Electromagnetic Waves in a Time Periodic Medium With Step-Varying Refractive Index

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***Abstract*— We present an exact mathematical framework for electromagnetic wave propagation in periodically time-modulated media, in which the permittivity is homogenous and modulated in a step-varying fashion. By using Hill's equation theory, we show that this problem has analytical solutions. We connect the dispersion relation, which exhibits $k-$gaps, with the Hill stability analysis, providing an alternative mathematical description for wave propagation in temporal crystals. Our analysis, which is exact and transposable to other kinds of waves or modulation schemes, provides general useful physical and mathematical insights, complementing the use of numerical techniques such as finite differences in the time domain, or harmonic balance schemes, with a more transparent and practical design tool. The present analytical transient mathematical analysis, in contrast with the existing frequency-domain numerical approaches, can exhibit the parametric properties of electromagnetic waves inside a time periodic medium. For this reason, it can be a useful tool for the design of active microwave and optical devices, which employ time periodic wave medium modulation to filter or parametrically amplify wave signals.**

***Index Terms*— Electromagnetic wave propagation, Hill's equation, parametric amplification, stability analysis, time-modulated media.**

I. Introduction

SINCE the pioneering works of Brillouin [1], wave propagation in spatial periodical devices has been of great interest to researchers and engineers. Spatially periodic structures have, indeed, been broadly used to filter and amplify and in general manipulate electromagnetic signals in a variety of modern technological applications including telecommunications, signal processing, and imaging. It is well-known that electromagnetic waves experience reflections from discontinuities. A periodic formulation of small discontinuities can trigger the creation of Bloch waves, which can be subject to Bragg reflections and bandgaps [1]–[9]. In addition to medium discontinuities in space, the effect of medium discontinuities on electromagnetic wave propagation in the time domain has also attracted the attention and curiosity of the research community [10]–[13]. Several studies about temporal crystals, also known as time periodic wave media and time-Floquet systems, have been reported over the years dating back to the 1950's [7], [14]–[30]. More recently, time-modulated media have been used to exhibit parametric amplification [24], [31], near-zero effective refractive index [32], nonreciprocity [33], nonreciprocal gain and frequency conversion [34], and electromagnetic energy accumulation [35]. Recent findings have also connected them with parity–time symmetric scattering theory [31]. They have also been implemented in combination with spatial periodicity to engineer band structures [36], [37].

One of the key properties of time periodic wave media, at the core of this paper, is the fact that they can exhibit complete gaps in the dispersion relation, in a way analogous to space periodic media. These gaps, however, are formed for the wave vector k (momentum gaps) instead of frequency ω. Contrary to what happens in space crystals: bandgaps support evanescent wave propagation, which are forced by passivity to describe fields that exponentially decay away from sources [38], [39], the active nature of temporal crystals allows the gaps to also support fields that grow exponentially in time. Thus, the absence of wave propagation in these $k-$gaps can be related to unstable wave solutions. Furthermore, such media do not conserve frequency and the same wave vector can be associated with an infinite number of operating frequencies, which is a very unique property. More specifically for a modulation of frequency Ω, one obtains a temporal Brillouin zone of width Ω, and a periodicity of the dispersion relation along the frequency axis [$k(\omega) = k(\omega + n\Omega)$, for $n \in \mathbb{Z}$], in close resemblance with the periodicity of the band structure along the k-axis in the spatial analogous case.

Yet, to date, such interesting properties have only been analyzed either numerically using finite-difference time-domain (FDTD) simulations, or semianalytically using opaque approximate methods involving truncation of the harmonic content of the field, which inherently limits our understanding of such systems and our ability to conveniently predict their behavior and design them for specific purposes. Conversely, here, we demonstrate an exact, fully analytical solution to the complex problem of wave propagation in a temporal crystal with periodic step-varying dielectric constant, which can be used directly as a transient mathematical tool to determine parametric or evanescent properties of time periodic electromagnetic media, without any approximations. This analytical treatment is based, on the Hill equation stability theory. We consider a nonmagnetic, isotropic, and nonstationary infinite medium, in which the dielectric permittivity is modulated in time with a step function. We show that this special problem

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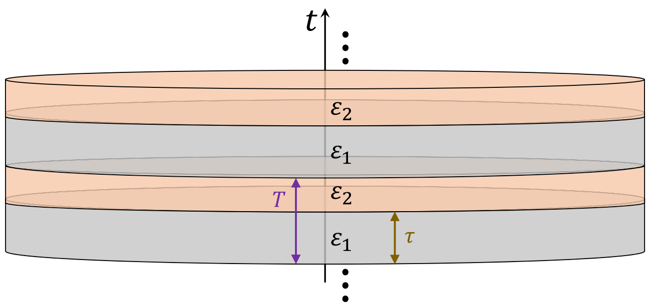

