Time Optimization in Spectrum Sensing: Interesting Cases

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***Abstract*—Traditionally spectrum sensing approaches do not take the historical traffic data into account. In such approaches, equal amount of time is allocated for spectrum sensing of the sub-bands in the band of interest. In this research paper, we formulate the problem of time optimal spectrum sensing taking the historical data into account. We solve the problem in practically interesting special cases. Effectively interesting integer programming problems are solved. In that effort it is shown that the variance of discrete random variable constitutes a quadratic form associated with a laplacian like matrix. Using this result, time optimal spectrum sensing is formulated as a multi-linear objective function optimization problem.**

***Index Terms*—Spectrum Sensing, Pareto Front, Integer Programming, Stochastic Optimization.**

I. Introduction

In recent years, cognitive radio technology [1] is proposed for making efficient utilization of electromagnetic spectrum. At the physical layer of cognitive radio networks, various techniques are proposed for spectrum sensing [2]. In earlier efforts of spectrum sensing, the temporal record/history of spectrum utilization has been completely ignored [3] [4] [5]. The impact of taking historical data into account in dynamic spectrum allocation is studied in [6] and [7]. These studies proves that the overall spectrum utilization increases if historical records are taken into considerations. Some researchers realized that such approach to spectrum sensing is sub-optimal[8][9]. In this research paper, we make precise mathematical formulation of time optimal spectrum sensing and propose interesting solutions that are practically useful.

II. Time Optimal Spectrum Sensing: Integer Linear Programming

Consider a band of EM spectrum available for wireless communication. Let this band be subdivided into sub-bands labeled $1, 2, \dots, M$. In traditional spectrum sensing based on, say, energy detection, all the sub-bands are scanned with a fixed, constant time irrespective of the historical data about packet traffic (refer fig. 1). It is logically clear that the sub bands which are expected to be heavily occupied(based on historical traffic data) can be scanned faster(with smaller sensing time) while the less occupied sub-bands can be scanned using larger sensing time. The total available time for spectrum sensing of the entire band is assumed to be constant, say L seconds. The sensing time allocated for each of the sub bands is assumed to be integer valued.

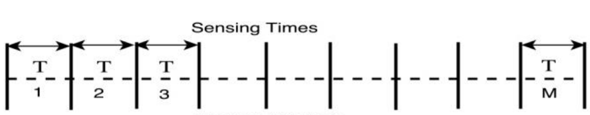

Fig. 1. Traditional Spectrum Sensing.

Remark 1. The time optimal spectrum sensing problem formulated below does not depend on the spectrum sensing approach.

As discussed earlier, we take the historical traffic data on various sub-bands into account for choosing the spectrum sensing time. In this direction the historical traffic data can be modeled as an one of the time series models. The prediction tool(model) can be chosen to be more sophisticated (artificial neural network based approach).

Let the predicted traffic in M sub-bands be denoted by $n_1, n_2, \dots, n_M$. We normalize the traffic in various sub-bands in the following manner

$$q_i = \frac{n_i}{\sum_{j=1}^{M} n_j} \text{ for } 1 \leq i \leq M \tag{1}$$

Thus $\{q_1, q_2, \dots, q_M\}$ is a probability mass function, associated with traffic data in various sub-bands(on some time units).

Now, we formulate the time-optimal spectrum sensing problem. Our goal is to allocate the total time for sensing the entire band (say L seconds) into time for sensing sub-bands(i.e.$T_1, T_2, \dots, T_M$) such that the average sensing time is minimized. i.e.

$$\text{minimize} \sum_{i=1}^{M} T_i q_i \text{ subject to } \sum_{i=1}^{M} T_i = L \tag{2}$$

We reason in [10] that if no constraints are imposed on T_i, then we have a trivial problem.

For example, smallest sensing time is at-least T_1 and other sensing times differ by at-least d time units. Allocation can be as following: $T_1, T_1 + d, T_1 + 2d, \dots, T_1 + (M-1)d, (L-S)$ with $(L-S) \geq T_1$, where $S = (T_1) + (T_1 + d) + (T_1 + 2d) + \dots, (T_1 + (M-1)d)$.

Thus we are naturally led to imposition of realistic (practical) constraints on the integer valued T_i's.

