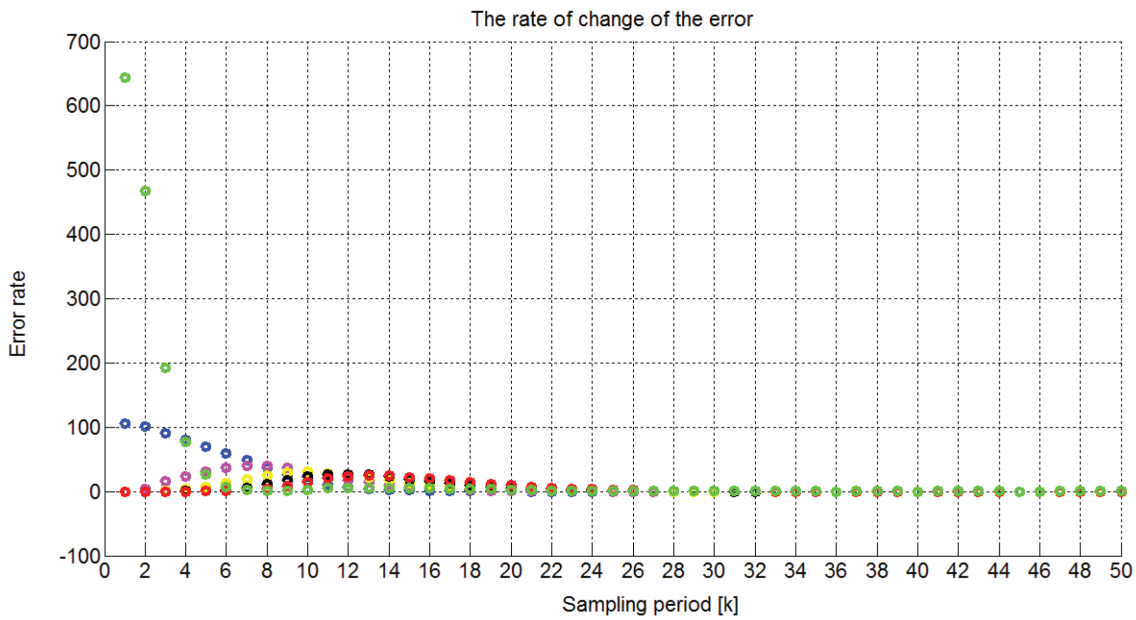

Figure 19: Rate of change of the error for observer initial conditions of 5.0 and model initial conditions of 1.0 using a constant control input of 1.0.