Fig. 1. Illustration of the infinite time periodic medium under study. The dielectric constant is uniform in space and is periodic in time, successively taking the values ε_1 and ε_2.

of electromagnetic theory can be mathematically modeled as a system of differential equations with time periodic coefficients, which support exact analytical solutions. We connect for the first time to our knowledge the Hill stability analysis with the band structure properties of such complex wave media and we validate our exact analytical solutions and remarks with the approximate prediction from a rigorous numerical approach using Floquet theorem [40] in the frequency domain. Thus, in contrast with the numerical studies of the literature, this paper provides an exact analytical mathematical framework for the analysis of such wave propagation problem, by mapping it to Hill's equation [41]–[46]. This treatment, which can also be easily applied to other modulation schemes (as shown in the Appendix), gives precious insights for the overall parametric behavior of time-modulated media which is lacking from the already established numerical approaches and highlights an interesting connection between the mathematical theory of Hill's differential equation and electromagnetic wave propagation in a time periodic medium.

II. Time-Domain Analysis: Electric Displacement Wave

Let us assume an infinite medium for which the dielectric permittivity ε is modulated in time due to the perfect synchronization of capacitor switches. Hence, ε is

$$\varepsilon(t) = \begin{cases} \varepsilon_1, & 0 \le t < \tau \\ \varepsilon_2, & \tau < t \le T \end{cases} \tag{1}$$

where $\varepsilon(t) = \varepsilon(t+T)$, as shown in Fig. 1. In the absence of sources, Maxwell's equations are

$$\begin{aligned} &\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0, \\ &\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \times \mathbf{H} = \partial_t \mathbf{D} \end{aligned} \tag{2}$$

and the constitution relations are $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \varepsilon(t)\mathbf{E}$. At this point, it is convenient to form the wave equation for the electric displacement field. Since we assume an infinite medium, we can truncate the problem to a 1-D scalar field. Applying Maxwell's equations, the constitution relations and the curl operator two times, we extract the wave equation

$$\frac{\partial^2 D}{\partial x^2} = \mu_0 \varepsilon(t) \frac{\partial^2 D}{\partial t^2}. \tag{3}$$

In order to find solutions of the wave equation, we use the separation of variables method, which implies that we search for solutions of the form: $D(x,t) = X(x)T(t)$. By plugging it into (3), we get a system of two uncoupled differential equations

$$\frac{d^2 X(x)}{dx^2} + k^2 X(x) = 0 \tag{4}$$

$$\frac{d^2 T(t)}{dt^2} + \frac{k^2}{\mu_0 \varepsilon(t)} T(t) = 0. \tag{5}$$

Equation (4) is well-known and gives solutions of the form $\exp(\pm jkx)$. In combination with (5), it can provide the full-wave profile of the field. In order to solve (5), we introduce the variable transformation $\xi = \Omega t/2$, where Ω is the time-modulation frequency, and the unit rectangular coefficient $\psi(\xi) = \psi(\xi + \pi)$ is shown as follows:

$$\psi(\xi) = \begin{cases} 1, & 0 < \xi < \tau'(= \Omega\tau/2) \\ -1, & \tau' < \xi < \pi. \end{cases} \tag{6}$$

Equation (5) becomes

$$\frac{d^2 T(\xi)}{d\xi^2} + \left[\frac{2k^2}{\Omega^2 \mu_0} \left(\frac{\varepsilon_2 + \varepsilon_1}{\varepsilon_1 \varepsilon_2} \right) - 2 \frac{k^2}{\Omega^2 \mu_0} \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 \varepsilon_2} \right) \psi(\xi) \right] T(\xi) = 0. \tag{7}$$

This differential equation is the well-known Hill's equation [41]

$$\frac{d^2 T(\xi)}{d\xi^2} + [a - 2q\psi(\xi)] T(\xi) = 0. \tag{8}$$

For the π periodic function $\psi(\xi)$, $a = 2k^2(\varepsilon_2 + \varepsilon_1)/(\Omega^2 \mu_0 \varepsilon_1 \varepsilon_2)$, and $q = k^2(\varepsilon_1 - \varepsilon_2)/(\Omega^2 \mu_0 \varepsilon_1 \varepsilon_2)$. Since we consider a step variation of the dielectric permittivity, $\psi(\xi)$ is not just π periodic but also unit rectangular and thus obeys the Meissner equation

$$\frac{d^2 T(\xi)}{d\xi^2} + \frac{4k^2}{\Omega^2 \mu_0 \varepsilon_1} T(\xi) = 0, \quad 0 \le \xi < \tau'(= \Omega\tau/2) \tag{9}$$

$$\frac{d^2 T(\xi)}{d\xi^2} + \frac{4k^2}{\Omega^2 \mu_0 \varepsilon_2} T(\xi) = 0, \quad \tau' \le \xi < \pi. \tag{10}$$

