Optimal Design of Multichannel Equalizers for the Structural Similarity Index

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***Abstract*— The optimization of multichannel equalizers is studied for the structural similarity (SSIM) criteria. The closed-form formula is provided for the optimal equalizer when the mean of the source is zero. The formula shows that the equalizer with maximal SSIM index is equal to the one with minimal mean square error (MSE) multiplied by a positive real number, which is shown to be equal to the inverse of the achieved SSIM index. The relation of the maximal SSIM index to the minimal MSE is also established for given blurring filters and fixed length equalizers. An algorithm is also presented to compute the suboptimal equalizer for the general sources. Various numerical examples are given to demonstrate the effectiveness of the results.**

***Index Terms*— Structural similarity, mean square error, linear equalizers, image restoration, optimal design.**

I. Introduction

As a technique to recover an original image from degraded observations, image restoration has been explored extensively and is now a mature research field [1]–[3]. Various effective methods have been proposed for image restoration [2]–[6]. Among them, the linear equalizer based on Wiener filter is a conventional yet benchmark method [2], [3]. Recently, lots of research focused on blind equalization, which arose from various signal processing applications [4]–[6]. The optimization criteria is usually measured by the mean squared error (MSE), defined as ℓ_2 norm of the error signal between the original signal and reconstructed signal [3]. While this measure is comfortable to use and often results in analytical solutions, it does not always reflect the true quality of the reconstructed images. A better quality-assessment measure, called structural similarity (SSIM) index, has been drawing much attention in recent years [7]–[9], and has found a variety of applications, ranging from image restoration [10]–[12] and image quantization [20] to image/video coding [21]–[25] and sparse representation [26], [27]. While SSIM index is generally non-convex, important mathematical properties have been shown in [9] that convexity, quasi-convexity and generalized convexity hold locally for some metrics derived from SSIM.

The SSIM index represents the quality of a distorted image by comparing the correlations in luminance, contrast, and structure, between the reference and distorted images [7]. In most existing works, SSIM has been used for quality evaluation and algorithm comparison purposes only. There have been efforts to use SSIM as optimization criteria in order to improve perceived image/video quality in a number of image processing problems. In particular, an algorithm was proposed in [11] for designing the optimal linear filter that maximizes the Stat-SSIM index, which is a statistical version of the SSIM index. It has been shown that the non-convex optimization problem can be transformed into a quasi-convex problem, which has a near closed-form solution and can be efficiently solved by a bisection procedure [11]. This idea was extended to multichannel image restoration in [12], where the algorithm is provided in detail only for two-channel systems. To our best knowledge, no explicit algorithm is available to deal with the SSIM-based restoration for general multi-channel images, although the MSE-based restoration from multichannel images has been studied in a number of applications [2]–[6].

In this paper, we propose a systematic method to design the optimal linear equalizer for multiple channels by using the SSIM criteria. We adopt the same assumption as [11], [12] that the blurring filter and the power spectral density (PSD) of the additive noise component is known at the receiver, under which the advanced technique can be used to estimate the cross covariance between observed image and the original image. Different to [11] and [12], this paper presents closed-form formulae explicitly, which reveal more insights into the SSIM based equalization problem. The main contributions of this paper are as follows. (i) The closed-form formulae of the optimal linear equalizer and the optimal SSIM performance are provided for general multi-channel systems when the mean of the source is zero. (ii) It is shown that the optimal equalizer designed by SSIM criteria and the one by MSE criteria differ only from a scalar factor, which turns out to be the inverse of the achievable SSIM index. (iii) The solution establishes the relation of the SSIM performance to the property of the blurring filters and the length of the equalization filters. (iv) A sub-optimal solution is given when the mean of the original image is not zero.

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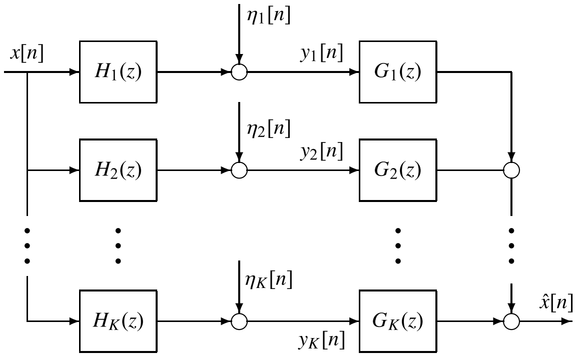

Fig. 1. A multichannel linear equalization system.

There have been various methods to design the optimal synthesis filter bank with MSE or MSE-reduced criteria [13]–[16]. It is natural to introduce SSIM index into the optimal design of filter banks. A preliminary result has been reported for a special class of filter banks with the filter length equal to the decimation number [17]. The proof idea of Theorem 1 in this paper comes from [17]. However, this paper is essentially different to [17]. The subband signals and the reconstructed signals are wide sense cyclo-stationary in filter bank systems instead of wide sense stationary (WSS) in this paper, due to the rate-changing operation [18], [19]. To deal with the wide sense cyclo-stationary property of general filter banks, one has to define the SSIM index for this new setup, which is still an open problem.

The remaining part of this paper is organized as follows. Section II introduces some basic concepts on SSIM index and gives the problem formulation. In Section III, we present the closed-form formula to compute the optimal SSIM equalizer when the mean of the source is zero. Direct relation is established between the optimal SSIM equalizer and the Wiener filter, which provides the optimal MSE performance. Section IV addresses the general sources, of which the mean may not be zero. An algorithm is given to compute the suboptimal solution. Various numerical examples are presented in Section V to demonstrate the effectiveness of the theoretical results. Finally concluding remarks are given in Section VI.

Throughout the paper, $\mathbb{R}^{P\times M}$ denotes the set of $P\times M$ matrices. For a matrix X, let X^T, X^*, X^{-1} and $X^\dagger$ denote its transpose, conjugate transpose, inverse and pseudo-inverse respectively. Denote $\mathbf{e}_n$ as the n-dimension column vector with all elements equal to 1. $\mathcal{E}(\cdot)$ denotes the expectation of a random variable.

II. Preliminaries and Problem Statement

In this section, we provide some preliminaries about the multi-channel equalization and the SSIM index. Please refer to [7] and [11] for more details.

The term multichannel image refers to multiple image frames of the same scene that are acquired by different sensors. The system model is shown in Fig. 1, where $H_i(z)$ and $G_i(z)$ denote the blurring filter and the equalization filter respectively, $x[n]$ and $\hat{x}[n]$ denote the original image and the reconstructed image respectively, $y_i[n]$ is the sub-image generated by the blurring filter $H_i(z)$, and $\eta_i[n]$ is the noise into the ith channel, $i=1,\ldots,K$. The goal is to design the equalizer $G_i(z)$ so that the SSIM index between $\hat{x}[n]$ and $x[n]$ is maximized. Throughout the paper, we assume that the source $x[n]$ is a wide sense stationary (WSS) process, and we also assume that the blurring filter and the power spectral density (PSD) of the additive noise component is known at the receiver [11]. Denote

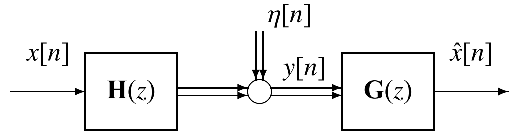

Fig. 2. Block diagram of Fig. 1.

$$\mathbf{H}(z)=\begin{bmatrix} H_1(z) & \cdots & H_K(z)\end{bmatrix}^T := \sum_{i=0}^{M-1} h[i]z^{-i}$$
$$\mathbf{G}(z)=\begin{bmatrix} G_1(z) & \cdots & G_K(z)\end{bmatrix} := \sum_{j=0}^{N-1} g[j]z^{-j}$$

where $h[i]\in\mathbb{R}^{K\times 1}$, $i=0,\ldots,M-1$ and $g[j]\in\mathbb{R}^{1\times K}$, $j=0,\ldots,N-1$ are the impulse response coefficients of the blurring filter and the equalizer respectively. The filter length of $H_i(z)$ can be different and M refers to the largest length among the blurring filters. However, we assume that the length of all the equalizers is equal to N. By denoting

$$y[n]=\begin{bmatrix} y_1[n] & \cdots & y_K[n]\end{bmatrix}^T$$
$$\eta[n]=\begin{bmatrix} \eta_1[n] & \cdots & \eta_K[n]\end{bmatrix}^T,$$

we can convert Fig. 1 to an equivalent block diagram shown in Fig. 2. Note that

$$y[n]=\sum_{i=0}^{M-1} h[i]x[n-i]+\eta[n]$$
$$\hat{x}[n]=\sum_{j=0}^{N-1} g[j]y[n-j].$$

In this paper, we use the simplified form of the SSIM index as defined in [10]–[12]:

$$\mathrm{StatSSIM}(x,\hat{x})=\frac{2\mu_x\mu_{\hat{x}}+c_1}{\mu_x^2+\mu_{\hat{x}}^2+c_1}\frac{2\sigma_{x\hat{x}}+c_2}{\sigma_x^2+\sigma_{\hat{x}}^2+c_2} \tag{1}$$

where μ_x and $\mu_{\hat{x}}$ are the means of the source and the reconstructed image, σ_x^2 and $\sigma_{\hat{x}}^2$ are the corresponding variances, and $\sigma_{x\hat{x}}$ is the covariance of x and $\hat{x}$.

