iterations and the computational time for FGMRES, respectively. We see that our preconditioners are effective also in this case, especially in the neighborhood of Wood's anomaly.

V. Conclusion

We summarize the results obtained in this communication as follows:

We extended the periodic FMM, which has been applied only to BIEM so far, to VIEM for doubly-periodic transmissions problems for Maxwell's equations. We then verified the proposed method in non-periodic problems by comparing the numerical solutions with the analytical results. The proposed method was then applied to periodic problems in which we found that the numerical solutions agreed with the BIEM solutions.

We also proposed two preconditioners. One is the Gram preconditioner in which we use the Gram matrix part of the discretized linear equation as the right preconditioner. The other is the $AP^{-1}BQ^{-1}$-preconditioner, which reduces the original integral operator essentially to a compact perturbation of an identity when the domain Ω is of infinite extent. This approach makes good use of matrices which we already have in the computation of the coefficient matrix for the original integral equation. With numerical examples, we verified that these preconditioners can reduce the number of iterations and the computational time for iterative solvers. We also found that our preconditioners work even in problems where no-preconditioned methods did not lead to convergence within reasonable numbers of iterations. In terms of the computational time, the Gram-preconditioned method is more efficient than the $AP^{-1}BQ^{-1}$-preconditioned method. These preconditioned methods remain effective even near Wood's anomalies where the non-preconditioned approach becomes inefficient.

References

[1] L. Greengard and V. Rokhlin, "A fast algorithm for particle simulations," *J. Comput. Phys.*, vol. 73, no. 2, pp. 325–348, Dec. 1987.

[2] Y. Otani and N. Nishimura, "An FMM for orthotropic periodic boundary value problems for Maxwell's equations," *Wave. Random. Complex*, vol. 19, no. 1, pp. 80–104, Feb. 2009.

[3] M. I. Sancer, K. Sertel, J. L. Volakis, and P. Van Alstine, "On volume integral equations," *IEEE Trans. Antennas Propag.*, vol. 54, no. 5, pp. 1488–1495, May 2006.

[4] M. M. Botha, "Solving the volume integral equations of electromagnetic scattering," *J. Comput. Phys.*, vol. 218, no. 1, pp. 141–158, Oct. 2006.

[5] H. Gan and W. C. Chew, "A discrete BCG-FFT algorithm for solving 3D inhomogeneous scatterer problems," *J. Electromagn. Wave*, vol. 9, no. 10, pp. 1339–1357, 1995.

[6] W. B. Ewe, L. W. Li, and M. S. Leong, "Fast solution of mixed dielectric/conducting scattering problem using volume-surface adaptive integral method," *IEEE Trans. Antennas Propag.*, vol. 52, no. 11, pp. 3071–3077, Nov. 2004.

[7] L. E. Sun and W. C. Chew, "A novel formulation of the volume integral equation for electromagnetic scattering," *Wave. Random. Complex*, vol. 19, no. 1, pp. 162–180, Feb. 2009.

[8] C. C. Lu, "Indoor radio-wave propagation modeling by multilevel fast multipole algorithm," *Microw. Opt. Techn. Let.*, vol. 29, no. 3, pp. 168–175, May 2001.

[9] C. C. Lu, "A fast algorithm based on volume integral equation for analysis of arbitrarily shaped dielectric radomes," *IEEE Trans. Antennas Propag.*, vol. 51, no. 3, pp. 606–612, Mar. 2003.

[10] Y. Shi and C. H. Chan, "Multilevel Green's function interpolation method for analysis of 3-D frequency selective structures using volume/surface integral equation," *J. Opt. Soc. Am. A*, vol. 27, no. 2, pp. 308–318, Feb. 2010.

[11] A. D. Baczewski, N. C. Miller, and B. Shanker, "Rapid analysis of scattering from periodic dielectric structures using accelerated Cartesian expansions," *J. Opt. Soc. Am. A*, vol. 29, no. 4, pp. 531–540, Apr. 2012.

[12] *Electromagnetic Theory of Gratings*, R. Petit, Ed. Berlin, Germany: Springer-Verlag, 1980.

[13] D. H. Schaubert, D. R. Wilton, and A. W. Glisson, "A tetrahedral modeling method for electromagnetic scattering by arbitrarily shaped Inhomogeneous dielectric Bodies," *IEEE Trans. Antennas Propag.*, vol. AP-32, no. 1, pp. 78–85, Jan. 1984.

[14] M. Costabel, E. Darrigrand, and H. Sakly, "The essential spectrum of the volume integral operator in electromagnetic scattering by a homogeneous body," *C. R. Acad. Sci. Paris, Ser. I*, vol. 350, pp. 193–197, Feb. 2012.

[15] K. Niino and N. Nishimura, "Preconditioning based on Calderon's formulae for periodic fast multipole methods for Helmholtz' equation," *J. Comput. Phys.*, vol. 231, no. 1, pp. 66–81, Jan. 2012.

[16] G. Chen and J. Zhou, *Boundary Element Methods*. London, U.K.: Academic Press, 1992.

[17] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd ed. Philadelphia, PA, USA: SIAM, 2003.