In order to find the solution of such systems, it is necessary to calculate a quantity known as the state transition matrix [41]–[46]. (This matrix is analogous to the $W-$form of the transfer matrix used in 1-D photonic crystals and in quantum particle propagation [47], [48].) The transition matrix $\mathbf{\Phi}(t,\tau)$ is defined by the Wronskian $\mathbf{W}(t)$ (Appendix)

$$\mathbf{\Phi}(t,\tau) = \mathbf{W}(t)\mathbf{W}(\tau)^{-1} \tag{11}$$

and can be easily constructed for the Meissner equation. In fact, for $0 \le \xi < \tau'$, we find the following expression for (12), as shown at the top of the following page.

For $\tau' \le \xi < \pi$, the transition matrix satisfies the property $\mathbf{\Phi}(\xi, 0) = \mathbf{\Phi}(\xi, \tau') \cdot \mathbf{\Phi}(\tau', 0)$ [41] (Appendix), as shown in (13) at the top of the following page.

For $\xi > \pi$, assuming that $\xi = m\pi + \xi'$ with $0 < \xi' < \pi$ and m is an integer, the transition matrix is defined as $\mathbf{\Phi}(\xi, 0) = \mathbf{\Phi}(\xi', 0) \cdot \mathbf{\Phi}(\pi, 0)^m$ [41]–[46] (Appendix). Knowing the transition matrix leads to the solution of the wave equation.

$$\mathbf{\Phi}(\xi,0) = \begin{pmatrix} \cos\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\xi\right] & \frac{\Omega\sqrt{\mu_0\varepsilon_1}}{2k}\sin\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\xi\right] \\ -\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\sin\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\xi\right] & \cos\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\xi\right] \end{pmatrix} \tag{12}$$

$$\mathbf{\Phi}(\xi,0) = \begin{pmatrix} \cos\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_2}}(\xi-\tau')\right] & \frac{\Omega\sqrt{\mu_0\varepsilon_2}}{2k}\sin\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_2}}(\xi-\tau')\right] \\ -\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_2}}\sin\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_2}}(\xi-\tau')\right] & \cos\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_2}}(\xi-\tau')\right] \end{pmatrix} \cdot \begin{pmatrix} \cos\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\tau'\right] & \frac{\Omega\sqrt{\mu_0\varepsilon_1}}{2k}\sin\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\tau'\right] \\ -\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\sin\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\tau'\right] & \cos\left[\frac{2k}{\Omega\sqrt{\mu_0\varepsilon_1}}\tau'\right] \end{pmatrix}. \tag{13}$$

For example, the wave has the following form, for the time period $(0, T)$:

$$D(x,t) = \cos\left(\frac{kt}{\sqrt{\mu_0\varepsilon_1}}\right)\left(A_0e^{-jkx}+B_0e^{jkx}\right) + \frac{\Omega\sqrt{\mu_0\varepsilon_1}}{2k}\sin\left(\frac{kt}{\sqrt{\mu_0\varepsilon_1}}\right)\left(C_0e^{-jkx}+D_0e^{jkx}\right) \quad 0<t<\tau \tag{14}$$

$$\begin{aligned} D(x,t) = &\left(\cos\left(\frac{k(t-\tau)}{\sqrt{\mu_0\varepsilon_2}}\right)\cos\left(\frac{k\tau}{\sqrt{\mu_0\varepsilon_1}}\right) - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sin\left(\frac{k(t-\tau)}{\sqrt{\mu_0\varepsilon_2}}\right)\sin\left(\frac{k\tau}{\sqrt{\mu_0\varepsilon_1}}\right)\right) \\ &\times\left(A_0e^{-jkx}+B_0e^{jkx}\right) \\ &+\left(\frac{\Omega\sqrt{\mu_0\varepsilon_1}}{2k}\cos\left(\frac{k(t-\tau)}{\sqrt{\mu_0\varepsilon_2}}\right)\sin\left(\frac{k\tau}{\sqrt{\mu_0\varepsilon_1}}\right) + \frac{\Omega\sqrt{\mu_0\varepsilon_2}}{2k}\sin\left(\frac{k(t-\tau)}{\sqrt{\mu_0\varepsilon_2}}\right)\cos\left(\frac{k\tau}{\sqrt{\mu_0\varepsilon_1}}\right)\right) \\ &\times\left(C_0e^{-jkx}+D_0e^{jkx}\right) \\ &\tau<t<T \end{aligned} \tag{15}$$