The problem can be stated as follows: Given the blurred M-tap filter $\mathbf{H}(z)=\sum_{i=0}^{M-1}h[i]z^{-i}$, the noise $\eta[n]$ with known PSD, and the observed image $y[n]$, to design the N-tap linear equalizer $\mathbf{G}(z)=\sum_{j=0}^{N-1}g[j]z^{-j}$ such that the StatSSIM index between $x[n]$ and $\hat{x}[n]$ is maximized. Here $x[n]\in\mathbb{R}$, $y[n]\in\mathbb{R}^K$, $h[i]\in\mathbb{R}^{K\times 1}$ and $g[j]\in\mathbb{R}^{1\times K}$ for any i,j. To compute the StatSSIM index, we need more notations. Denote

$$\mathbf{g} = \begin{bmatrix} g[0] & g[1] & \cdots g[N-1] \end{bmatrix} \in \mathbb{R}^{1\times NK}$$
$$\mathbf{y}[n] = \begin{bmatrix} y^T[n] & y^T[n-1] \cdots y^T[n-N+1] \end{bmatrix}^T \in \mathbb{R}^{NK\times 1}$$
$$\underline{\eta}[n] = \begin{bmatrix} \eta^T[n] & \eta^T[n-1] \cdots \eta^T[n-N+1] \end{bmatrix}^T \in \mathbb{R}^{NK\times 1}$$
$$\mathbf{x}(n:n-j) = \begin{bmatrix} x[n] & x[n-1] & \cdots & x[n-j] \end{bmatrix}^T \in \mathbb{R}^{(j+1)\times 1}.$$

Then we have

$$\mathbf{y}[n] = \mathbf{H}(N,M)\mathbf{x}(n:n-N-M+2) + \underline{\eta}[n],$$
$$\hat{x}[n] = \mathbf{g}\mathbf{y}[n] = \mathbf{g}\mathbf{H}(N,M)\mathbf{x}(n:n-N-M+2) + \mathbf{g}\underline{\eta}[n],$$

where $\mathbf{H}(N,M) \in \mathbb{R}^{NK\times(N+M-1)}$ is given by

$$\mathbf{H}(N,M) = \begin{bmatrix} h[0] & h[1] & & \cdots & & 0 \\ 0 & h[0] & \ddots & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & & h[0] & \cdots & h[M-1] \end{bmatrix}. \quad (2)$$

Since $x[n]$ is WSS, we can define

$$C_{xx}(i) = \mathcal{E}\left((x[n-i]-\mu_x)(x[n]-\mu_x)\right),$$

for $i = 1, 2, \ldots$. Note that $C_{xx}(0) = \sigma_x^2$. Denote

$$\Sigma_{\mathbf{x}} = \begin{bmatrix} C_{xx}(0) & C_{xx}(1) & \cdots & C_{xx}(N+M-2) \\ C_{xx}(1) & C_{xx}(0) & \cdots & C_{xx}(N+M-3) \\ \vdots & & \ddots & \vdots \\ C_{xx}(N+M-2) & \cdots & \cdots & C_{xx}(0) \end{bmatrix}$$
$$\Sigma_\eta = \mathcal{E}\left(\underline{\eta}[n]\underline{\eta}^T[n]\right) \in \mathbb{R}^{NK\times NK}$$

Denote the first column of $\Sigma_{\mathbf{x}}$ as $\Sigma_{\mathbf{x}}(1)$. By direct computation, we have

$$\mu_{\hat{x}} = \mathcal{E}\left(\hat{x}[n]\right) = \mathbf{g}\mathbf{H}(N,M)\mathbf{e}_{N+M-1}\mu_x = \left(\sum_{j=0}^{N-1} g[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right)\mu_x, \quad (3)$$

$$\begin{aligned}\sigma_{x\hat{x}} &= \mathcal{E}\left((x[n]-\mu_x)(\hat{x}[n]-\mu_{\hat{x}})\right) \\ &= \mathcal{E}\left((\hat{x}[n]-\mu_{\hat{x}})(x[n]-\mu_x)\right) \\ &= \mathcal{E}\left((\mathbf{g}\mathbf{H}(N,M)\mathbf{x}(n:n-N-M+2)-\mu_{\hat{x}})(x[n]-\mu_x)\right) \\ &= \mathbf{g}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) \end{aligned} \quad (4)$$

and

$$\begin{aligned}\sigma_{\hat{x}}^2 &= \mathcal{E}\left((\hat{x}[n]-\mu_{\hat{x}})^2\right) \\ &= \mathbf{g}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}\mathbf{H}^T(N,M)\mathbf{g}^T + \mathbf{g}\Sigma_\eta\mathbf{g}^T \\ &= \mathbf{g}\mathbf{Q}\mathbf{g}^T \end{aligned} \quad (5)$$

where

$$\mathbf{Q} = \mathbf{H}(N,M)\Sigma_{\mathbf{x}}\mathbf{H}^T(N,M) + \Sigma_\eta. \quad (6)$$

It follows from the above derivations that the StatSSIM in (1) can be written as

$$\text{StatSSIM}(x,\hat{x}) = S_1(x,\hat{x})S_2(x,\hat{x}) \quad (7)$$

where

$$S_1(x,\hat{x}) = \frac{2\mu_x^2\left(\sum_{j=0}^{N-1} g[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right) + c_1}{\mu_x^2\left(1 + \left(\left(\sum_{j=0}^{N-1} g[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right)\right)^2\right) + c_1} \quad (8)$$

and

$$S_2(x,\hat{x}) = \frac{2\mathbf{g}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) + c_2}{\sigma_x^2 + \mathbf{g}\mathbf{Q}\mathbf{g}^T + c_2}. \quad (9)$$

Now that the problem becomes to find $\mathbf{g}$ such that StatSSIM$(x,\hat{x})$ is maximized.

III. Analytical Formulae for Zero Mean Sources

In this section, we deal with the sources with zero means, of which StatSSIM$(x,\hat{x}) = S_2(x,\hat{x})$ since $S_1(x,\hat{x}) = 1$. The zero mean assumption is not so restrict in the sense that the mean can be subtracted from the original signal before process and compensated to the reconstructed signal later [12]. We shall show that the optimal equalizer that maximizes the SSIM index is equal to the one that minimizes the MSE criteria multiplied by a scalar factor, of which the analytical formula is derived. We shall also establish the relationship between the maximum SSIM index $S_{2max}(x,\hat{x})$ and the optimal MSE value. For this purpose, define

$$b = \Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1), \quad (10)$$

where $\mathbf{H}(N,M)$ and $\mathbf{Q}$ are given by (2) and (6) respectively, $\Sigma_{\mathbf{x}}(1)$ denotes the first column of $\Sigma_{\mathbf{x}}$.

Theorem 1: Assume that $x[n]$ and the noise $\eta[n]$ are two zero mean WSS random processes and are uncorrelated with each other. Given the blurring filter $\mathbf{H}(z)$, let b be defined by (10). The equalization filter $\mathbf{G}(z) = \sum_{j=0}^{N-1} g[j]z^{-j}$ that maximizes $S_2(x,\hat{x})$ in (9) is unique and given by

$$\mathbf{g}_{opt} = \gamma\,\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1} \quad (11)$$

where

$$\gamma = \frac{\sqrt{c_2^2 + 4b(\sigma_x^2 + c_2)} - c_2}{2b}. \quad (12)$$

Moreover, $S_{2\max}(x,\hat{x}) = \frac{1}{\gamma}$.