A New Unconditionally Stable Scheme for FDTD Method Using Associated Hermite Orthogonal Functions

Zheng-Yu Huang, Li-Hua Shi, Bin Chen, and Ying-Hui Zhou

***Abstract*—An unconditionally stable solution using associated Hermite (AH) functions is proposed for the finite-difference time-domain (FDTD) method. The electromagnetic fields and their time derivatives in time-domain Maxwell's equations are expanded by these orthonormal basis functions. By applying Galerkin temporal testing procedure to these expanded equations the time variable can be eliminated from the calculations. A set of implicit equations is derived to calculate the magnetic filed expansion coefficients of all orders of AH functions for the temporal variable. And the electrical field coefficients can be obtained respectively. With the appropriate translation and scale parameters, we can find a minimum-order basis functions subspace to approach a particular electromagnetic field. The numerical results have shown that the proposed method can reduce the CPU time to 0.59% of the traditional FDTD method while maintaining good accuracy.**

***Index Terms*—Associated Hermite (AH) basis functions, electromagnetic field, finite difference time domain (FDTD), unconditionally stable.**

I. Introduction

The finite-difference time-domain (FDTD) method is a conditionally stable numerical technique to analyze transient electromagnetic problems [1]. Its conditional stability means that the time step size should be limited by the well-known Courant-Friedrich-Lecy (CFL) stability condition. To mode fine structures, such as thin material and slot, many efficient technologies like sub-gridding [2], spatial filtering [3] and some less time consuming unconditionally stable methods, such as alternating-direction implicit (ADI) method [4]–[6], Crank-Nicolson scheme [7], [8], and weighted Laguerre polynomials (WLP) FDTD method [9]–[12], have been proposed and applied in various electromagnetic computational problems.

In this communication, AH functions are applied as temporal basis and testing functions to obtain an unconditionally stable FDTD

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The authors are with the National Key Laboratory on Electromagnetic Environmental Effects and Electro-optical Engineering, PLA University of Science and Technology, Nanjing 210007, China (e-mail: huangzynj@163.com; shilhnj@163.com).

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solution. AH basis functions are derived from orthogonal Hermite polynomials with a translated and scaled Gaussian function. AH function has similar representation form in frequency-domain and time-domain. This property allows it to perform simultaneous extrapolation in time and frequency domains. It is widely used as the expansion of time domain waveforms in many fields such as biomedical engineering [13], signal analysis and processing [14]. Compared to other orthogonal basis, AH functions have the most compact time-frequency support (TFSs) [15] and therefore it is suitable to expand transient signals with less order of polynomials [16]. One of the most important applications of AH expansion in computational electromagnetic is to use it for extrapolation of the full responses from early time and low-frequency data [16]–[20]. However, it has not been combined with FDTD to form an unconditionally stable method, like WLP-FDTD.

The main purpose of our communication is to apply AH expansion as temporal basis to FDTD calculation. First, for the case of 2-D time-domain Maxwell's equations, electromagnetic fields and their time derivatives are expanded by these basis functions. Then, using the Galerkin [9] method for a temporal testing scheme, an explicit recurrence relation, from $q-1$th and qth order expansion coefficients of AH functions to $q+1$th order coefficients, is deduced. But as the zeroth and first order coefficients can not be calculated independently, we can not calculate these coefficients orders by orders directly like other basis expansion method [9]. To deal with this problem, we take all orders of coefficients at one point in computational domain as an unknown vector variable. Then, we establish a set of implicit equations for vector variables in whole computational domains. To ensure the solvability of the equation set, we introduce an initial condition. Finally, we solve the equation set to obtain the coefficients of AH functions for electric and magnetic fields.

Since the equations set has a banded coefficients matrix in which all elements are nested sub-matrixes, we use a banded matrix decomposition principle [21] to solve the equation set indirectly. Our method only requires the coefficients matrix to be assembled and decomposed once. The minimum order or number of basis functions is dependent on the time duration and the bandwidth of the analysed fields. Therefore the order-selection of the AH functions has been discussed. Moreover, unlike the alternately solving method in [9], our method allows the whole coefficients of magnetic fields be calculated separately and then the electric ones.

II. Formulations and AH-FDTD Method

A. AH Functions

Associated Hermite basis functions are an orthonormal set of basis functions $\{\phi_n(t) = (2^n n! \pi^{1/2})^{-1/2} e^{-t^2/2} H_n(t)\}$, $(n = 0, 1 \ldots)$ [13], where $H_n(t) = (-1)^n e^{t^2} (d^n/dt^n)(e^{-t^2})$ is Hermite polynomials [22]. Unlike the Laguerre basis functions mentioned in [9] and [23], the AH basis functions are not causal [13]. However, they can be transformed to the causal form by virtue of a proper time-translating parameter and then used to span the causal electromagnetic responses [16], [17]. The transformed basis function set $\{\bar{\phi}_n(\tilde{t}) = (2^n n! \sigma \pi^{1/2})^{-1/2} e^{-\tilde{t}^2/2} H_n(\tilde{t})\}$ are also orthogonal with respect to the transformed time variable $\tilde{t} = (t - T_f)/\sigma$. Where T_f is a time-translating parameter. σ is a time-scaling parameter. By controlling these two parameters, the time-frequency support of the AH functions $\{\bar{\phi}_n(\tilde{t})\}$ space can be changed flexibility. So arbitrary locally time-supported functions can be spanned by these transformed basis functions, including the causal electromagnetic responses.