where the values A_0, B_0, C_0, and D_0 are defined by the initial field at $t = 0$. An important remark is that in this time periodic structure the field condition at $t = 0$ is equivalent with the boundary and initial field conditions of spatially periodic electromagnetic propagation problems. As an illustration, we show in Fig. 2(a)–(c) the electric displacement oscillations for a specific case with time-symmetric field initial conditions ($A_0 = 1$ and $B_0 = C_0 = D_0 = 0$), with $\varepsilon_1 = 4\varepsilon_0$, $\varepsilon_2 = \varepsilon_0$, and $\tau = T/2$, and for three different parameter scenarios, $\omega = 0.68\ \Omega$, $\omega = 2\ \Omega$, and $\omega = 2.5\ \Omega$. These plots highlight how the modulation generates other frequencies, even after a short number of periods, consistent with the Floquet theorem [40]. It is also apparent that different harmonics may undergo different time-behaviors, being either stable, attenuated, or parametrically amplified. More specifically, the physical and engineering importance of this time-domain analysis is explicitly illustrated in Fig. 2. As we show in Section III, propagation at the conditions of Fig. 2(b) and (c) is unstable and more specifically is at the 4th and 5th $k-$gap of the band structure, respectively, but different initial conditions result to an attenuated electric displacement field [Fig. 2(b)] and to a parametrically amplified electric displacement field [Fig. 2(c)]. These results are analytical without using any time marching numerical method such as FDTD and cannot be obtained by harmonic balance techniques

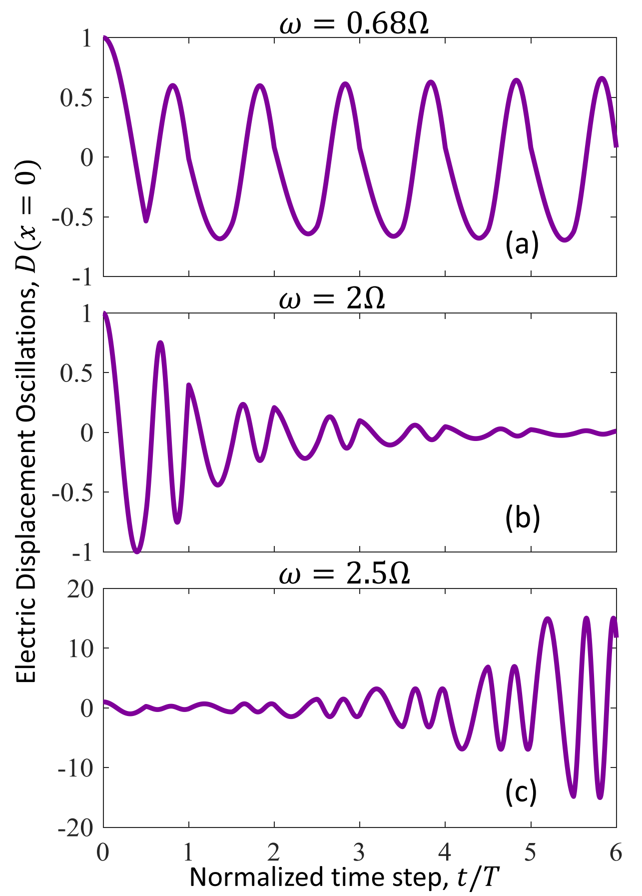

Fig. 2. Field oscillations for the time periodic medium with $\varepsilon_1 = 4\varepsilon_0$, $\varepsilon_1 = \varepsilon_0$, and $\tau = T/2$ with frequency excitation. (a) $\omega = 0.68\ \Omega$, (b) $\omega = 2\ \Omega$, and (c) $\omega = 2.5\ \Omega$.