Proof: Since $\mathbf{Q}$ is positive definite, it follows from singular value decomposition (SVD) that there exist unitary $\mathbf{U}$ and diagonal matrix $\Sigma_{\mathbf{Q}} > 0$ such that $\mathbf{Q} = \mathbf{U}\Sigma_{\mathbf{Q}}\mathbf{U}^T$. Denote

$$\mathbf{q} = \mathbf{g}\mathbf{U}\Sigma_{\mathbf{Q}}^{\frac{1}{2}}. \quad (13)$$

we know that $S_1(x,\hat{x}) = 1$ since $x[n]$ is zero mean. Hence equation (7) can be written as

$$\begin{aligned}\text{StatSSIM}(x,\hat{x}) = S_2(x,\hat{x}) &= \frac{2\mathbf{g}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) + c_2}{\sigma_x^2 + \mathbf{g}\mathbf{U}\Sigma_{\mathbf{Q}}\mathbf{U}^T\mathbf{g}^T + c_2} \\ &= \frac{2\mathbf{q}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}\mathbf{U}^T\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) + c_2}{\sigma_x^2 + \mathbf{q}\mathbf{q}^T + c_2}. \end{aligned} \quad (14)$$

Define $\bar{\mathbf{q}} = \frac{1}{\alpha}\mathbf{q}$, where $\alpha^2 = \mathbf{q}\mathbf{q}^T$. The optimization problem can be written as

$$\max_{\alpha,\bar{\mathbf{q}}} S_2(x,\hat{x}) = \max_{\alpha,\bar{\mathbf{q}}} \frac{2\alpha\bar{\mathbf{q}}\Sigma_Q^{-\frac{1}{2}}\mathbf{U}^T\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1)+c_2}{\sigma_x^2+\alpha^2+c_2} \quad (15)$$

subject to $\bar{\mathbf{q}}\bar{\mathbf{q}}^T = 1$. Note that for any α, the optimal argument $\bar{\mathbf{q}}$ that maximizes $2\alpha\bar{\mathbf{q}}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}\mathbf{U}^T\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1)$ is uniquely given by

$$\bar{\mathbf{q}}_{opt} = \frac{1}{\sqrt{b}}\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{U}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}, \quad (16)$$

where b is defined by (10). Moreover, the resulting maximum is

$$2\alpha\frac{1}{\sqrt{b}}\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) = 2\alpha\sqrt{b} \quad (17)$$

Then the optimization problem (15) turns out to be

$$\max_{\alpha} S_2(x,\hat{x}) = \max_{\alpha}\frac{2\sqrt{b}\alpha+c_2}{\alpha^2+\sigma_x^2+c_2}, \quad (18)$$

and $\bar{\mathbf{q}}_{opt}$ is given by (16). By some basic calculus techniques, the optimal parameter α can be computed easily

$$\alpha_{opt} = \frac{\sqrt{c_2^2+4b(\sigma_x^2+c_2)}-c_2}{2\sqrt{b}}, \quad (19)$$

and the resulting maximum is given by

$$S_{2\max}(x,\hat{x}) = \frac{2b}{\sqrt{c_2^2+4b(\sigma_x^2+c_2)}-c_2} = \frac{1}{\gamma},$$

where γ is defined by (12). It follows from (13) and (16) that the optimal $\mathbf{g}_{opt}$ is given by

$$\begin{aligned}\mathbf{g}_{opt} &= \mathbf{q}_{opt}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}\mathbf{U}^T = a_{opt}\bar{\mathbf{q}}_{opt}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}\mathbf{U}^T\\ &= \frac{a_{opt}}{\sqrt{b}}\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{U}\Sigma_{\mathbf{Q}}^{-1}\mathbf{U}^T\\ &= \gamma\,\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}.\end{aligned}$$

This completes the proof. □

Remark 1: Since the optimal equalizer is unique, the value $1/\gamma$ should be the same as γ defined in [11] for single channel systems, of which an iterative algorithm is given. Theorem 1 tells us that γ can be computed analytically by (12) not only for single channel systems, but also for general multi-channel systems.

Theorem 1 provides not only the optimal SSIM solution, but also some useful insights into the design procedure. For instance, we know intuitively that the longer the length of the equalization filters, the better performance of SSIM can be achieved for fixed $H(z)$, $\Sigma_{\mathbf{x}}$ and Σ_η. Some interesting problems that arise naturally include what the performance limitation is as N tends to infinity, and how the designer set a good tradeoff between N and the achievable SSIM index. These problems can be partly solved by drawing the relation of γ and N by using (10) and (12), which is illustrated by different examples in Section V.

It is well-known that $\mathbf{g}_{MSE} = \Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}$ is the optimal Wiener solution, which minimizes the MSE. In [8], the authors show the advantage of SSIM index compared with MSE by various applications in image processing. Theorem 1 shows that there is a direct connection between the equalizer with maximal SSIM index and the one with minimal MSE. Indeed, the only difference between $\mathbf{g}_{opt}$ in (11) and $\mathbf{g}_{MSE}$ is a scalar γ, which surprisingly turns out to be the inverse of the optimal SSIM index. This means that the SSIM restored image is merely a contrast adjustment of the traditional MSE-based restoration.[1] Since the scalar γ is always greater than 1, and usually not a constant in practical applications, the SSIM restored image is always brighter than the MSE-based restoration, although the variation of the brightness of each pixel depends on the corresponding γ.

In the problem of finding the optimal sparse representation of static systems, it has been shown in [27] that the optimal SSIM-based representation is a scaling of the optimal L_2-based representation. However, the technique can not be used to the multichannel equalization problem which essentially deals with dynamic systems.

The following theorem gives the direct relation between the optimal SSIM index and MSE.

Theorem 2: Assume that $x[n]$ and the noise $\eta[n]$ are two independent WSS random processes with zero mean. Given the blurring filter $H(z)$, let $S_{\max}$ and $E_{\min}$ be respectively the maximal SSIM index and the minimal MSE achieved by N-tap linear equalizers. Then the following equalities hold,

$$S_{\max} = \frac{2(\sigma_x^2-E_{\min})}{\sqrt{c_2^2+4(\sigma_x^2-E_{\min})(\sigma_x^2+c_2)}-c_2} \quad (20)$$

$$E_{\min} = (1-S_{\max})(c_2S_{\max}+\sigma_x^2S_{\max}+\sigma_x^2). \quad (21)$$

Proof: Since $\mu_x = 0$ and $\mu_{\hat{x}} = 0$, by using (4) and (5), we have

$$\begin{aligned}E_{\min} &= \mathcal{E}(\hat{x}[n]-x[n])^2 = \sigma_{\hat{x}}^2 - 2\sigma_{\hat{x}x}+\sigma_x^2\\ &= \mathbf{g}_{MSE}\mathbf{Q}\mathbf{g}_{MSE}^T - 2\mathbf{g}_{MSE}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1)+\sigma_x^2\\ &= \sigma_x^2 - b,\end{aligned} \quad (22)$$

where b is given by (10). The last equality in (22) follows from the fact that $\mathbf{g}_{MSE} = \Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}$. By substituting $b = \sigma_x^2 - E_{\min}$ to (12), we get (20). (21) can be verified directly by using (20). □

Apart from Theorem 1, the SSIM performance limitation can also be computed directly by using (20) once we know the MSE performance limitation, which has been extensively studied in signal processing community as well as control community [31]–[33].

Remark 2: Theoretically, an entire image is modeled as a wide sense stationary (WSS) process, which means that the optimal equalizer (11) is a non-adaptive solution. In practice, however, the equalizer $\mathbf{g}_{opt}$ is computed adaptively by using the estimation of $\Sigma_{\mathbf{x}}$ estimated from $\mathbf{y}$ locally on a sliding window.

The PSD of the source can be estimated under the assumption that the blurring filters and the PSD of the noise are

[1]This insight was motivated by an anonymous reviewer.

known at the receiver [11], [28], [29]. For single channel systems, an efficient estimation method is recommended for the application of image restoration in [11]. In fact, $\Sigma_{\mathbf{x}}$ can also be estimated directly by using matrix manipulations for multichannel systems. Note that

$$\begin{aligned} C_{\mathbf{yy}} &= \mathcal{E}\left(\left(\mathbf{y}[n] - \mu_y \mathbf{e}_{NK}\right)\left(\mathbf{y}[n] - \mu_y \mathbf{e}_{NK}\right)^T\right) \\ &= \mathbf{H}(N,M)\Sigma_{\mathbf{x}}\mathbf{H}^T(N,M) + \Sigma_\eta, \end{aligned} \tag{23}$$

where $\mathbf{H}(N,M)$ is defined by (2). $\Sigma_{\mathbf{x}}$ can be determined uniquely from (23) if $\mathbf{H}(N,M)$ is of full column-rank. For multichannel systems with $K > 1$, the full column-rank condition is satisfied when $H_i(z), i = 1, \dots K$ have no common zeros and $N > M$ [34]. From this point of view, we may contribute the improvement of multichannel equalizers compared with single channel equalizers to two aspects, of which the first one is that a larger SSIM index can be achieved by (12), and the second that a better estimation of $\Sigma_{\mathbf{x}}$ can be reached.