1) T_i's are in arithmetic progression (refer fig. 2a). i.e. $T_1, T_1+d, \ldots, T_1+(M-1)d$. These times must add up to total sensing time, L. Thus, we have

$$2MT_1 + dM(M-1) = 2L \tag{3}$$

Remark 2. In the above equation M and L are known. T_1 and d are unknown variables. Since T_1 and d are always constrained to be integers, we have a linear diophantine equation of the form $aT_1 + bd = 2L$, where $a = 2M$ and $b = M(M-1)$. There are standard techniques for solving such an algebraic equation [11].

2) T_i's are in Geometric progression (refer fig. 2b). i.e. $T_1, (T_1)(d), (T_1)(d^2), \ldots, (T_1)(d^{M-1})$. They must add up to total sensing time L.

$$T_1 \frac{(d^M - 1)}{d-1} = L \tag{4}$$

As discussed earlier M, L are known and T_1, d are unknown. Thus we need to solve the following algebraic equation in integers

$$T_1 d^M - Ld - (T_1 - L) = 0 \tag{5}$$

Goal: To solve the above algebraic equation for T_1, d suppose we assume that $d = 2$. Thus we have to decide 'T_1' for $T_1(2^M - 1) - L = 0$

Thus, for a given M; $T_1, (2^M - 1)$ must be divisors of L. If not, no solution exists. Suppose M is such that $2^M - 1$ is a prime i.e. a Mersenne prime. If L happens to be a prime number, no solution exists. (It should be noted that M must necessarily be a prime for $2^M - 1$ to be a Mersenne prime). Thus in this case for a solution to exist L must be such that its prime factorization contains the Mersenne prime $2^M - 1$. For a given M if L is divisible by $2^M - 1$, we have

$$T_1 = \frac{L}{(2^M - 1)} \tag{6}$$

Significance of this solution: Energy detection is facilitated by the use of FFT of certain length/size. Typically the FFT sizes are power of 2. Thus, d can be chosen to be a power of 2, leading to explicit solution for T_1, i.e.

$$T_1 = L \frac{d-1}{(d^M - 1)} \tag{7}$$

Factoring L gives all the possibilities for T_1. A short computation will give the desired solutions, if any.

Justification of AP/GP for sensing times: As the probabilities decrease, the increase in sensing times assume values in an AP i.e. the rate of increase of sensing times is linear, or sensing times increase geometrically (implemented by an FFT of suitable frequency resolution.) e.g. $a, 2a, 4a, 8a, 16a, \ldots$.

Remark 3. It should be noted that in all above cases, the real, positive solutions of diophantine equation are such that as T_1 increases, d decreases.

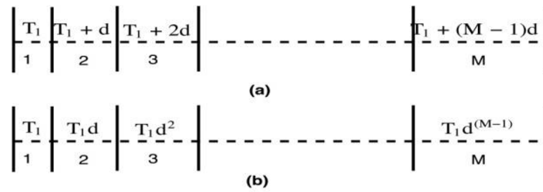

Fig. 2. Spectrum Sensing when sensing times are in a) A.P. b) G.P.

III. Numerical Experiments

A. Sensing times are in A.P.

We now consider case 1 in section 2. We invoke the following theorem on computing the solution of linear Diophantine equation [11], when sensing times are in A.P.

Theorem: The linear Diophantine equation $ax + by = c$ has a solution if and only if $d|c$ (d divides c), where d is the greatest common divisor(G.C.D) of (a, b). Furthermore if (x_0, y_0) is a solution for this equation, then the set of solutions of the equations consist of all integer pairs (x, y), where $x = x_0 + t(b/d)$ and $y = y_0 - t(a/d)$,where $t = \ldots, -2, -1, 0, 1, 2, \ldots$

Remark 4. We can compute any one solution discussed in the above theorem using Euclidean (G.C.D. Computation) algorithm.

Only non-negative integers $\{T_1, d\}$ should be selected as the solution.