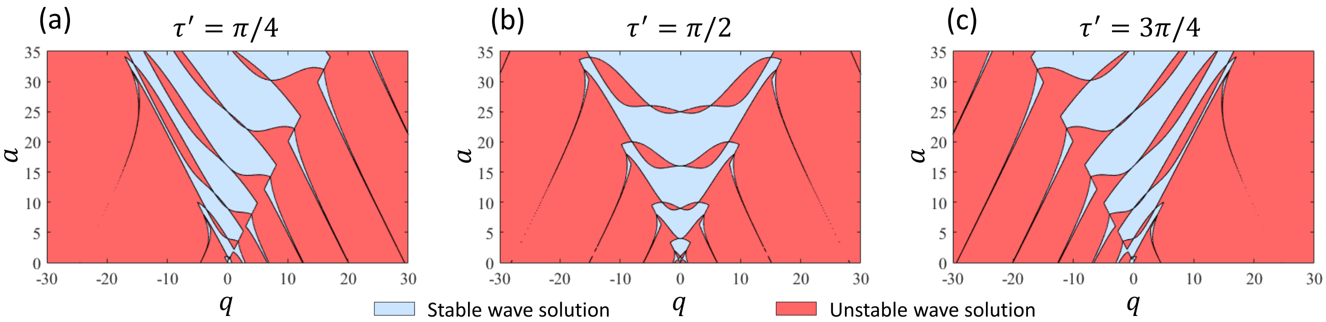

Fig. 3. Illustrations of the Hill stability charts for (a) $\tau' = \pi/4$, (b) $\tau' = \pi/2$, and (c) $\tau' = 3\pi/4$. Blue color: stable wave states. Red: unstable wave states.

since such techniques can only approximate steady stable wave states.

III. Stability Analysis

In Section II, we found the solution of the electric displacement wave from an exact, strict mathematical analysis in the time domain. It is important to mention that the wave solution can be short-time stable, but as time passes the wave signal can exponentially increase or decrease and result in parametric amplification or perfect absorption, depending on the symmetry of the excitation and the modulation. Lagrange stability analysis is mathematically linked with the existence or the absence of unitarity in the state transition matrix. It has been proven [41], [42] (Appendix) that the stability is satisfied for operating regions when

$$|\mathrm{tr}(\mathbf{\Phi}(\pi, 0))| < 2. \tag{16}$$

For our specific case, the condition (16) becomes

$$\Big|\cos[(a+2q)^{1/2}(\pi-\tau')]\cos[(a-2q)^{1/2}\tau'] - \frac{1}{2}\left(\sqrt{\frac{a+2q}{a-2q}} + \sqrt{\frac{a-2q}{a+2q}}\right) \times \sin[(a+2q)^{1/2}(\pi-\tau')]\sin[(a-2q)^{1/2}\tau']\Big| < 1 \tag{17}$$

where a and q are defined in (7). As it is shown in (17), the stability analysis is extremely dependent on all a, q, and τ'. Even though the next step for the stability analysis is to consider numerical values for its parameters, we can extract some general information for some points of operation. In the case that $\varepsilon_1 - \varepsilon_2 \to 0$, the depth of modulation converges also to 0, i.e., $q \to 0$. If we also assume that $\tau' = \pi/2$, i.e., $\tau = T/2$, then we find marginal instability for $a = n^2$, $n \in \mathbb{Z}$. For this choice of parameters, the effective dielectric permittivity is easily found from the general formula $\varepsilon_{\text{eff}} = T/\int_0^T dt/\varepsilon(t)$ as: $\varepsilon_{\text{eff}} = 2\varepsilon_1\varepsilon_2/(\varepsilon_2 + \varepsilon_1)$ and the system becomes a parametric amplifier (or a perfect absorber) for the operating frequency

$$\omega = \frac{k}{\sqrt{\mu_0 \varepsilon_{\text{eff}}}} = n\frac{\Omega}{2}. \tag{18}$$

The physical interpretation of this result is that at these specific frequencies (even multiples of the excitation frequency), the energy provided to modulate the medium can be optimally coupled to the existing wave, leading to parametric wave pumping, or on the contrary brutally force the attenuation of the existing field. In order to determine which of these conditions occur one has to proceed to the transient analysis of Section II. (Harmonic balance techniques that are most often used and are defined in Section IV can only approximate a group of field harmonics for the steady-state wave conditions and cannot exhibit the parametric characteristics of the wave states.) The mathematical interpretation of this exotic phenomenon is that by choosing this operating frequency, the eigenvalues of the state transition matrix collide to unity and the system experiences $t-$multiplied instability; this means that no matter how small the difference of the dielectric permittivity, the system provides either gain of parametric nature to the wave signal or coherent perfect absorption. These special operating conditions are associated with the Floquet diabolic and exceptional points, as shown in related studies [49]. The analysis of the general Hill equations predicts that marginal instability always occurs for $a = n^2$, $n \in \mathbb{Z}$, when $q \to 0$, independent of the time modulation of the dielectric permittivity. Before moving to the quantitative study of the Hill stability chart, it is also useful to mention that the stability analysis of our system is symmetric for $\tau' = \pi/2$, meaning that substituting $q \to -q$ will not change the stability chart. In Fig. 3, we provide the stability chart for a representative group of values of τ'. As we expect for $\tau' = \pi/2$, the chart is symmetric since the modulation is also symmetric. As the modulation becomes asymmetric, the stability chart follows this asymmetry. Another important remark is that the density of stable regions are reduced as the absolute value of q (hence the time modulation depth) increases. Even though the stability charts present the stable and unstable regions for every possible combination of values of a and q, the representative wave state parameters for our problem form the geometric locus of a line, starting from (0, 0) point and continuing with an angle $\theta = \arctan(2(\varepsilon_1 + \varepsilon_2)/(\varepsilon_1 - \varepsilon_2))$ [similar Hill stability charts and geometric loci can be obtained for general periodic $\varepsilon(t)$]. Note that for an angle of $\theta = 90°$, the geometric locus will not intersect with any region with instability; this means obviously that when $\varepsilon_1 = \varepsilon_2$ the dispersion relation has no discontinuities and any value of k is allowed. As the angle θ deviates from 90°, the line intersects with instability regions. This phenomenon directly explains the existence of forbidden values of k, known as $k-$gaps in the dispersion relation.