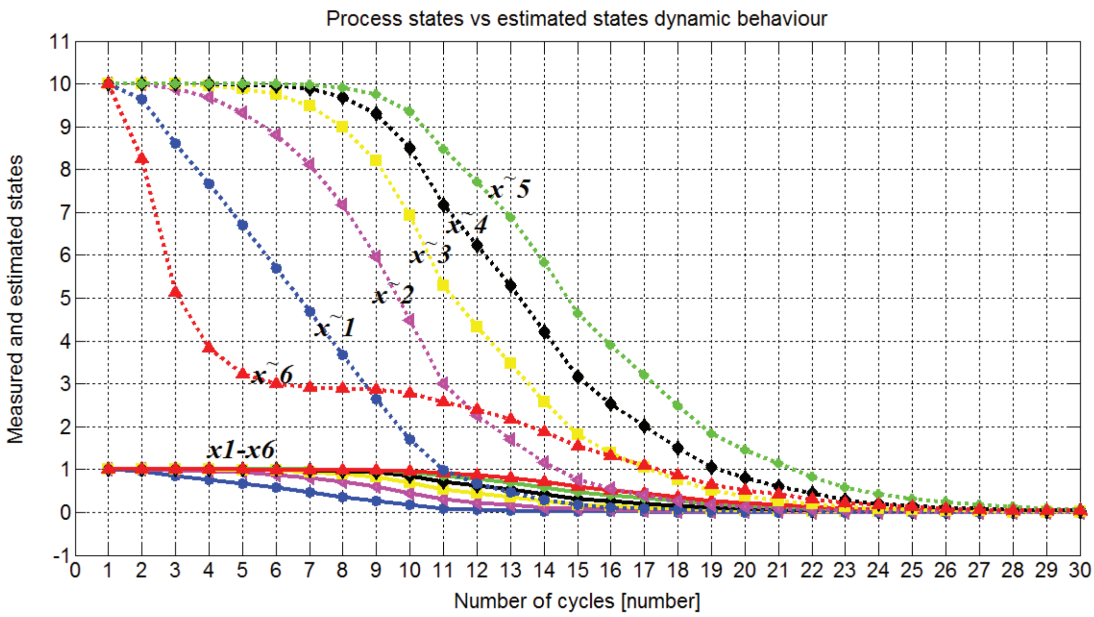

Figure 20: Model states and estimated states for observer initial conditions of 10.0 and model initial conditions of 1.0 using a constant input signal of 1.0.

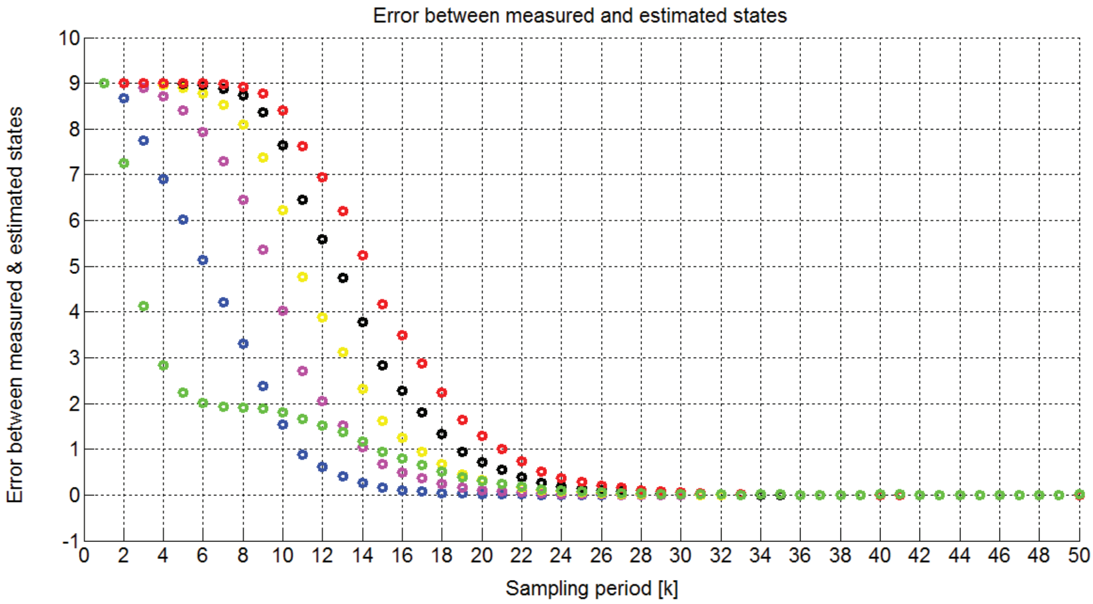

Figure 21: Error between process states and estimated states for observer initial conditions of 10.0 and model initial conditions of 1.0 using a constant control input of 1.0.