Examples: Case of linear Diophantine Equation

case 1: Suppose for scanning $M = 10$ bands, total sensing time L is 100ms. With traditional solution 10ms of time will be allocated to each band. With our approach placing values in $aT_1 + bd = 2L$, where $a = 2M$, $b = M(M-1)$

$$\begin{aligned} &20T_1 + 90d = 200 \\ &\text{GCD } (20, 90) = 10. \text{ (200 is divisible by 10.)} \\ &T_1 = 1 + t(90/10) \\ &d = 2 - t(20/10) \end{aligned} \tag{8}$$

For $t = \ldots - 2, -1, 0, 1, 2, \ldots$ there are multiple solutions but there is only one interesting solution is with $t = 0, T_1 = 1$ and $d = 2$. Therefore, our time allocation is as following

Band	i	ii	iii	iv	v	vi	vii	viii	ix	x
Time(ms)	1	3	5	7	9	11	13	15	17	19

case 2: In above case only one interesting solution was available. There are multiple solutions possible if GCD(a,b) in $aT_1 + bd = 2L$ is a. Total sensing time $L = 180ms$ and total no. of bands $M = 9$. Again in conventional solution the allocation will be to allocate 20ms to each band.

$$\begin{aligned} &18T_1 + 72d = 360 \\ &\text{GCD } (18, 72) = 18. \text{ (360 is divisible by 18.)} \\ &T_1 + 4d = 20 \end{aligned} \tag{9}$$

One solution can be $d = 5$ and T_1 =0

$$\begin{aligned} T_1 &= 0 + t(72/18) \\ d &= 5 - t(18/18) \end{aligned} \tag{10}$$

Only for $t = 1, 2, 3, 4$ there are values with $\{T_1, d\} > 0$. So there are multiple solutions. Sensing times can be allocated as following

Band $\{T_1, d\}\downarrow$	i	ii	iii	iv	v	vi	vii	viii	ix
Time(ms) {4,4}	4	8	12	16	20	24	28	32	36
Time(ms) {8,3}	8	11	14	17	20	23	26	29	32
Time(ms) {12,2}	12	14	16	18	20	22	24	26	28
Time(ms) {16,1}	16	17	18	19	20	21	22	23	24

B. Sensing times are in G.P.

Now for case 2 in section 2. Suppose for scanning $M = 10$ bands, total sensing time is $L = 1023ms$. With traditional solutions there may be various assignments. With our approach, let's say common ratio $d = 2$. After placing values in equation (7), $T_1 = 1$. Therefor, our time allocation with respect to each band is as following

Band	i	ii	iii	iv	v	vi	vii	viii	ix	x
Time(ms)	1	2	4	8	16	32	64	128	256	512

Thus, as per the allocated time, FFT bins can be allocated for energy detection based sensing.

Similarly if total sensing time $L = 341$ms and number of bands $M = 5$, the possible allocations are

Band $\{T_1, d\}\downarrow$	i	ii	iii	iv	v
Time(ms) {1,4}	1	4	16	64	256
Time(ms) {11,2}	11	22	44	88	176

Hence, if the sensing times are in G.P., for some cases multiple allocations are possible.

Remark 5. The solution for $\{T_1, d\}$ in all cases, is always a matching pair.

Problem: From above examples we can find that, the number of solutions i.e. $\{T_1, d\}$ in above cases can be strictly more than one and we are not clear which solution should be utilized.

Goal: We would like to arrive at solutions that minimizes both the mean and variance of sensing time random variable. Suppose even after such optimization procedure, we arrive at multiple solutions. Heuristically, some solutions are eliminated on the basis of $\{T_1, d\}$ that are too low or too high.

IV. Time Optimal Spectrum Sensing : Stochastic Optimization

$\{q_1, q_2, q_3, \ldots, q_M\}$ are unsorted probabilities representing traffic data in various sub-bands. $\{p_1, p_2, p_3, \ldots, p_M\}$ are sorted decreasing probabilities (arranged in order from highly occupied to less occupied). If Z is spectrum sensing time random variable and T_j's are sorted sensing time values, the mean and variance can be

$$\text{Mean} = E[Z] = \sum_{i=1}^{M} T_i q_i = \sum_{j \in R} T_j p_j \tag{11}$$

$$E[Z^2] = \sum_{j \in R} {T_j}^2 p_j$$

$$\text{Variance} = var[Z] = E[Z^2] - (E[Z])^2 \tag{12}$$

where R is a suitable index set.

In above equation minimization of mean(E[Z]) maximizes variance[Z]. Our goal is to minimize mean as well as variance (joint optimization problem). We would like to arrive at a pareto optimal solution.