Evidently, as the modulation depth increases, θ decreases and the sizes of the $k-$gaps are also increased. From (17) and Fig. 3, it is evident that a critical angle θ_c exists, below which the unstable regions occupy most of the space, and hence most wave numbers are not stable. This phenomenon is linked with the coexistence of unstable regions for the Meissner equation [41]. This critical angle is occurred for $a = \pm 2q$ resulting to $\theta_c = \arctan(2) = 63.43°$ or $\theta_c = 180° - \arctan(2) = 116.57°$. These remarks provide us with a criterion for operating with parameters that lead to possibly denser stable regions. More specifically, it yields a relation between the parameters of the amplitudes of the time periodic medium, namely, ε_1, ε_2. This condition indicates, after some elementary mathematical algebra, that when ε_1 and ε_2 are not of the same sign, the operating parameters are most likely to land in instability, although stability is still possible.

IV. Frequency-Domain Analysis: Dispersion Relation

The frequency-domain analysis is the most commonly employed to deal with time periodic media. It applies the Floquet theorem [40] directly and provides numerical results about the steady state (yet without providing much insights about the wave physics, the parametric nature of the wave and the breaking of the energy conservation of the electromagnetic wave in such active media). This numerical frequency-domain analysis is limited to approximating the steady-state responses. In this section, we apply such an independent numerical method in order to compare it with our exact solution and validate the band structure properties with those that are established by the Hill stability analysis. Since the periodic modulation of the permittivity can be expanded in Fourier series as: $\varepsilon(t) = \sum \tilde{\varepsilon}_n e^{jn\Omega t}$, the Floquet theorem leads us to the conclusion that the field solution is of the form

$$E(k, \omega, t) = \left(\sum E_n(\omega) e^{jn\Omega t}\right) e^{jkx}. \quad (19)$$

Plugging (19) into the wave equation

$$\frac{\partial^2 E}{\partial x^2} - \mu_0 \frac{\partial^2(\varepsilon(t)E)}{\partial t^2} = 0 \quad (20)$$

we get, after using the orthogonality relations of the Fourier coefficients [30]

$$\sum \left((\omega - m\Omega)^2 \tilde{\varepsilon}_{m-n} - \frac{k^2}{\mu_0}\, \delta_{mn}\right) E_n(\omega) = 0 \quad (21)$$

where δ_{mn} is the delta function and $m, n \in \mathbb{Z}$. Equation (21) forms a $\mathbf{A}\mathbf{x} = 0$ linear system. It is straightforward to show that $k(\omega) = k(\omega + n\Omega)$, and $d_k\omega|_{n\Omega/2} = \infty$, for $n \in \mathbb{Z}$ (also observed in [30]). By truncating the system to a finite number of harmonics and setting $\det(\mathbf{A}) = 0$, we can get nontrivial solutions and extract the dispersion relation. Numerically, this means that for a fixed frequency ω, the appropriate value of k which forces the determinant to singularity gives us the dispersion relation. In our special case, the Fourier coefficients of the dielectric permittivity used in this calculation are