The time support T_q and frequency support W_q of the AH functions $\bar{\phi}_q(\tilde{t})$ can be approximated by the following empirical formula [24]

$$T_q \approx 2\sigma(\sqrt{\pi q/1.7} + 1.8) \tag{1}$$
$$W_q \approx (\sqrt{\pi q/1.7} + 1.8)/(2\pi\sigma). \tag{2}$$

A causal function $u(x,t)$, such as the electric or magnetic field function, can be expanded by

$$u(r,t) = \sum_{n=0}^{\infty} u_n(r)\bar{\phi}_n(\tilde{t}) \tag{3}$$

By using the time derivation of the nth order AH function [13]

$$\frac{d}{dt}\bar{\phi}_n(\tilde{t}) = \begin{cases} -\frac{1}{\sigma}\sqrt{\frac{1}{2}}\bar{\phi}_1(\tilde{t}) & (n=0) \\ \frac{1}{\sigma}\sqrt{\frac{n}{2}}\bar{\phi}_{n-1}(\tilde{t}) - \frac{1}{\sigma}\sqrt{\frac{n+1}{2}}\bar{\phi}_{n+1}(\tilde{t}) & (n \geq 1) \end{cases} \tag{4}$$

we can deduce the first derivative of $u(x,t)$ with respect to t

$$\frac{\partial}{\partial t}u(r,t) = \frac{1}{\sigma}\sum_{n=0}^{\infty}\left(u_{n+1}(r)\sqrt{\frac{n+1}{2}} - u_{n-1}(r)\sqrt{\frac{n}{2}}\right)\bar{\phi}_n(\tilde{t}) \tag{5}$$

where $u_{-1}(r) = 0$. The derivation of (5) can be found in Appendix 1.

B. FDTD With AH Functions Subspace

With simple and lossless media, the time-domain Maxwell's equations for 2-D TE_z model case is

$$\frac{\partial}{\partial t}E_x(r,t) = \frac{1}{\varepsilon(r)}\frac{\partial}{\partial y}H_z(r,t) - \frac{J_x(r,t)}{\varepsilon(r)} \tag{6}$$
$$\frac{\partial}{\partial t}E_y(r,t) = -\frac{1}{\varepsilon(r)}\frac{\partial}{\partial x}H_z(r,t) - \frac{J_y(r,t)}{\varepsilon(r)} \tag{7}$$
$$\frac{\partial}{\partial t}H_z(r,t) = \frac{1}{\mu(r)}\frac{\partial}{\partial y}E_x(r,t) - \frac{1}{\mu(r)}\frac{\partial}{\partial x}E_y(r,t) \tag{8}$$

where ε is the electric permittivity and μ is the magnetic permeability. Applying (3) and (5) to (6)–(8), the field functions in these equations can be expanded by

$$\frac{1}{\sigma}\sum_{n=0}^{\infty}\left(E_x^{n+1}(r)\sqrt{\frac{n+1}{2}} - E_x^{n-1}(r)\sqrt{\frac{n}{2}}\right)\bar{\phi}_n(\tilde{t}) = \frac{1}{\varepsilon(r)}\sum_{n=0}^{\infty}\frac{\partial}{\partial y}H_z^n(r)\bar{\phi}_n(\tilde{t}) - \frac{J_x(r,t)}{\varepsilon(r)} \tag{9}$$

$$\frac{1}{\sigma}\sum_{n=0}^{\infty}\left(E_y^{n+1}(r)\sqrt{\frac{n+1}{2}} - E_y^{n-1}(r)\sqrt{\frac{n}{2}}\right)\bar{\phi}_n(\tilde{t}) = -\frac{1}{\varepsilon(r)}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}H_z^n(r)\bar{\phi}_n(\tilde{t}) - \frac{J_y(r,t)}{\varepsilon(r)} \tag{10}$$

$$\frac{1}{\sigma}\sum_{n=0}^{\infty}\left(H_z^{n+1}(r)\sqrt{\frac{n+1}{2}} - H_z^{n-1}(r)\sqrt{\frac{n}{2}}\right)\bar{\phi}_n(\tilde{t}) = \frac{1}{\mu(r)}\sum_{n=0}^{\infty}\frac{\partial}{\partial y}E_x^n(r)\bar{\phi}_n(\tilde{t}) - \frac{1}{\mu(r)}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}E_y^n(r)\bar{\phi}_n(\tilde{t}). \tag{11}$$

By using the orthogonal property of the AH functions, we use the temporal Galerkin testing procedure [9] to eliminate the time-dependent terms $\bar{\phi}_n(\tilde{t})$. Then, we get

$$\frac{1}{\sigma}\left(E_x^{q+1}(r)\sqrt{\frac{q+1}{2}} - E_x^{q-1}(r)\sqrt{\frac{q}{2}}\right) = \frac{1}{\varepsilon(r)}\frac{\partial}{\partial y}H_z^q(r) - \frac{J_x^q(r)}{\varepsilon(r)} \tag{12}$$

$$\frac{1}{\sigma}\left(E_y^{q+1}(r)\sqrt{\frac{q+1}{2}} - E_y^{q-1}(r)\sqrt{\frac{q}{2}}\right) = -\frac{1}{\varepsilon(r)}\frac{\partial}{\partial x}H_z^q(r) - \frac{J_y^q(r)}{\varepsilon(r)} \tag{13}$$

$$\frac{1}{\sigma}\left(H_z^{q+1}(r)\sqrt{\frac{q+1}{2}} - H_z^{q-1}(r)\sqrt{\frac{q}{2}}\right) = \frac{1}{\mu(r)}\frac{\partial}{\partial y}E_x^q(r) - \frac{1}{\mu(r)}\frac{\partial}{\partial x}E_y^q(r) \tag{14}$$

where

$$J_x^q(r) = \int_{-T_s/2}^{T_s/2} J_x(r,t)\bar{\phi}_q(\tilde{t})d\tilde{t}, \quad q = 0, 1\ldots \tag{15}$$

$$J_y^q(r) = \int_{-T_s/2}^{T_s/2} J_y(r,t)\bar{\phi}_q(\tilde{t})d\tilde{t}, \quad q = 0, 1\ldots \tag{16}$$