A. T_i's are in Arithmetic Progression

$$E[Z] = \sum_{j=1}^{M} [T_1 + (j-1)d] p_j$$

$$= T_1 + \sum_{j=1}^{M} (j-1) p_j d$$

$$E[Z] = T_1 + (\mu)(d) \tag{13}$$

where $\mu = \sum_{j=1}^{M} (j-1) p_j$

$$\begin{aligned} E[Z^2] &= \sum_{j=1}^{M} (T_j)^2 p_j \\ &= \sum_{j=1}^{M} [T_1 + (j-1)d]^2 p_j \\ &= {T_1}^2 + (\alpha) d^2 + (2T_1 d)(\mu) \end{aligned} \tag{14}$$

where $\alpha = \sum_{j=1}^{M} (j-1)^2 p_j$

$$\begin{aligned} var[Z] &= {T_1}^2 + (\alpha) d^2 + (2T_1 d)\mu - ({T_1}^2 + \mu^2 d^2 + 2\mu T_1 d) \\ &= (\alpha)(d^2) - \mu^2 d^2 \\ &= (\alpha - \mu^2) d^2 \end{aligned} \tag{15}$$

Optimal choice of $\{T_1, d\}$ are coupled. Thus, the problem boils down to minimize $E[Z]$ as well as $var[Z]$. Again we need to pick T_1 and d from possibly non-unique solutions for $\{T_1, d\}$ (determined by Diophantine equation). T_1 does not effect $var[Z]$ and only affects $E[Z]$. So choose minimum possible positive solution for T_1.

1) Mean Sensing Time Minimization Only: First, let's consider the case where only mean sensing time minimization is required. After solving case 2 in section 3, we get multiple solutions lets say $\{(a_1, d_1), \ldots, (a_l, d_l), \ldots, (a_K, d_K)\}$, which are positive and real integers. If $a_1 < \cdots < a_l < \cdots < a_K$, then $d_1 > \cdots > d_l > \cdots > d_K$.

Lemma 1. (a_1, d_1) is the best solution which minimizes the expected value $E[Z]$

Proof: Please refer [10] section(4) for detailed proof.

Remark 6. It should be noted that if only the average sensing time needs to be minimized, then the feasible solution pair $\{T_1, d\}$ for which T_1 is minimum should be utilized.

2) Simultaneous Mean and Variance Minimization: Now, for simultaneous minimization of $E[Z]$ and $var[Z]$ with respect to d (treating T_1 as constant.)

$$\begin{aligned} E[Z] &= f(d) = T_1 + (\mu)(d) \\ var[Z] &= (\alpha - \mu^2)d^2 \\ &\text{hence}(\alpha - \mu^2) \geq 0 \end{aligned} \tag{16}$$

Lemma 2. Unique optimal solution for d exists where $E[Z] = var[Z]$.

Proof: For an optimal solution

$$\begin{aligned} E[Z] &= var[Z] \\ T_1 + (\mu)(d) &= (\alpha - \mu^2)d^2 \\ (\alpha - \mu^2)d^2 - \mu d - T_1 &= 0 \\ ad^2 + bd + c &= 0 \end{aligned} \tag{17}$$

where $a = (\alpha - \mu^2), b = -\mu, c = -T_1$

$$\begin{aligned} b^2 - 4ac &> 0 \text{ for d to be real.} \\ \mu^2 - 4(\alpha - \mu^2)(-T_1) &> 0 \\ \mu^2 + 4(\alpha - \mu^2)T_1 &> 0 \text{ since } (\alpha - \mu^2) > 0 \end{aligned} \tag{18}$$

The zeros are distinct, thus we are interested in the value of d in the first quadrant. Thus,a unique optimal solution for d is achieved. Q.E.D. Note that closest integer value of d is utilized for optimal solution.

Remark 7. We expect the optimization problem formulated in the time-optimal spectrum sensing to arise in other applications. The above lemma provides solution.