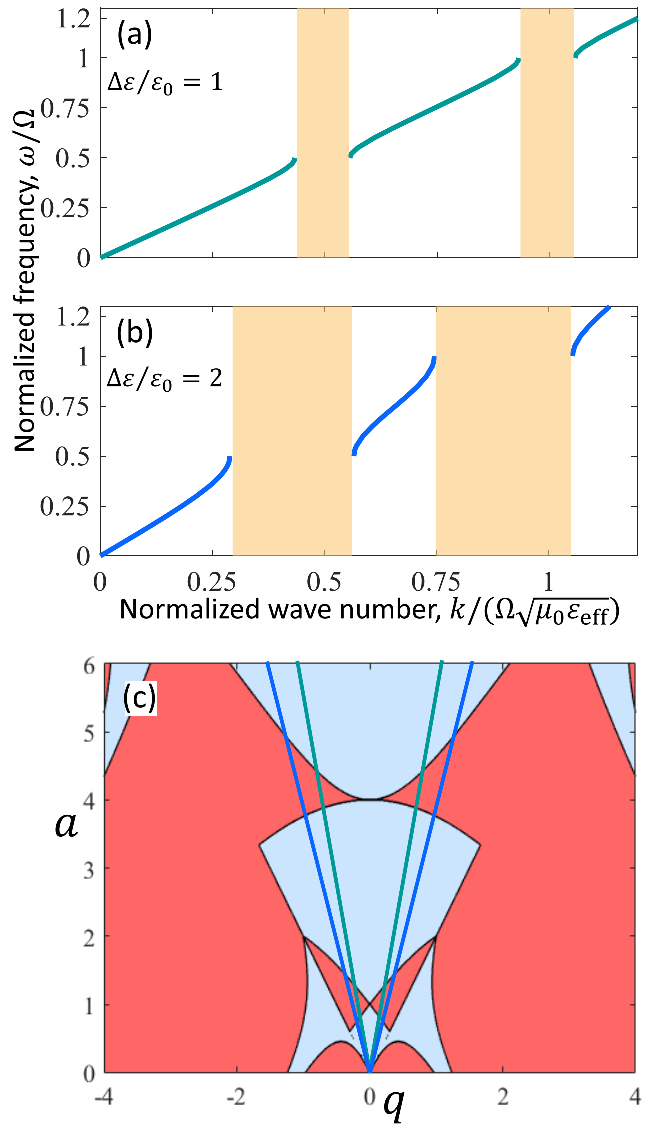

Fig. 4. (a) Dispersion relation of the time periodic medium with $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = \varepsilon_0$, and $\tau = T/2$. (b) Dispersion relation of the time periodic medium with $\varepsilon_1 = 3\varepsilon_0$, $\varepsilon_2 = \varepsilon_0$, and $\tau = T/2$. (c) Stability chart with the geometric loci (lines) associated with the operating parameters of (a) and (b) colored with green and blue, respectively.

$$\tilde{\varepsilon}_n = \begin{cases} \dfrac{j}{2\pi n}(e^{-j2n\pi\tau/T} - 1)(\varepsilon_1 - \varepsilon_2), & n \neq 0 \\ \varepsilon_2 + (\varepsilon_1 - \varepsilon_2)\tau/T, & n = 0. \end{cases} \quad (22)$$

Fig. 4 shows the obtained dispersion relation for some representative values of the time modulation of the medium. In Fig. 4(a), we present the dispersion relation for the case $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = \varepsilon_0$ and $\tau = T/2$. These parameters form an angle $\theta_1 \approx 80.54°$ at the stability chart, as shown in Fig. 4(c) (green line). In Fig. 4(b), we present the dispersion relation for the case $\varepsilon_1 = 3\varepsilon_0$, $\varepsilon_2 = \varepsilon_0$, and $\tau = T/2$. For this example, the angle at the stability chart has to be lower ($\theta_2 \approx 75.96°$) since the modulation depth is higher [blue line at Fig. 4(c)]. For the symmetric modulation of $\tau = T/2$, it is evident that wave propagation has the same dispersion when the values of ε_1 and ε_2 are swapped. This is directly linked with the fact that the stability chart is also symmetric for this specific condition ($\tau' = \Omega\tau/2 = \pi/2$). When this happens, the operating line at the stability chart has an angle $\theta' = 180° - \theta$. As expected from the exact analytical solution, the dispersion relation exhibits $k-$gaps. These $k-$gaps are found exactly at the parametric amplification condition (18), in full agreement with the exact time-domain analytical solution of Section III. Indeed, if we look closely at Fig. 4, we find that the nth $k-$gap encountered at the dispersion relation is related with the unstable region that is near the marginal instability at $a = n^2$. As expected, the gap is larger for an increased modulation depth since its line locus has a lower slope and intersects more with the unstable regions, as shown in Fig. 4(c). Going back to the transient analysis of Section II and the field oscillations of Fig. 2, the power of this analytical transient analysis for time periodic media is evident. The conditions for wave propagation of Fig. 2(b) and (c) are amid the momentum gaps of the dispersion relation. More precisely, they are at the parametric amplification conditions with $n = 4$ and $n = 5$ (4th and 5th $k-$gaps). The frequency-domain analysis cannot predict the parametric behavior of the electric displacement wave oscillations, whereas the analytical model based on the transition matrix can obtain the exact results. Depending on the initial conditions, the wave can be evanescent [Fig. 2(b)] or parametrically amplified [Fig. 2(c)].