For a compactly supported source, we can use a finite time interval T_s to replace limit of infinity, and then perform the integrations in (15) and (16) numerically. This interval should cover the range of interest of waveforms. Then we can rewrite (12)–(14) in a matrix form

$$E_x|_{i,j}^{q+1} - \sqrt{\frac{q}{q+1}}E_x|_{i,j}^{q-1} = \sqrt{\frac{2}{q+1}}\frac{\sigma}{\varepsilon_{i,j}\Delta\bar{y}_j}\left(H_z|_{i,j}^q - H_z|_{i,j-1}^q\right) - \sqrt{\frac{2}{q+1}}\frac{\sigma}{\varepsilon_{i,j}}J_x|_{i,j}^q \tag{17}$$

$$E_y|_{i,j}^{q+1} - \sqrt{\frac{q}{q+1}}E_y|_{i,j}^{q-1} = -\sqrt{\frac{2}{q+1}}\frac{\sigma}{\varepsilon_{i,j}\Delta\bar{x}_i}\left(H_z|_{i,j}^q - H_z|_{i-1,j}^q\right) - \sqrt{\frac{2}{q+1}}\frac{\sigma}{\varepsilon_{i,j}}J_y|_{i,j}^q \tag{18}$$

$$H_z|_{i,j}^{q+1} - \sqrt{\frac{q}{q+1}}H_z\Big|_{i,j}^{q-1} = \sqrt{\frac{2}{q+1}}\frac{\sigma}{\mu_{i,j}\Delta y_j}\left(E_x|_{i,j+1}^q - E_x|_{i,j}^q\right) - \sqrt{\frac{2}{q+1}}\frac{\sigma}{\mu_{i,j}\Delta x_i}\left(E_y|_{i+1,j}^q - E_y|_{i,j}^q\right) \tag{19}$$

For a signal with compact support, its time-frequency domain can be covered by an AH basis function subspace spanned by $\{\bar{\phi}_0, \bar{\phi}_1, \bar{\phi}_2 \cdots \bar{\phi}_{Q-1}\}$ [13]. (17)–(19) reflect an explicit recurrence relation from $q-1$th and qth order expansion coefficients of AH functions to $q+1$th order coefficients. If the zeroth and first order coefficients can be calculated, the reminders can also be obtained. However, it is difficult to realize in this way, for its zeroth order coefficients could not be obtained separately and firstly like the way proposed in [9]. Instead, we propose another approach to obtain a solution.

Firstly, initial conditions of electromagnetic fields are introduced and expanded by $\{\bar{\phi}_q(\tilde{t_0})\}(q = 0, 1\ldots Q-1)$. For instance, the initial condition of magnetic field can be represented as $H_z(r,t_0) = \sum_{q=0}^{Q-1} H_z^q(r)\bar{\phi}_q(\tilde{t_0})$.

Then all orders of coefficients at one point in the computational domain are taken as an unknown vector variable and a set of implicit equations with vector variables including initial conditions can be assembled. (17)–(19) can be rewritten in a form of implicit equations set by

$$[\alpha][E_x]_{i,j} = \bar{C}_y^E\Big|_{i,j}[\beta]([H_z]_{i,j} - [H_z]_{i,j-1}) - \frac{\sigma}{\varepsilon_{i,j}}[\beta][J_x]_{i,j} + \left[E_x^{t_0}\right]_{i,j} \tag{20}$$

$$[\alpha][E_y]_{i,j} = -\bar{C}_x^E\Big|_{i,j}[\beta]([H_z]_{i,j} - [H_z]_{i-1,j}) - \frac{\sigma}{\varepsilon_{i,j}}[\beta][J_y]_{i,j} + \left[E_y^{t_0}\right]_{i,j} \tag{21}$$

$$[\alpha][H_z]_{i,j} = \bar{C}_y^H\Big|_{i,j}[\beta]([E_x]_{i,j+1} - [E_x]_{i,j}) - \bar{C}_x^H\Big|_{i,j}[\beta]([E_y]_{i+1,j} - [E_y]_{i,j}) + \left[H_z^{t_0}\right]_{i,j} \tag{22}$$

where

$$[\alpha] = \begin{bmatrix} & 1 & & & \\ -\sqrt{\frac{1}{2}} & & \ddots & & \\ & \ddots & & 1 & \\ & & -\sqrt{\frac{Q-2}{Q-1}} & & 1 \\ \bar{\phi}_0(\tilde{t_0}) & \bar{\phi}_1(\tilde{t_0}) & & \cdots & \bar{\phi}_{Q-1}(\tilde{t_0}) \end{bmatrix}$$