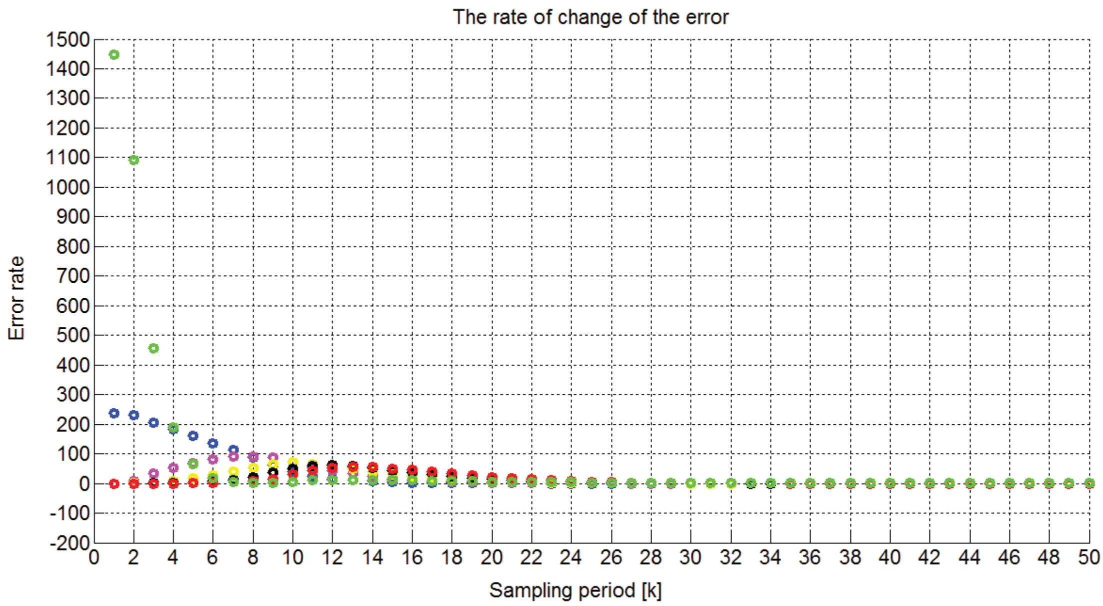

Figure 22: Rate of change of the error for observer initial conditions of 10.0 and model initial conditions of 1.0 using a constant control input of 1.0.

These results are based on a system with constant observer gain matrix L. It can be clearly seen from the Figures 5–22 that the observer fully tracks the system and converges at a relatively short time period, the 30th cycle for the last stage. Each cycle is 17min. In general, the process of ion exchange is a slow process. In the case where the observer and the system have the same initial conditions, the observer converges even at a shorter period, just before the 20th cycle. This case also shows that the observer is stable. These results further show that the observer is performing very well. At higher observer initial conditions, compared to that of the system (Figure 20), the observer seems to be not smooth in its tracking, but does converge in time compared to other cases where the initial conditions are not far from that of the system. The process system response times have also been examined over various constant control signals and different initial conditions, and these cases are presented by Table III and Table IV. The system responses considered are the rise time (Tr), delay time (Td) and settle time (Ts) for the first, the third and the sixth stages of the ion exchange process column. Both the model and the observer (estimated) system responses are presented in Table III – Table IV.

TABLE III. RISE TIME VS OBSERVER CHANGING INITIAL CONDITION

Process Rise Time vs. Observer Changing Initial Conditions						
Input Signal $u(k)=0$						
Observer Init. Cond.	State $x1$		State $x3$		State $x6$	
	x_1	$\hat{x}_1$	x_3	$\hat{x}_3$	x_6	$\hat{x}_6$
-1	9.319	9.319	9.170	9.170	11.42	1.622
0	8.846	0	8.940	0	11.42	#NA
0.5	8.079	8.079	8.993	8.993	11.42	1.622
1	4.061	4.061	4.625	4.625	4.237	4.237
Input Signal $u(k)=1$						
-1	9.351	9.290	9.331	9.015	11.56	1.624
0	8.870	8.814	9.100	10.22	11.57	#NA
0.5	8.111	7.138	9.130	9.264	11.53	8.831
1	4.078	4.078	4.693	4.693	4.294	4.294
Input Signal $u(k)=5$						
-1	9.483	9.173	10.07	8.638	12.13	1.633
0	8.964	8.823	10.02	10.85	12.21	#NA
0.5	8.238	8.380	9.847	11.43	12.08	9.550
1	4.145	4.145	4.964	6.309	6.300	6.309
Input Signal $u(k)=10$						
-1	9.651	9.026	11.61	8.249	12.91	1.643
0	9.236	8.841	11.58	10.69	13.13	#NA
0.5	8.413	8.850	11.47	#NA	12.96	10.75
1	4.231	4.231	5.400	5.400	6.728	6.724
Input Signal $u(k)=20$						
-1	10.04	8.863	#NA	7.731	18.56	1.667
0	9.783	8.947	#NA	10.89	18.21	#NA
0.5	8.723	10.30	13.26	12.86	17.95	17.10
1	4.407	4.407	20.96	20.96	20.96	20.96