Theorem 1 can not be applied when there is no noise η, for in such situation Q may not be invertible. In the following, we will show that $S_2(x,\hat{x}) = 1$ can always be achieved if $\mathbf{H}(N,M)$ satisfies a rank condition when there is no noise. Let

$$\mathbf{H}(N,M) = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} V = \mathbf{U}_1\Sigma_1\mathbf{V} \tag{24}$$

be the singular value decomposition (SVD) of $\mathbf{H}(N,M)$, where $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ is an $NK \times NK$ unitary matrix, Σ_1 is diagonal with nonnegative diagonal elements, and V is an $(N+M-1) \times (N+M-1)$ unitary matrix [30].

Theorem 3: Assume that $x[n]$ is a WSS process with zero mean and invertible $\Sigma_{\mathbf{x}}$. Assume further that $rank\{\mathbf{H}(N,M)\} = N+M-1$ and there is no noise. Then $S_2(x,\hat{x}) = 1$ and all optimal equalizers $G(z) = \sum_{j=0}^{N-1} g[j]z^{-j}$ can be parameterized by

$$\mathbf{g}_{opt} = [\,1 \quad 0 \quad \cdots \quad 0\,]\mathbf{H}^\dagger(N,M) + \mathbf{q}_2\mathbf{U}_2^*, \tag{25}$$

where U_2 is given in (24) and $\mathbf{q}_2 \in \mathbb{R}^{1\times(NK-N-M+1)}$ is an arbitrary row vector.

Proof: Since $rank\{\mathbf{H}(N,M)\} = N+M-1$, we know that Σ_1 in (24) is invertible. Define

$$\mathbf{q}_1 = \mathbf{g}\mathbf{U}_1\Sigma_1,$$

where $\mathbf{U}_1$ is given in (24). Note that all $\mathbf{g}$ can be written as

$$\mathbf{g} = \mathbf{q}_1\Sigma_1^{-1}\mathbf{U}_1^* + \mathbf{q}_2\mathbf{U}_2^*, \tag{26}$$

where $\mathbf{U}_2$ is given by (24) and $\mathbf{q}_2 \in \mathbb{R}^{1\times(NK-N-M+1)} = \mathbf{g}\mathbf{U}_2$. By direct computation, we have

$$\mathbf{g}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) = \mathbf{g}\mathbf{U}_1\Sigma_1\mathbf{V}\Sigma_{\mathbf{x}}(1) = \mathbf{q}_1\mathbf{V}\Sigma_{\mathbf{x}}(1) \tag{27}$$

$$\mathbf{g}\mathbf{Q}\mathbf{g}^T = \mathbf{g}\mathbf{U_1}\mathbf{\Sigma_1}\mathbf{V}\Sigma_{\mathbf{x}}\mathbf{V}^*\mathbf{\Sigma_1}\mathbf{U_1^*}\mathbf{g}^T = \mathbf{q}_1\mathbf{V}\Sigma_{\mathbf{x}}\mathbf{V}^*\mathbf{q}_1^*. \tag{28}$$

Substituting (27)–(28) to (9), we have

$$S_2(x,\hat{x}) = \frac{2\mathbf{q}_1\mathbf{V}\Sigma_{\mathbf{x}}(1) + c_2}{\sigma_x^2 + \mathbf{q}_1\mathbf{V}\Sigma_{\mathbf{x}}\mathbf{V}^*\mathbf{q}_1^* + c_2}. \tag{29}$$

Following a similar procedure in the proof of Theorem 1, we know that the optimal $\mathbf{q}_1$ maximizing (29) is uniquely given by

$$\mathbf{q}_{1opt} = \gamma\,\Sigma_{\mathbf{x}}^T(1)\Sigma_{\mathbf{x}}^{-1}\mathbf{V}^* = \gamma\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}\mathbf{V}^*,$$

where γ is defined as (12). Note that the parameter b can be computed directly as follows

$$\begin{aligned} b &= \Sigma_{\mathbf{x}}^T(1)\mathbf{V}^*\mathbf{V}\Sigma_{\mathbf{x}}^{-\mathbf{1}}\mathbf{V}^*\mathbf{V}\Sigma_{\mathbf{x}}(1) \\ &= \Sigma_{\mathbf{x}}^T(1)\Sigma_{\mathbf{x}}^{-\mathbf{1}}\Sigma_{\mathbf{x}}(1) = \sigma_x^2. \end{aligned}$$

It is easy to verify that $\gamma = 1$ by substituting b to γ. Hence

$$\begin{aligned} \mathbf{g}_{1opt} &= \mathbf{q}_{1opt}\Sigma_1^{-1}\mathbf{U}_1^* = [\,1 \ 0 \ \cdots \ 0\,]V^*\Sigma_1^{-1}U_1^* \\ &= [\,1 \ 0 \ \cdots \ 0\,]\mathbf{H}^\dagger(N,M) \end{aligned} \tag{30}$$

is an optimal equalizer with $S_2(x,\hat{x}) = 1$. By substituting (30) to (26), we know that all equalizers such that $S_2(x,\hat{x}) = 1$ can be written as the form of (25). This completes the proof. □

The full column rank condition in Theorem 3 is related to the zeros of the blurring filter, which is a common condition in the problem of blind equalization and identification [34]–[36].

We can see from (29) that the eigenvectors corresponding to the zero eigenvalue of Q has no effect to $S_2(x,\hat{x})$. The extra freedom of $\mathbf{q}_2$ in (25) can be used to make $S_1(x,\hat{x})$ larger when the mean of the source is not zero. Moreover, we may get extra design freedom by putting aside the effect to $S_2(x,\hat{x})$ of the eigenvector of Q if its corresponding eigenvalue is small. This idea will be employed to deal with general sources with non-zero mean in the following section.

IV. Suboptimal Equalizers for General Sources

In this section, we address the problem of finding $\mathbf{g}$ such that $S(x,\hat{x})$ in (7) is maximized for general images, of which the mean may not be zero. We first characterize the equalizer maximizing $S_1(x,\hat{x})$ alone.

Theorem 4: For the blurring filter $H(z) = \sum_{i=0}^{M-1} h[i]z^{-i}$ and the equalizer $G(z) = \sum_{j=0}^{N-1} g[j]z^{-j}$, $S_1(x,\hat{x}) = 1$ if and only if

$$\left(\sum_{j=0}^{N-1} g[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right) = 1. \tag{31}$$

Proof: Note that $1 + a^2 \geq 2a$ for arbitrary a, and the equality holds if and only if $a = 1$. Hence $S_1(x,y) \leq 1$, and the equality holds when $\left(\sum_{j=0}^{N-1} g[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right) = 1$. □

Note that (31) is a one-dimensional equality constraint, which means that it can be easily satisfied by one free parameter. From this point of view, Theorem 3 can be extended to general non-zero mean sources.