$$[\beta] = \begin{bmatrix} \sqrt{\frac{2}{1}} & & & \\ & \sqrt{\frac{2}{2}} & & \\ & & \ddots & \\ & & & \sqrt{\frac{2}{Q-1}} \\ & & & & 0 \end{bmatrix}$$

$$\bar{C}_x^E\Big|_{i,j} = \frac{\sigma}{\varepsilon_{i,j}\Delta\bar{x}_i}, \quad \bar{C}_y^E\Big|_{i,j} = \frac{\sigma}{\varepsilon_{i,j}\Delta\bar{y}_j}$$

$$\bar{C}_x^H\Big|_{i,j} = \frac{\sigma}{\mu_{i,j}\Delta x_i}, \quad \bar{C}_y^H\Big|_{i,j} = \frac{\sigma}{\mu_{i,j}\Delta y_j}$$

$[E_x]_{i,j}, [E_y]_{i,j}$ and $[H_z]_{i,j}$ are Q-tupple representations in AH subspace for the electric and the magnetic fields. For an example, $[H_z]_{i,j} = [H_z|_{i,j}^0 \cdots H_z|_{i,j}^{Q-2} H_z|_{i,j}^{Q-1}]_{1\times Q}^T$. $[E_x^{t_0}]_{i,j}$, $[E_y^{t_0}]_{i,j}$ and $[H_z^{t_0}]_{i,j}$ can be represented as an uniform formation $[P^{t_0}]_{i,j} = [0\cdots 0\ P(i,j,t_0)]_{1\times Q}^T$, $P = E_x, E_y$ or H_z.

To eliminate the electric field, we apply (20)–(21) to (22)

$$b_l[H_z]_{i-1,j} + b_r[H_z]_{i+1,j} + a[H_z]_{i,j} + b_u[H_z]_{i,j+1} + b_d[H_z]_{i,j-1} = [J]_{i,j} \tag{23}$$

where

$$b_l = \bar{C}_x^E\Big|_{i,j}\bar{C}_x^H\Big|_{i,j}[\beta][\alpha]^{-1}[\beta]$$

$$b_r = \bar{C}_x^E\Big|_{i+1,j}\bar{C}_x^H\Big|_{i,j}[\beta][\alpha]^{-1}[\beta]$$

$$b_u = \bar{C}_y^E\Big|_{i,j+1}\bar{C}_y^H\Big|_{i,j}[\beta][\alpha]^{-1}[\beta]$$

$$b_d = \bar{C}_y^E\Big|_{i,j}\bar{C}_y^H\Big|_{i,j}[\beta][\alpha]^{-1}[\beta]$$

$$a = -(b_u + b_d + b_l + b_r + [\alpha])$$

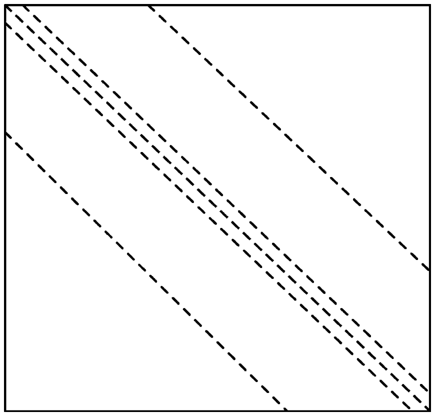

Fig. 1. The five-diagonal banded coefficient matrix $[A]$.

$$\begin{aligned}[J]_{ij} =\ & \bar{C}_y^H\Big|_{i,j}[\beta][\alpha]^{-1}[\beta]\left(\frac{\sigma}{\varepsilon_{i,j+1}}[J_x]_{i,j+1} - \frac{\sigma}{\varepsilon_{i,j}}[J_x]_{i,j}\right)\\ &- \bar{C}_x^H\Big|_{i,j}[\beta][\alpha]^{-1}[\beta]\left(\frac{\sigma}{\varepsilon_{i+1,j}}[J_y]_{i+1,j} - \frac{\sigma}{\varepsilon_{i,j}}[J_y]_{i,j}\right)\\ &- \bar{C}_y^H\Big|_{i,j}[\beta][\alpha]^{-1}\left(\left[E_x^{t_0}\right]_{i,j+1} - \left[E_x^{t_0}\right]_{i,j}\right)\\ &+ \bar{C}_x^H\Big|_{i,j}[\beta][\alpha]^{-1}\left(\left[E_y^{t_0}\right]_{i+1,j} - \left[E_y^{t_0}\right]_{i,j}\right) - \left[H_z^{t_0}\right]_{i,j}.\end{aligned}$$

From (23), we can find that each magnetic field vector variable has a relationship with the adjacent four magnetic field vector variables. If the magnetic field vector variables $[H_z]_{i,j}$ in the whole computational domain are calculated, the electronic field vector variables $[E_x]_{i,j}$ and $[E_y]_{i,j}$ can be obtained from (20)–(21).

Rewriting (23) as a nested matrix equation, we have

$$[A]\{[H_z]\} = \{[J]\} \tag{24}$$

where $\{[H_z]\}$ is an unknown vector assembled with vector variable elements that is consist of magnetic field coefficients of all orders for all points. $\{[J]\}$ is a vector related with excitation source coefficients for all points. $[A]$ is a banded sparse coefficient matrix, and each row of it has five nonzero Q order sub-matrixes, which have the same invariant part of $[\beta][\alpha]^{-1}[\beta]$. The shape of the banded matrix $[A]$ is shown in Fig. 1. In the case of the boundary edges, $[A]$ should be modified according to certain conditions. It should be note that each nonzero element is a Q order matrix.