TABLE IV. SETTLE TIME VS OBSERVER CHANGING INITIAL CONDITION

Process Settle Time vs. Observer Changing Initial Conditions						
Input Signal $u(k)=0$						
Observer Init. Con.	State $x1$		State $x3$		State $x6$	
	x_1	$\hat{x}_1$	x_3	$\hat{x}_3$	x_6	$\hat{x}_6$
-1	16.00	16.00	23.00	22.90	26.50	28.50
0	28.20	16.40	40.00	21.00	45.00	42.00
0.5	14.30	14.40	19.20	21.20	27.70	36.00
1	11.00	16.00	22.00	22.00	0.000	0.000
Input Signal $u(k)=1$						
-1	16.20	15.90	24.00	22.50	29.80	28.80
0	31.50	21.30	39.70	42.80	46.50	59.00
0.5	14.50	14.50	22.30	20.30	27.20	28.00
1	11.50	11.50	14.60	14.60	25.50	29.10
Input Signal $u(k)=5$						
-1	17.00	15.20	32.00	26.00	36.00	32.00
0	39.00	37.50	47.50	38.50	54.00	48.00
0.5	15.55	19.30	21.10	21.50	30.40	29.70
1	12.00	12.00	21.00	21.00	28.00	28.00
Input Signal $u(k)=10$						
-1	16.00	13.40	23.50	18.20	29.80	28.30
0	37.50	37.50	41.50	36.00	59.00	59.00
0.5	17.74	27.50	21.88	21.60	28.40	35.00
1	15.50	15.50	25.60	25.60	26.00	26.00
Input Signal $u(k)=20$						
-1	16.00	16.10	23.00	17.80	29.30	28.00
0	31.50	25.00	41.50	35.00	58.00	51.00
0.5	22.00	14.50	21.70	22.00	35.00	29.80
1	11.75	11.75	13.30	13.30	30.00	25.30

7.5 Discussion

The observer has shown that it converges within a reasonable time and maintains the error rate close to zero for the rest of the observed period, Figure 5 – Figure 22. The observer has also shown to be sensitive to the input signal. The input signal values should also be kept within normalized values (0–1.0), otherwise overshoot will be experienced in some process stages. The observer gain matrix values tend to be very high; this could be associated with the determinant calculation that involves a very large number of computational values. Error results showed the success of the observer. The error tracked the difference between estimated and measured states correctly as expected. The error rate showed fairly fast tracking capabilities, thus confirming a well designed observer.

8. CONCLUSION

The observer design for a bilinear model of the Continuous Countercurrent Ion Exchange (CCIX) process has been developed. The observer design has been developed based on real data of the ion exchange process obtained from experiments conducted previously in an ion exchange process of the same type. The data have been normalized for this exercise and simulation results from Simulink and Matlab showed the design to be competently conclusive. The observer converges even if different initial conditions are applied (see Figures 5–22). The results show that the bilinear type observer is applicable in this type of a bilinear process models. The influence of the process initial conditions in comparison to the observer initial conditions has been presented through system responses as indicated by the figures and the tables, Tables III – IV.

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