Theorem 5: Assume that $x[n]$ is a WSS process with non-zero mean and invertible $\Sigma_{\mathbf{x}}$, and there is no noise η. For the blurring filter $H(z)$ with $\sum_{i=0}^{M-1} h[i] \neq 0$, $S(x,\hat{x}) = 1$ can always be achieved provided $NK > N + M - 1$ and $rank\{\mathbf{H}(N,M)\} = N+M-1$. Moreover, all optimal equalizers $G(z) = \sum_{j=0}^{N-1} g[j]z^{-j}$ can be parameterized by

$$\mathbf{g}_{opt} = [\,1 \quad 0 \quad \cdots \ 0\,]\mathbf{H}^\dagger(N,M) + \mathbf{q}_2\mathbf{U}_2^*, \tag{32}$$

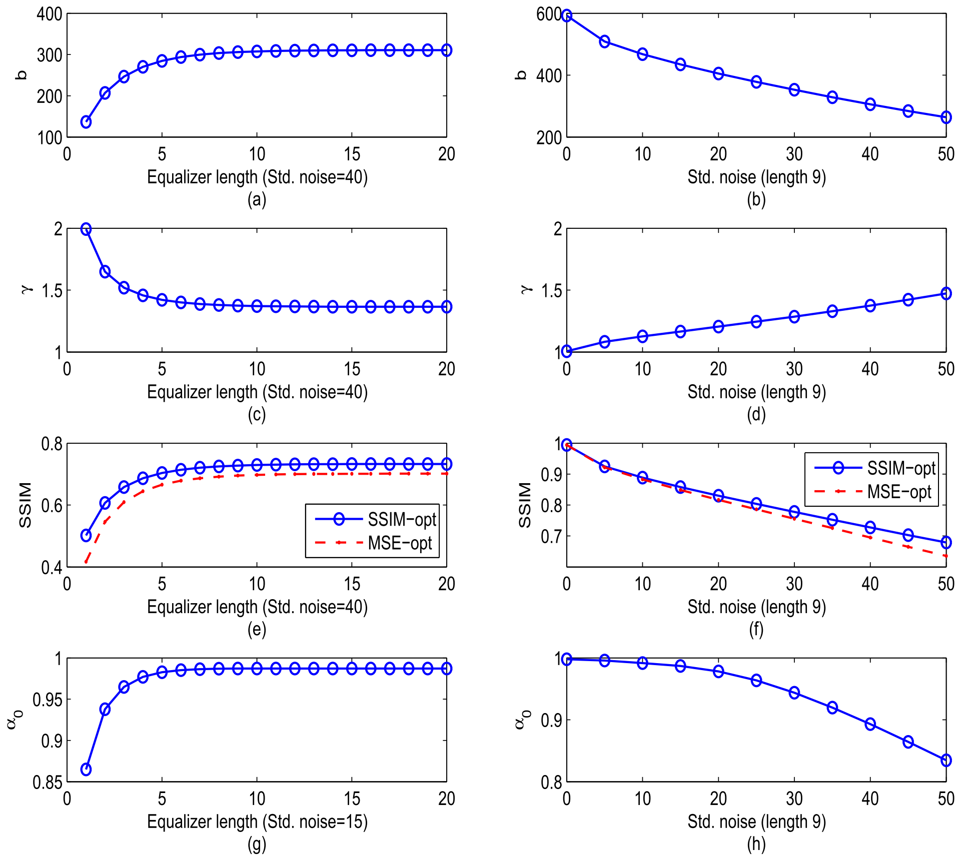

Fig. 3. b, γ and the maximal SSIM index of single channel systems.

Algorithm 1 Calculation of a Suboptimal SSIM Equalizer for General Sources

A-I Decompose $\mathbf{Q}$ as

$$\mathbf{Q} = \left[\begin{array}{cc} \mathbf{U}_1 & \mathbf{u}_2 \end{array}\right] \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & \underline{\sigma} \end{array}\right] \left[\begin{array}{c} \mathbf{U}_1^* \\ \mathbf{u}_2^* \end{array}\right],$$

where $\underline{\sigma}$ and $\mathbf{u}_2$ are the smallest eigenvalue and the corresponding eigenvector respectively.

A-II Compute b, γ and $\mathbf{g}$ by equations (10)-(12).

A-III Set $\alpha_0 = \left(\sum_{j=0}^{N-1} g[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right)$, and compute $S_0 = \text{StatSSIM}(x, \hat{x})$ by substituting $\mathbf{g}$ to equations (7)-(9).

A-IV Set $S = S_0, \alpha = \alpha_0$.

A-V Choose q_2 such that $g_{new} = \mathbf{g} + q_2\mathbf{u}_2^*$ satisfies $\left(\sum_{j=0}^{N-1} g_{opt}[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right) = \frac{\alpha+1}{2}$.

A-VI Compute $S_{new} = \text{StatSSIM}(x, \hat{x})$ by substituting $\mathbf{g}_{new}$ to equations (7)-(9).

A-VII If $S_{new} > S$, set $\mathbf{g} = \mathbf{g}_{new}, S = S_{new}, \alpha = \frac{\alpha+1}{2}$ and go to A-V, else save S and $\mathbf{g}$ as outputs.

where U_2 is defined in (24) and $\mathbf{q}_2 \in \mathbb{R}^{1\times(NK-N-M+1)}$ is such that

$$\left(\sum_{j=0}^{N-1} g_{opt}[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right) = 1. \tag{33}$$

Proof: Since $NK > N + M - 1$ and $rank\{\mathbf{H}(N, M)\} = N + M - 1$, it follows from Theorem 3 that $S_2(x, \hat{x}) = 1$ can always be achieved by $\mathbf{g}_{opt} = [\,1\ 0 \cdots 0\,]\,\mathbf{H}^{\dagger}(N, M) + \mathbf{q}_2\mathbf{U}_2^*$, where $\mathbf{q}_2 \in \mathbb{R}^{1\times(NK-N-M+1)}$ is non-empty free vector. Taking $\mathbf{q}_2$ such that (33) holds we get the equalizer with $S_1(x, \hat{x}) = 1$. Therefore, $\mathbf{g}_{opt}$ given by (32) makes $S(x, \hat{x}) = S_1(x, \hat{x})S_2(x, \hat{x}) = 1$. □

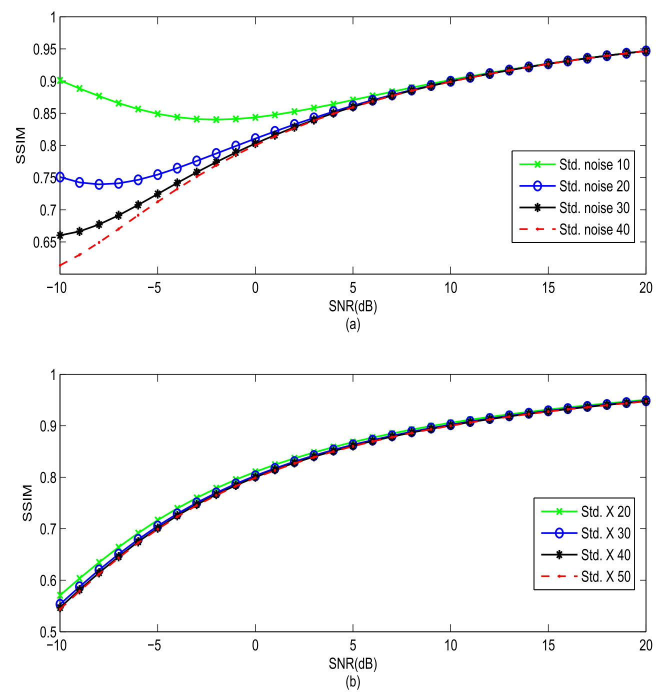

Fig. 4. The SSIM index w.r.t SNR ($N = 9$).

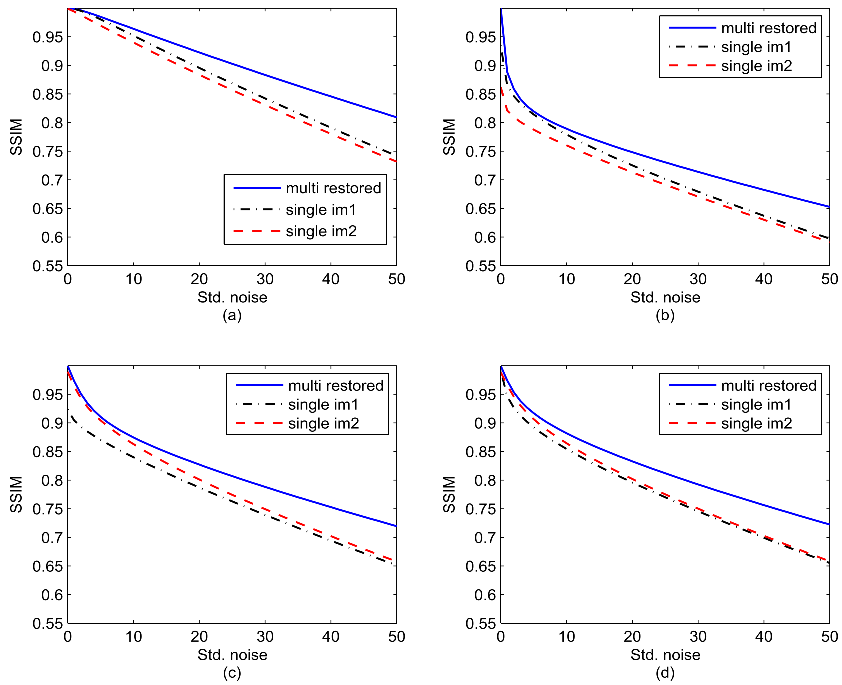

Fig. 5. The SSIM index w.r.t. the noise strength of two channel systems ($N = 11$).