For the PEC boundary condition, the rows and columns matrix elements of $[A]$ should be replaced with Q order null sub-matrix except diagonal sub-matrix elements. Also the respective vector variables of the exciting vector $\{[J]\}$ should be Q order null vectors.

For the first-order dispersive boundary condition (DBC), we choose the absorbing boundary (ABC) from [25]. Then, at $x = 0$ or $x = X$, we have

$$\left(\frac{\partial}{\partial x} \pm \frac{1}{\nu}\frac{\partial}{\partial t}\right)E_y(r,t) = 0. \tag{25}$$

Applying (3) and (5) to (25) and eliminate the temporal terms to obtain (at $x = X$)

$$\frac{\partial}{\partial x}E_y^q(r) + \frac{1}{\nu\sigma}\left(\sqrt{\frac{q+1}{2}}E_y^{q+1}(r) - \sqrt{\frac{q}{2}}E_y^{q-1}(r) = \right)0. \tag{26}$$

Using the averaging technique and the central difference scheme [9]

$$\begin{aligned}E_y|_{I+1/2,j}^q &= \frac{E_y|_{I+1,j}^q + E_y|_{I+1,j}^q}{2}\\ \frac{\partial}{\partial x}E_y|_{I+1/2}^q &= \frac{E_y|_{I+1,j}^q - E_y|_{I,j}^q}{\Delta x_I}\end{aligned} \tag{27}$$

rewriting (26) as a discrete form at point, we have

$$\begin{aligned}&-\sqrt{\frac{q}{q+1}}E_y\Big|_{I+1}^{q-1} + \frac{2\nu\sigma}{\Delta x_I}\sqrt{\frac{2}{q+1}}E_y\Big|_{I+1}^{q} + E_y|_{I+1}^{q+1}\\ &\quad = \sqrt{\frac{q}{q+1}}E_y\Big|_{I}^{q-1} + \frac{2\nu\sigma}{\Delta x_I}\sqrt{\frac{2}{q+1}}E_y\Big|_{I}^{q} - E_y|_{I}^{q+1}.\end{aligned} \tag{28}$$

Assembling all of Q orders from (28) as a matrix equation

$$\left(\frac{2\nu\sigma}{\Delta x_I}[\beta] + [\alpha]\right)[E_y]_{I+1,j} = \left(\frac{2\nu\sigma}{\Delta x_I}[\beta] - [\alpha]\right)[E_y]_{I,j} \tag{29}$$

we have a similar ABC matrix equation at $x = 0$

$$\left(\frac{2\nu\sigma}{\Delta x_1}[\beta] + [\alpha]\right)[E_y]_{1,j} = \left(\frac{2\nu\sigma}{\Delta x_1}[\beta] - [\alpha]\right)[E_y]_{2,j}. \tag{30}$$

Introducing (29), (30) and the PEC boundary condition to (20)–(22), we can adapt (24) into a completeness matrix equation as following

$$[\tilde{A}]\{[H_z]\} = \{\widetilde{[J]}\} \tag{31}$$

where $[\tilde{A}]$ is an adapted banded sparse coefficient matrix. Its rows related to boundary condition have no more than five nonzero Q order sub-matrix elements. We use a banded matrix decomposition principle [18] to indirectly obtain the results of (31). We can first perform the lower-upper (LU) decomposition of $[\tilde{A}]$, and then calculate the coefficients of the magnetic field by using the back-substitution method. The decomposition and the back-substitution routine are performed only once. Then, the electric fields' coefficients can be obtained from (20)–(21).

Finally, we can obtain $E_x(r,t)$, $E_y(r,t)$ and $H_z(r,t)$ from all of the expansion coefficients of the electric and magnetic fields. For instance, $H_z(r,t)$ can be reconstructed as

$$H_z(r,t) = \sum_{q=0}^{Q-1} H_z^q(r)\bar{\phi}_q(\tilde{t}). \tag{32}$$

III. Numerical Example

To validate our numerical method, an experiment with 2-D parallel plate waveguide is performed. The rectangular waveguide with a thin PEC slot and partially filled with dielectric material is shown in Fig. 2. The width of this waveguide is 0.08 m, and the length is 1.2 m. There are 140 × 8 non-uniform cells in the computational domain. In the PEC slot area, the PEC plate is divided into small cells, and the minimum cell size is 0.6 μm × 0.0045 m. The thickness of dielectric material is 0.04 m, and the relative permittivity with no loss is 2. A sinusoidal-modulated Gaussian pulse is chosen as the y direction excitation source

$$J_y(t) = e^{-\frac{(t-t_c)^2}{t_d^2}}\sin(2\pi f_c(t-t_c)) \tag{33}$$

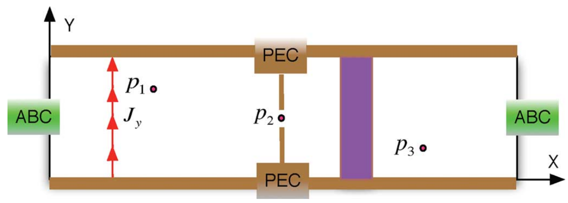

Fig. 2. Computational domain of 2-D parallel plate waveguide with the thin PEC slot of the thickness 1.2 μm and the distance 0.9 cm, and the partly filled dielectric material of the thickness 0.04 m.