B. T_i's are in Geometric Progression

$$E[Z] = \sum_{j=1}^{M} T_j p_j = \sum_{j=1}^{M}(T_1 d^{j-1})p_j = T_1(\sum_{j=1}^{M} d^{j-1}p_j) \tag{19}$$

$$E[Z^2] = \sum_{j=1}^{M}(T_j)^2 p_j = \sum_{j=1}^{M}({T_1}^2 d^{2(j-1)})p_j$$

$$var[Z] = {T_1}^2[\sum_{j=1}^{M} d^{2j-2}p_j] - {T_1}^2[\sum_{j=1}^{M} d^j p_j]^2$$

$$= {T_1}^2[(\sum_{j=1}^{M} d^{2j-2}p_j) - (\sum_{j=1}^{M} d^j p_j)^2]$$

Let's

$$\begin{aligned} E[Z] &= T_1 f(d) \\ var[z] &= {T_1}^2 R(d) \end{aligned} \tag{20}$$

where $f(d) = \sum_{j=1}^{M} d^{j-1}p_j$ and

$$\begin{aligned} R(d) &= (\sum_{j=1}^{M} d^{2j-2}p_j) - (\sum_{j=1}^{M} d^j p_j)^2 \\ &= f(d^2) - [f(d)]^2 \end{aligned}$$

1) Mean Sensing time minimization only: Let's consider the case where only mean sensing time minimization is required. After solving GP case in section 3, we get multiple solutions lets say $\{(a_1, d_1), \ldots, (a_l, d_l), \ldots, (a_K, d_K)\}$, which are positive and real integers. If $a_1 < \cdots < a_l < \cdots < a_K$, then $d_1 > \cdots > d_l > \cdots > d_K$.

Lemma 3. (a_1, d_1) is the best solution which minimizes the expected value $E[Z]$.

Proof: Please refer [10] section(4) for detailed proof.

2) Simultaneous Mean and Variance Minimization: The optimal choice of minimal T_1 will be optimal for both $E[Z]$ and $var[Z]$. But minimization of $E[Z]$ with respect to d will maximize $var[z]$. Thus we are interested in Pareto Optimal Solution i.e. jointly optimal choice for d for minimizing $E[Z]$ as well as $var[z]$. We now prove that if $f(d)$ is minimized, $R(d)$ is maximized.

Claim: If $f(t)$ is minimized, then $f(t^2)$ is maximized.

Suppose we consider the unconstrained optimization/minimization of f(t) then

$$\begin{aligned} &\text{Let } K(t) = t^2 \Rightarrow f(t^2) = f(K(t)) \\ &\frac{df(K(t))}{dt} = \frac{df}{dk}\frac{dk}{dt} = (\frac{df}{dt})(2t) \\ &\frac{d^2f(K(t))}{dt^2} = \frac{d^2f}{dt^2}(2t) + (\frac{df}{dt})(2) \\ &\frac{df(K(t))}{dt} = 0 \text{ if and only if } \frac{df}{dt} = 0 \end{aligned} \tag{21}$$

Further, the minima of $f(.)$ are in the left half plane. Suppose they are real valued, e.g. to

$$\begin{aligned} &\frac{d^2f}{dt^2}|_{t=t_0} > 0 \text{ with } t_0 < 0 \\ &\frac{d^2f(K(t))}{dt^2}|_{t=t_0} = \frac{d^2f}{dt^2}(2t)|_{t=t_0} + 0 \\ &\text{Since } t < t_0 \\ &\frac{d^2f(K(t))}{dt^2}|_{t=t_0} < 0 \end{aligned} \tag{22}$$

$f(t^2)$ is maximized, when $f(t)$ is minimized. Thus, we look for Pareto optimal solution for d i.e. denoted t here. So use closest solution of (4) pair of T_1, d to Pareto optimal solution.

Pareto Optimal Solution: (Fixed Point Equation)

$$\begin{aligned} E[Z] &= T_1 f(d) \\ E[Z^2] &= {T_1}^2 g(d) \\ E[Z] &= var[Z] \\ &\text{note that g(d) = f}(d^2) \\ T_1 f(d) &= {T_1}^2 g(d) - {T_1}^2 (f(d))^2 \\ T_1 f(d^2) - T_1(f(d))^2 - f(d) &= 0 \end{aligned} \tag{23}$$

Since $f(d)$ is a polynomial in d, we have a polynomial equation which has multiple zeros.

Q: How can we determine optimal d?