Assume that $x[n]$ and the noise $\eta[n]$ are two WSS random processes and are uncorrelated with each other. Assume further that the mean of $\eta[n]$ is zero. Given the blurring filter $\mathbf{H}(z)$, let $\mathbf{Q}$ be defined by (6). It follows from the proof of Theorem 1 and Theorem 5 that if one eigenvalue of $\mathbf{Q}$ is small, we may ignore its effect on $S_2(x, \hat{x})$ and use the freedom to maximize $S_1(x, \hat{x})$. This idea brings about Algorithm 1 which provides a suboptimal solution for non-zero mean sources.

Algorithm 1 computes a suboptimal equalizer $\mathbf{g}$ with SSIM index S. Moreover, we can obtain an upper and lower bound of the maximal SSIM index $\text{StatSSIM}_{\max}(x, \hat{x})$. This is stated by the following corollary.

Corollary 1: Let γ, $\mathbf{g}$ and S be computed by Algorithm 1, then we have

$$\frac{2\alpha_0 + c_1/\mu_x^2}{1 + \alpha_0^2 + c_1/\mu_x^2}\frac{1}{\gamma} \leq S \leq \text{StatSSIM}_{\max}(x, \hat{x}) \leq \frac{1}{\gamma}. \quad (34)$$

Theorem 1 and Theorem 4 show that $S_{2\max} = 1/\gamma$ and $S_{1\max} = 1$ can be achieved by different optimal equalizers respectively. The iteration in Algorithm 1 is actually searching a tradeoff between S_1 and S_2. It follows from (34) that the SSIM index of the suboptimal solution by Algorithm 1 is determined by α_0, c_1/μ_x^2 and $1/\gamma$. The value of α_0 and S_1 will be computed for $\mathbf{g}_{opt}$ with $S_{2\max} = 1/\gamma$ in numerical examples.

V. Numerical Examples

In this section, we will present various numerical examples to illustrate the usage of the proposed theoretical results.

To avoid the influence of the estimation error of $C_{\mathbf{xx}}$, we use a first-order Autoregressive (AR) model with correlation $\rho = 0.95$ as the model of the source image in Example 1–3. The first-order AR model is a good representation of many images. In three examples, we set $\sigma_x^2 = 600$ and $c_2 = 58.5225$ if they are not specified explicitly.

Example 1: This example demonstrates how the value of b, γ and the optimal SSIM index vary with respect to the equalizer length N and the noise strength. The blurring filter is the Gaussian filter with length 5 and standard deviation 2. The noise is white Gaussian with the standard deviation $\sigma_n = 40$. The subplots in the left column of Fig. 3 show the relation with the equalizer length N. We can see that the performance improves as the length N of the equalizer increases. However, the improvement is not significant for $N \geq 9$ as shown by Fig. 3e. This suggests a good choice of the equalizer length between the restoration performance and the implementation complexity. The subplots in the right column of Fig. 3 show the relation with the noise strength. Generally the SSIM index decreases as the standard deviation of the noise increases. The SSIM performance of the optimal MSE equalizer is also drawn in Fig. 3e–3f, both of which show that the optimal SSIM equalizer does give a better performance. However, the performance difference between SSIM index and MSE is small when the noise standard deviation small as shown by Fig. 3f. Fig. 3g–3h plot the value of α_0 defined in Algorithm 1, which determines the value of S_1. We see that α_0 is closer to 1 as the equalizer length gets larger.

The relation between the optimal SSIM index and the SNR ($N = 9$) is shown in Fig. 4. In Fig. 4a, the SNR increases by increasing σ_x for fixed σ_n, while in Fig. 4b, the SNR increases by reducing σ_n for fixed σ_x. For fixed blurring filters, we can see from Fig. 4 that the SSIM performance is determined by the SNR uniquely when the SNR is high.

Example 2: This example considers the two channel systems. The following four sets of blurring filters are chosen:

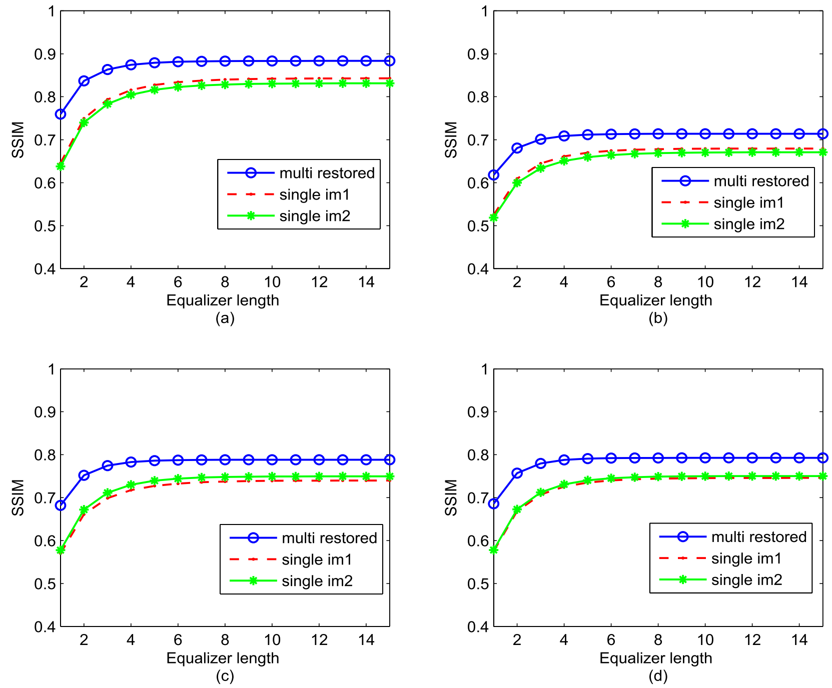

Fig. 6. The SSIM index w.r.t. the equalizer length of two channel systems ($\sigma_n = 30$).

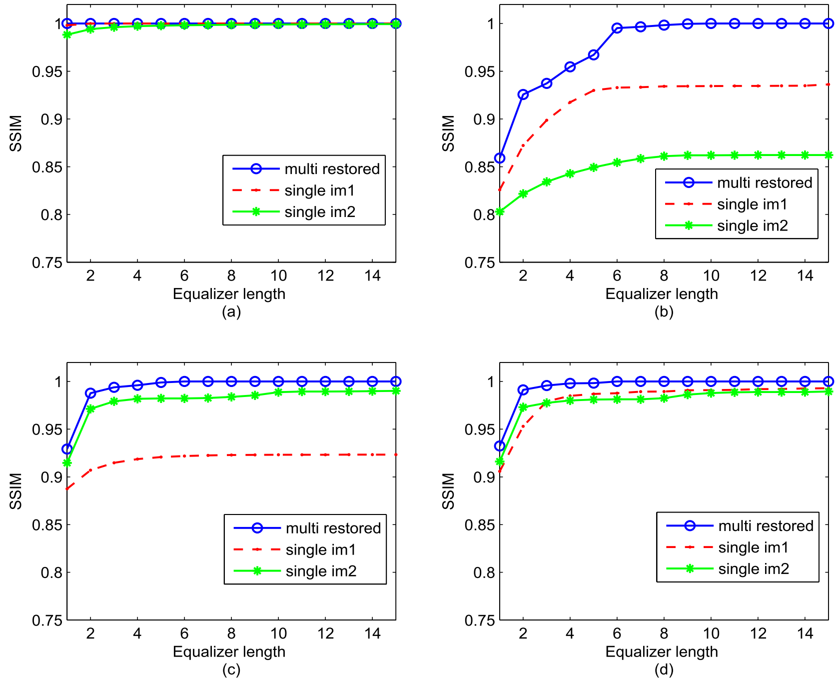

Fig. 7. The SSIM index w.r.t. the equalizer length of two channel systems ($\sigma_n = 0$).

(a) $H_1(z) = \frac{1+0.25z^{-1}}{1.25}$, $H_2(z) = \frac{1+0.98z^{-1}}{1.98}$; (b) $H_1(z) = fir1(10, 0.15)$, $H_2(z) = fir1(10, 0.35)$; (c) Gaussian blurring filters with length 7 and $\sigma_{H_1} = 1, \sigma_{H_2} = 5$; (d) Gaussian blurring filters with length 7 and $\sigma_{H_1} = 2, \sigma_{H_2} = 15$. *fir1* is a Matlab function used to design specified FIR filters. The blurring filters in the first set are minimum phase filters, while those in other sets are non-minimum phase filters. The simulation results are shown by Fig.5–7, in which each subplot with label from (a) to (d) corresponds to the set with the same label.