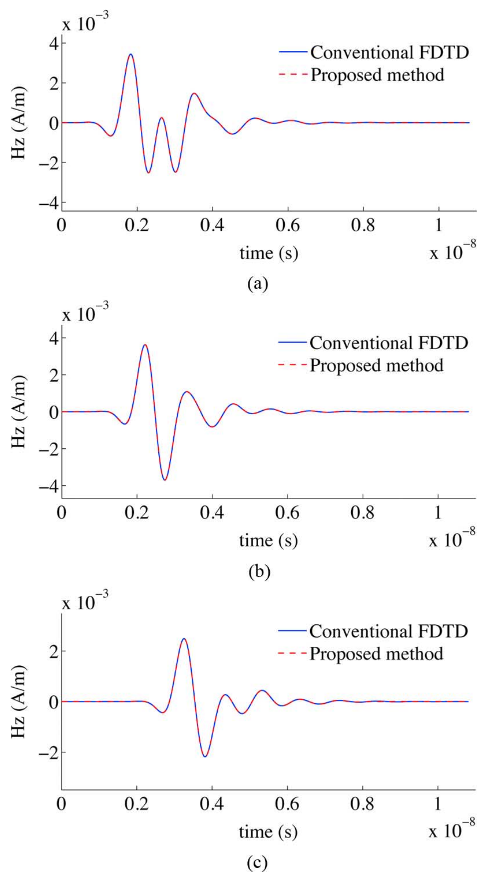

Fig. 3. The comparison of transient magnetic fields at (a) p_1, (b) p_2 and (c) p_3.

where $t_d = 1/(2f_c)$, $t_c = 4t_d$, and $f_c = 0.6$ GHz. The time duration of interest for the analyzed fields is chosen as $T_s = 10.81$ ns and the bandwidth is limited up to the frequency $W_s = 3$ GHz. Then, according to (1)–(2), we expand the signal by a set of AH functions $\{\bar{\phi}_q(\tilde{t})\}$ $(q = 0, 1 \ldots Q-1)$ under the condition of $T_q > T_s$, $W_q > W_s$. Then we can find the minimum required order of $Q = 32$ and the rang of $\sigma = 5.12 \times 10^{-10}$ from the condition above.

The time-step size is set as 1.98 fs for the conventional FDTD method because of the CFL condition requirement, while for our method, the time-step size is set as 8 ps. This value is small enough to evaluate (15)–(16) to numerically calculate the AH functions' coefficients of the excitation pulse.

The results of the magnetic field at p_1, p_2 and p_3 in Fig. 2 are shown in Fig. 3. The agreement between the proposed method and conventional FDTD method is quite good. From Fig. 4, we can deduce more details from the relative errors compared with the conventional FDTD at these points. The values of relative error for three curves are very small, almost below -40 dB. The relative error here is defined as $R = 20\log_{10}[|H_z^a(t) - H_z^c(t)|/\max[H_z^c(t)]]$, where $H_z^a(t)$ and $H_z^c(t)$ are the magnetic fields tested in the proposed method and the conventional FDTD method respectively. Table I provides the information of time-step size and the computing resources for the numerical simulations.

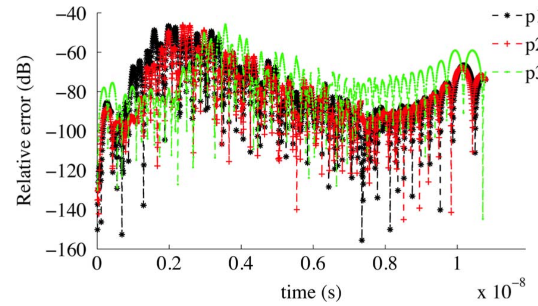

Fig. 4. Relative error of the proposed method to conventional FDTD.

TABLE I
COMPARISON OF THE CPU RESOURCES FOR THE PARALLEL PLATE WAVEGUIDE

	Δt	Memory	CPU times
Conventional FDTD	1.98 fs	0.98 Mb	412.31 s
Proposed method	8.0 ps	77.8 Mb	2.43 s

The proposed method needs to assemble a banded nested matrix only once and then perform a decomposition and back-substitution routine. Although its basic computing elements are Q order matrixes, with a relatively longer time and larger storage space, the time step can be set as 4040 times that of the conventional FDTD, which is not limited by the CLF stability condition and unconditional stable. The total memory storage for the proposed method is increased to 77.8 Mb, almost 80 times of one iteration of conventional FDTD, while the total CPU time for the proposed method can be reduced to about 0.59% of the conventional FDTD method, with the accuracy still being guaranteed.

IV. CONCLUSION

This communication proposes an unconditionally stable solution for the FDTD algorithm based on AH basis functions. The method is free from CFL stability condition for it has eliminated the time variable in calculation. Instead, we have derived a set of implicit equations for orders-vector variables to indirectly solve the Maxwell's equations. Compared with conventional FDTD method, the proposed method requires fewer CPU time while rendering in an efficient solution for it has eliminated the time iteration. In the numerical simulation of the 2-D TEz case with fine structures, the new method is very efficient, and the results agree well with that of the conventional FDTD method. Further study will focus on the more general three-dimensional cases, and the way to reduce the memory consumption should also be investigated.