Choose smallest real d that is feasible.
Example: Let M=3

$$f(d) = \sum_{j=1}^{3} d^{(j-1)} p_j = p_1 + dp_2 + d^2 p_3$$
$$g(d) = f(d^2) = \sum_{j=1}^{3} d^{2j-2} p_j = p_1 + d^2 p_2 + d^4 p_3 \tag{24}$$

Replace values in equation

$$T_1 f(d^2) - T_1 (f(d))^2 - f(d) = 0 \tag{25}$$

Let $T_1 = 1, p_1 = .5, p_2 = .3, p_3 = .2$

$$(p_1 + d^2 p_2 + d^4 p_3) - (p_1 + dp_2 + d^2 p_3)^2 - (p_1 + dp_2 + d^2 p_3) = 0$$

After solving equations values for d = 2.43, -1.26, -0.20±0.68i

In the above example, the solution contains only one positive real value that is of our interest. Rest of the values are not useful. Take the closest integer value of d which is real positive optimal solution. Similarly following table shows real and unique positive value of d for different orders with different probabilities

M	2	4	6	8	10
d	4.00	1.75	1.47	1.33	1.27

C. Generalization:

The most general choice of sensing times (increasing numbers) leads to the constrained partition problem. Further the sensing times must minimize the mean as well as variance of the sensing time random variable.

The above discussion naturally leads to the following more interesting optimization problems (related to joint optimization of moments of a discrete random variable.) Let Z be a random variable assuming values $\{T_1, T_2, \ldots, T_M\}$ with probabilities $\{q_1, q_2, \ldots, q_M\}$ respectively.

$$E[Z^2] = \sum_{i=1}^{M} T_i^2 q_i \tag{26}$$

Let $\{T_i\}_{i=1}^M$ be the unknowns and $\{q_i\}_{i=1}^M$ are known constants. Then the mean and variance of the random variable are given by

$$\begin{aligned} E[Z] &= \sum_{i=1}^{M} T_i q_i = f(T_1, T_2, \ldots, T_M) = f(T) \\ var[Z] &= E[Z^2] - (E[Z])^2 \\ &= g(T_1, T_2, \ldots, T_M) = g(T) \end{aligned} \tag{27}$$

Goal: To see if we can optimize $E[Z]$, $var[z]$ jointly.

Q: Do we have an interesting functional equation arising in the joint optimization of $E[Z]$, $var[Z]$?

$$\begin{aligned} E[Z] &= var[Z] \\ E[Z] &= E[Z^2] - (E[Z])^2 \\ E[Z^2] - E[Z] - (E[Z])^2 &= 0 \\ \text{letting } E[Z^2] &= h(T_1, T_2, \ldots, T_M) \\ &= f(T_1^2, T_2^2, \ldots, T_M^2) \end{aligned} \tag{28}$$

The multivariate functional equation that must be solved is given by

$$\begin{aligned} f(T_1^2, T_2^2, \ldots, T_M^2) - f(T_1, T_2, \ldots, T_M) \\ -(f(T_1, T_2, \ldots, T_M))^2 = 0 \end{aligned} \tag{29}$$

Is there a solution to such a functional equation? Mostly it constitutes the Pareto Front(Non-Dominating solution set).

V. Time Optimal Spectrum Sensing: Linear and Quadratic Programming (Hybrid Programming)

In this section we attempt to provide the most general solution to the problem of time optimal spectrum sensing.

Objective Functions:

$$\begin{aligned} C &= [p_1, p_2, \ldots, p_M]^T \\ T &= [T_1, T_2, \ldots, T_M]^T \\ D &= diag\{p_1, p_2, \ldots, p_M\} \\ E[Z] &= C^T T = T^T C = <C, T> \\ var[Z] &= T^T DT - (C^T T)^2 = T^T DT - (T^T C)^2 \\ &= T^T DT - T^T CC^T T = T^T (D - C^T C) T \\ &= T^T GT \text{ ,where } G = D - CC^T \end{aligned} \tag{30}$$

Inferences:

- G is a laplacian like matrix in the sense that the diagonal elements are positive and off diagonal elements are negative. Further, the sum of rows is zero just like laplacian matrix in graph theory. It is positive semi-definite.
- -G is a symmetric generator matrix.

Example:

$$\begin{aligned} G &= D - CC^T \\ G &= \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} - \begin{bmatrix} {p_1}^2 & p_1 p_2 & p_1 p_3 \\ p_1 p_2 & {p_2}^2 & p_2 p_3 \\ p_1 p_3 & p_2 p_3 & {p_3}^2 \end{bmatrix} \\ &= \begin{bmatrix} p_1(1-p_1) & -p_1 p_2 & -p_1 p_3 \\ -p_1 p_2 & p_2(1-p_2) & -p_2 p_3 \\ -p_1 p_3 & -p_2 p_3 & p_3(1-p_3) \end{bmatrix} \end{aligned} \tag{31}$$

Goal: To arrive at non-trivial T vector which minimizes the variance of random variable Z.