Fig. 5 shows that the SSIM index decreases as σ_n increases. In Fig. 5, the length of all equalizers is set to 11. For each set of blurring filters, the two channel equalizer has better SSIM performance than the single channel equalizer. This performance improvement of two channel systems is more significant as σ_n increases (the SNR decreases). The relationship between the SSIM index and the equalizer length is shown in Fig. 6 ($\sigma_n = 30$) and Fig. 7 ($\sigma_n = 0$). It can be seen from Fig. 6 that the two channel equalizer gives much

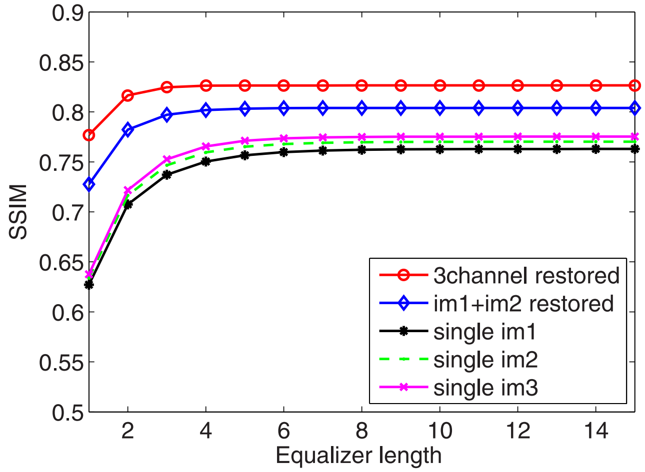

Fig. 8. The SSIM index w.r.t. the equalizer length of three-channel systems.

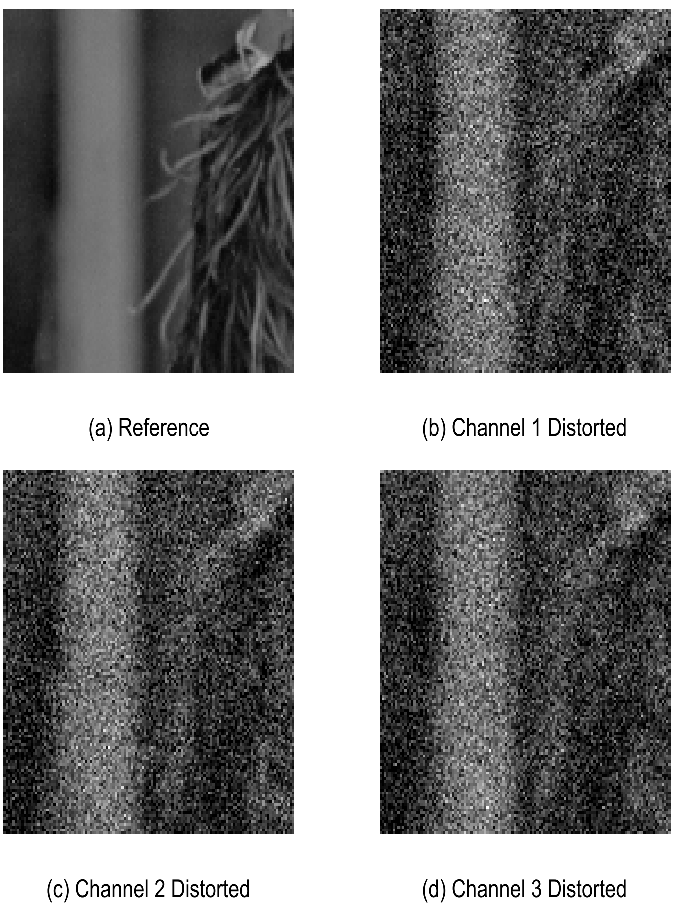

Fig. 9. The original image and distorted images: (a) Original image (b) Distorted image with $\sigma_{H_1} = 2, \sigma_n = 35$, MSE = 1298.975465, SSIM = 0.421610. (c) Distorted image with $\sigma_{H_2} = 3, \sigma_n = 35$, MSE = 1291.302170, SSIM = 0.414036. (d) Distorted image with $\sigma_{H_3} = 7, \sigma_n = 35$, MSE = 1350.026872, SSIM = 0.393555.

better SSIM performance in each case. One can determine the equalizer length which provides a good tradeoff between the SSIM performance and the implementation complexity. For this particular example, $N = 6$ is a good choice.

For minimum phase blurring filters, the maximal SSIM index with either single channel equalization or two channel equalization tends to 1 when the equalizer length increases as shown by Fig. 7a. As shown in Fig. 7b–7d, the SSIM index can never reach 1 by single channel FIR equalizer for the non-minimum phase blurring filter. However, the SSIM index tends to 1 by the two channel equalizer as the filter length increases. SSIM is almost 1 for blurring filters set (b), set (c) and set (d) when the equalizer length N is equal to 8, 5

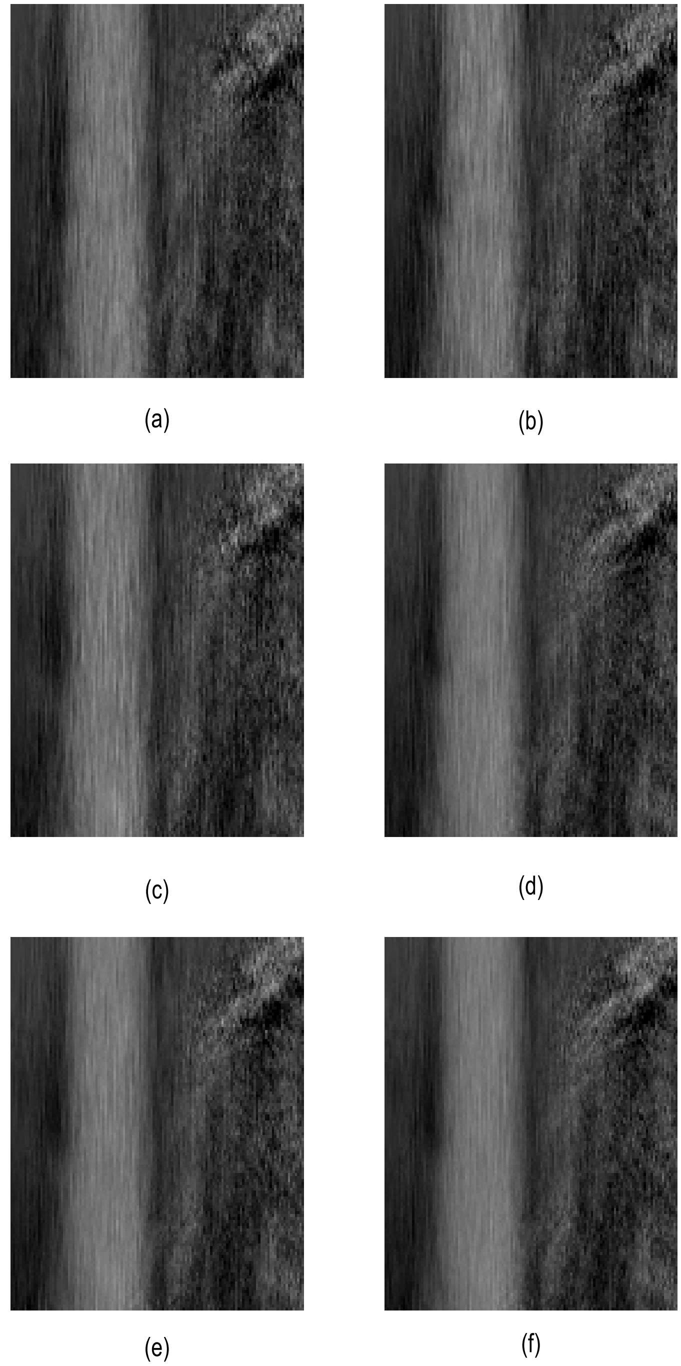

Fig. 10. Images restored by using different sub-channel images: (a) Image restored by channel 1, MSE=267.155018, SSIM=0.718756. (b) Image restored by channel 2, MSE=279.178332, SSIM=0.703635. (c) Image restored by channel 3, MSE=301.266248, SSIM=0.694187. (d) Image restored by channel 1,2, MSE=223.739171, SSIM=0.760284. (e) Image restored by channel 2,3, MSE=242.486797, SSIM=0.747186. (f) Image restored by three channels, MSE=214.094281, SSIM=0.774641.