V. Conclusion

In this paper, we have developed the proper mathematical tools to deal with wave propagation in time periodic media. We used the Hill differential equation to model this electromagnetic theoretical problem and we described the complex wave phenomena associated with the parametric gain in light of the Hill stability analysis. Moreover, we analyzed the dispersion relation of such structures with detail and connected it with the classical Hill analysis of stability for time-varying systems. We believe that this paper enriches the pre-existing literature on time-varying wave phenomena by providing in-depth exact mathematical and physical insight for wave propagation in general time periodic media. As we described, the design and synchronization process of an active time periodic medium device can be multifaceted because the same operating conditions could result in parametric amplification or in attenuated fields. This analytical mathematical framework could be of great assistance to the engineering community. It can be used directly and is a powerful design tool for active microwave and optical devices, which employ time periodic wave medium modulation to filter wave signals ($k-$gap devices), since it has the ability to detect the parametric oscillating properties, in contrast with the existing frequency-domain numerical solvers. Our mathematical approach can be also useful for other waves (acoustic, mechanical, elastic, and gravity) and other modulation schemes (as shown in the Appendix), and may become a precious tool for the design and understanding of time modulated wave systems.

Appendix: General Solutions of Electric Displacement Field for Time Periodic Dielectric Permittivity

Equations (3)–(5) characterize the electric displacement field in a wave medium with a time-dependent $\varepsilon = \varepsilon(t)$. In the case, ε is also periodic: $\varepsilon(t) = \varepsilon(t+T)$ the differential equation of the electric displacement field is

$$\frac{d^2 D(t)}{dt^2} + f(t)D(t) = 0 \tag{23}$$

where $f(t) = k^2/(\mu_0 \varepsilon(t))$ is also T periodic. We define the state vector as

$$\tilde{D} = \begin{bmatrix} D(t) \\ d_t D(t) \end{bmatrix}. \tag{24}$$

The general problem of (23) can be formulated as

$$d_t \tilde{D} = \mathbf{V}(t)\tilde{D} \tag{25}$$

where

$$\mathbf{V}(t) = \mathbf{V}(t+T) = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix}. \tag{26}$$

Since the differential equation is of the second order, the Wronskian is a 2×2 matrix, formed by the independent homogenous solutions $\tilde{D}_1$ and $\tilde{D}_2$

$$\mathbf{W}(t) = [\tilde{D}_1 \quad \tilde{D}_2]. \tag{27}$$

After basic algebraic manipulations, we get

$$\tilde{D} = \mathbf{W}(t)\mathbf{W}(0)^{-1}\tilde{D}(0). \tag{28}$$

Hence we define the transition matrix $\mathbf{\Phi}(t_2, t_1)$, which connects the electric displacement solution at time t_1 to the one at time t_2 as

$$\mathbf{\Phi}(t_2, t_1) = \mathbf{W}(t_2)\mathbf{W}(t_1)^{-1}. \tag{29}$$

Plugging the transition matrix to (25), we derive some interesting properties. For instance, it is easily found that the transition matrix is a homogenous solution of the differential equation and $\mathbf{\Phi}(mT, 0) = \mathbf{\Phi}(T, 0)^m$ for m integer (which is a property directly linked with the Floquet theory). After this analysis, the problem is simplified in finding this transition matrix. The transition matrix is

$$\mathbf{\Phi}(t_N, 0) = \mathbf{\Phi}(t_N, t_{N-1}) \cdots \mathbf{\Phi}(t_1, 0) \tag{30}$$

for: $t_N > t_{N-1} > \cdots t_1 > 0$.

In order to check the stability of the solution the eigenvalues λ_1 and λ_2 of $\mathbf{\Phi}(T, 0)$ have to take an absolute value less than unity. It is straightforward to show that

$$\lambda_{1,2} = \mathrm{tr}(\mathbf{\Phi}(T,0))/2 \pm \sqrt{[\mathrm{tr}(\mathbf{\Phi}(T,0))/2]^2 - 1}. \tag{31}$$

By requiring the stability condition $|\lambda| < 1$, we get the formula used directly in (16). For more details, we refer the reader to [41]–[46].

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