Function of interest for arriving at solutions where $E[Z] = var[Z]$:

$$\begin{aligned} J(T) &= var[Z] - E[Z] \\ &= T^T GT - T^T C \\ &= \begin{bmatrix} T^T & 1 \end{bmatrix} \begin{bmatrix} G & 0 \\ -C^T & 0 \end{bmatrix} \begin{bmatrix} T \\ 1 \end{bmatrix} \end{aligned} \tag{32}$$

We investigate the structure of matrix G before providing a solution to the problems proposed above.

Remark 8. $Ge = 0$ where $e = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T$
$G = G^T$
We use Laplacian and Laplacian like matrix interchangeably.

A. Most General Problem Solution

We now consider the problem of minimization of $E[Z]$ as well as $var[Z]$ of the random variable Z. In most general case T_i's constitutes the integer components of a partition of the total sensing time, L i.e.

$$\sum_{i=1}^{M} T_i = L \tag{33}$$

This constitutes the most general constraint without imposing any other constraint on T_i's.

We now reason that the quadratic programming associated with minimization of $var[z]$ can be reduced to M linear programming problems.From above discussion

$$var[Z] = T^T G T$$

By cholesky decomposition

$$\begin{aligned} G &= NN^T \\ T^T G T &= T^T N N^T T \\ &= (T^T N)(N^T T) \\ &= Y^T Y, \text{ where } Y = N^T T \\ &= \sum_{i=1}^{M} {y_i}^2 \end{aligned} \tag{34}$$

Minimizing $var[z]$, reduces to minimizing y_i for $1 \le i \le M$. But Y is a linear objective function in T. Thus, minimization of $var[Z]$ amounts to solving M linear programming problems.

In nutshell, our optimization problem reduces to solving $M+1$ integer linear programming problem (including minimization of mean) i.e. $E[Z] = T^T C$ subject to the constraint in (33).

Remark 9. Without imposing any other constraint on T_i's, we are led to trivial solution(as discussed earlier) in optimizing mean and variance of sensing time random variable. Thus, realistic constraint are imposed in the multi-linear objective optimization problem to get interesting solutions.

B. Properties of laplacian type matrix arising in variance expression of a Discrete Random Variable Z:

Theme: properties of Laplacian matrix arising in variance optimization of a discrete random variable.

Q: Can (linear algebraic) properties of matrix G be capitalized to derive new results on variance minimization?

Goal: To study properties of laplacian like matrix $G = D - CC^T$ where $D = diag\{p_1, p_2, \ldots, p_M\}$, $C = [p_1, p_2, \ldots, p_M]^T$, $G = G^T$

1) Eigen values are all real.
2) G is positive semidefinite, with an eigen value at zero ($e \ldots$ all ones vector). e is in the null space of G.
3) 0 is the smallest eigen value and all other eigen values lie on real axis.
4) Bounds on spectral radius of G

$$\begin{aligned} \sum_{j=1}^{n} G_{ij} &= \sum_{j=1}^{n} D_{ij} - p_j \sum (p_1 + \ldots + p_{j-1} + p_j + \ldots + p_M) \\ &= p_j - p_j = 0 \end{aligned} \tag{35}$$

$$\sum_{j=1}^{n} |G_{ij}| = p_j(1-p_j) + |p_j(p_1 + \ldots + p_{j-1} + p_{j+1} + \ldots + p_M)|$$

$$\begin{aligned} &= p_j(1-p_j) + |p_j(1-p_j)| \\ &= 2p_j(1-p_j) \end{aligned} \tag{36}$$

From linear algebra, we have that

$$\min_j \{2p_j(1-p_j)\} \le \text{Spectral radius(G)} \le \max_j \{2p_j(1-p_j)\}$$

All eigen values of G lie in the interval [0,1).

VI. Conclusion and Future Work

In the paper, information theoretic and integer linear programming approach for time optimal spectrum sensing is discussed. The problem is also formulated as stochastic optimization problem and solved. We expect the optimization problem formulated here to arise in other applications. By associating channel states, a more general model based on Quasi Birth and Death process is being developed and analyzed. Using standard results in queuing theory, various performance measures can be computed and interpreted.

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