TABLE I
OPTIMAL SSIM INDEX OF DIFFERENT RESTORATIONS ON DIFFERENT IMAGES ($\sigma_{H_1} = 2$, $\sigma_{H_2} = 3$, $\sigma_{H_3} = 7$, $\sigma_n = 35$)

SSIM restored	Channel 1	Channel 2	Channel 3	Channel 1&2	Channel 2&3	Three channels
Lena-region	0.718756	0.703635	0.694187	0.760284	0.747186	0.774641
Lena	0.719319	0.710864	0.705339	0.761722	0.755402	0.781372
Barbara	0.650579	0.647809	0.641732	0.690196	0.685135	0.704407
Boat	0.625919	0.615864	0.607479	0.659203	0.649023	0.671828
Goldhill	0.585816	0.576319	0.566121	0.639159	0.628722	0.662341

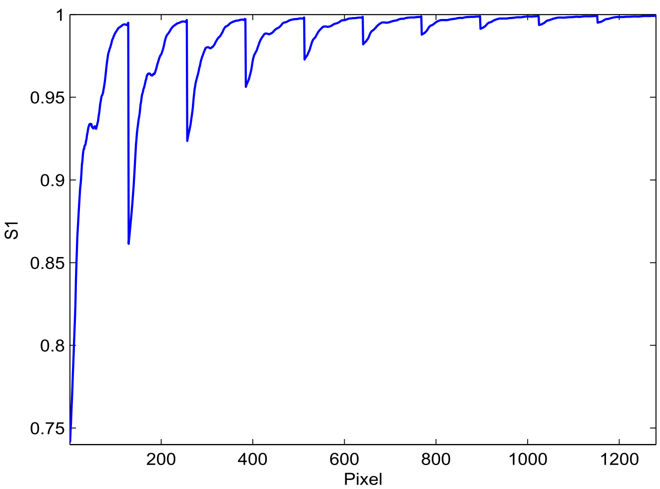

Fig. 11. The local value of S_1 of pixels from the first 10 rows with $c_1 = 6.5025$.

and 6 respectively. As we said before, the smallest equalizer length with good SSIM performance depends on the position of the non-minimum phase zeros of the blurring filters.

Example 3: This example considers a three-channel system with $\sigma_x^2 = 600$ and $\sigma_n^2 = 625$. The blurring filters are Gaussian filters with length 7 and $\sigma_{H_1} = 1, \sigma_{H_2} = 2$, $\sigma_{H_3} = 15$, respectively. The relationship between the SSIM index and the equalizer length is plotted in Fig. 8, which shows that the three-channel equalizer achieves much better SSIM performance than two-channel equalizers and single-channel equalizers. The computational burden is not heavy since only matrix manipulation is involved in Equation (11)-(12).

In the following three examples, we apply the method to real images. In practical applications, the SSIM index of an entire image is the average of all SSIM value computed locally along a sliding window. Therefore, the optimal $\mathbf{g}_{opt}$ is computed adaptively by using $\Sigma_{\mathbf{x}}$ estimated from $\mathbf{y}$ within a fixed window. In all three examples, the window is of size 35×35, and $c_2 = 58.5225$.

Example 4: In this example, we use a region of the Lena image as the test image, which is blurred by three different length 7 Gaussian filters with $\sigma_{H_1} = 2$, $\sigma_{H_2} = 3$, and $\sigma_{H_3} = 7$. The noise is Gaussian white with standard deviation $\sigma_n = 35$. The original image and distorted images of three channels are shown in Fig. 9. Fig. 10 shows the restored images by single channel blurred image and different combinations of sub-channel images. The optimal SSIM index is given in Table I. Both Fig. 10 and Table I show that multiple channel restoration outperforms the single channel restoration. Fig. 11 draws the local value of S_1 of the image restored by channel 1 only for pixels of the first ten rows. Here we choose $c_1 = 6.5025$. We can see that S_1 varies between 0.75 and 0.99. We find that

Fig. 12. Images restoration: (a) Original image. (b) Distorted image with $\sigma_{H_1} = 2, \sigma_n = 25$, MSE = 899.902475, SSIM = 0.588095. (c) Distorted image with $\sigma_{H_2} = 5, \sigma_n = 25$, MSE = 935.477916, SSIM = 0.571881. (d) Image restored by the SSIM-optimal filter of Channel 1, MSE = 628.557731, SSIM = 0.689969. (e) Image restored by the MSE-optimal filter of Channel 1,2, MSE = 499.992954, SSIM = 0.670820. (f) Image restored with the SSIM-optimal filter of Channel 1,2, MSE=609.016684, SSIM = 0.712847.

TABLE II

OPTIMAL SSIM INDEX OF DIFFERENT RESTORATIONS ON DIFFERENT IMAGES ($\sigma_{H_1} = 2, \sigma_{H_2} = 5, \sigma_{H_3} = 7, \sigma_n = 25$)

SSIM restored	Channel 1	Channel 2	Channel 3	Channel 1&2	Channel 2&3	Three channels
Lena-region	0.771767	0.750969	0.745187	0.795597	0.781671	0.803091
Lena	0.765644	0.753439	0.752718	0.796972	0.790106	0.811148
Barbara	0.689969	0.682998	0.681324	0.712847	0.710320	0.720719
Boat	0.664766	0.645722	0.643246	0.683162	0.672878	0.690225
Goldhill	0.639794	0.623363	0.622403	0.678107	0.669111	0.695227

Fig. 13. Images restoration: (a) Original image. (b) Distorted image with $\sigma_{H_1} = 2, \sigma_n = 35$, MSE = 1273.888322, SSIM = 0.351978. (c) Distorted image with $\sigma_{H_2} = 3, \sigma_n = 35$, MSE = 1280.202630, SSIM = 0.345316. (d) Image restored with the SSIM-optimal filter by Channel 1, MSE = 226.380905, SSIM = 0.719319. (e) Image restored with the MSE-optimal filter by Channel 1,2, MSE = 164.625729, SSIM = 0.716074. (f) Image restored with the SSIM-optimal filter by Channel 1,2, MSE = 189.378151, SSIM = 0.761722.

S_1 is small at the pixels near the left boundary of each row. One reason is that $\Sigma_{\mathbf{x}}$ can not be estimated accurately for those pixels.

Example 5: In this example, we consider the Barbara image of size 512×512 distorted by three length 7 Gaussian filters with $\sigma_{H_1} = 2$, $\sigma_{H_2} = 5$ and $\sigma_{H_3} = 7$. The noise is Gaussian white with standard deviation $\sigma_n = 25$. The original image, two distorted images and some reconstructed images are shown in Fig. 12. The SSIM values including other test images are given by Table II. We can see that the SSIM optimal equalizer outperforms the MSE-based equalizer, and the multiple channel restoration outperforms the single channel restoration.

Example 6: In this example, we use the Lena image of size 512×512 as the test image. Three length 7 Gaussian filters with $\sigma_{H_1} = 2$, $\sigma_{H_2} = 3$ and $\sigma_{H_3} = 7$ are adopted as the blurring filter. The noise is Gaussian white with standard deviation $\sigma_n = 35$. The original image and some distorted and reconstructed images are shown in Fig. 13. The SSIM values including other test images are given in Table I. We can draw the same conclusion as Example 5 that the multiple channel restoration outperforms the single channel restoration, and the SSIM optimal equalizer has better performance.

VI. CONCLUDING REMARKS

The optimal design of multichannel equalizers with SSIM criteria has been studied. Under the assumption that the source image is zero mean, closed-form formulae have been presented for the optimal equalizer and the achieved SSIM index. The results show that the equalizer with maximal SSIM index is the one with minimal MSE multiplied by a scalar γ, which turns out to be the inverse of the achievable maximum of the SSIM index. Direct relationship between the SSIM index and MSE has been established. An algorithm to compute the suboptimal solution for general sources is proposed. Finally various numerical examples have been given to illustrate the effectiveness of the theoretical results.

Much research can be done on SSIM index oriented optimization by using the idea developed in this paper. Firstly, it is interesting to apply the SSIM index into the design of general filter banks, which has wide applications in subband coding and compression. The main difficulty is to deal with the cyclo-stationary property of the signals. Secondly, new blind equalizers with the maximal SSIM index may be designed based on MSE-optimal equalizers [4], [5].

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