APPENDIX

The derivation of (5) from (3) and (4) is as following

$$\begin{aligned}\frac{\partial}{\partial t}u(r,t) &= \sum_{n=0}^{\infty} u_n(r)\frac{d}{dt}\bar{\phi}_n(\tilde{t}) \\ &= u_0(r)\frac{d}{dt}\bar{\phi}_0(\tilde{t}) + \sum_{n=1}^{\infty} u_n(r)\frac{d}{dt}\bar{\phi}_n(\tilde{t})\end{aligned}$$

Using the time derivation in (4), we have

$$\begin{aligned}
&u_0(r)\frac{d}{dt}\bar{\phi}_0(\tilde{t})+\sum_{n=1}^{\infty}u_n(r)\frac{d}{dt}\bar{\phi}_n(\tilde{t})\\
&\quad= u_0(r)\left(-\sqrt{\frac{1}{2\sigma^2}}\bar{\phi}_1(\tilde{t})\right)\\
&\quad+\sum_{n=1}^{\infty}u_n(r)\left(\sqrt{\frac{n}{2\sigma^2}}\bar{\phi}_{n-1}(\tilde{t})-\sqrt{\frac{n+1}{2\sigma^2}}\bar{\phi}_{n+1}(\tilde{t})\right)\\
&\quad\overset{m=n-1}{=} u_0(r)\left(-\sqrt{\frac{1}{2\sigma^2}}\bar{\phi}_1(\tilde{t})\right)\\
&\quad+\sum_{m=0}^{\infty}u_{m+1}(r)\left(\sqrt{\frac{m+1}{2\sigma^2}}\bar{\phi}_m(\tilde{t})-\sqrt{\frac{m+2}{2\sigma^2}}\bar{\phi}_{m+2}(\tilde{t})\right)\\
&\quad=\sum_{m=0}^{\infty}u_{m+1}(r)\sqrt{\frac{m+1}{2\sigma^2}}\bar{\phi}_m(\tilde{t})\\
&\quad+\underline{u_0(r)\left(-\sqrt{\frac{1}{2\sigma^2}}\bar{\phi}_1(\tilde{t})\right)+\sum_{m=0}^{\infty}u_{m+1}(r)\left(-\sqrt{\frac{m+2}{2\sigma^2}}\bar{\phi}_{m+2}(\tilde{t})\right)}
\end{aligned}$$

where

$$\begin{aligned}
&u_0(r)\left(-\sqrt{\frac{1}{2\sigma^2}}\bar{\phi}_1(\tilde{t})\right)\\
&\quad+\sum_{m=0}^{\infty}u_{m+1}(r)\left(-\sqrt{\frac{m+2}{2\sigma^2}}\bar{\phi}_{m+2}(\tilde{t})\right)\\
&\quad\overset{k=m+2}{=} u_0(r)\left(-\sqrt{\frac{1}{2\sigma^2}}\bar{\phi}_1(\tilde{t})\right)\\
&\quad+\sum_{k=2}^{\infty}u_{k-1}(r)\left(-\sqrt{\frac{k}{2\sigma^2}}\bar{\phi}_k(\tilde{t})\right)\\
&\quad=\sum_{k=0}^{\infty}u_{k-1}(r)\left(-\sqrt{\frac{k}{2\sigma^2}}\bar{\phi}_k(\tilde{t})\right)\quad(\text{Let } u_{-1}(r)=0)
\end{aligned}$$

then we have

$$\begin{aligned}
\frac{\partial}{\partial t}u(r,t) &= \sum_{m=0}^{\infty}u_{m+1}(r)\sqrt{\frac{m+1}{2\sigma^2}}\bar{\phi}_m(\tilde{t})\\
&\quad+\sum_{k=0}^{\infty}u_{k-1}(r)\left(-\sqrt{\frac{k}{2\sigma^2}}\bar{\phi}_k(\tilde{t})\right)\\
&\overset{m=n,k=n}{=}\sum_{n=0}^{\infty}\left(u_{n+1}(r)\sqrt{\frac{n+1}{2\sigma^2}}-u_{n-1}(r)\sqrt{\frac{n}{2\sigma^2}}\right)\bar{\phi}_n(\tilde{t})
\end{aligned}$$

REFERENCES

[1] A. Taflove and S. C. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*. Boston, MA, USA: Artech House, 2000.

[2] K. Xiao, J. P. David, and L. D. James, "A three-dimensional FDTD subgridding algorithm with separated temporal and spatial interfaces and related stability analysis," *IEEE Trans. Antennas Propag.*, vol. 55, no. 07, pp. 1981–1990, Jul. 2007.

[3] C. Chang and D. S. Costas, "A spatially filtered finite-difference time-domain scheme with controllable stability beyond the CFL limit: Theory and applications," *IEEE Trans. Microw. Theory Tech.*, vol. 61, no. 01, pp. 351–359, Jan. 2013.

[4] T. Namiki, "A new FDTD algorithm based on alternating-direction implicit method," *IEEE Trans. Microw. Theory Tech.*, vol. 47, pp. 2003–2007, Oct. 1999.

[5] S. G. Garcia, T. W. Lee, and S. C. Hagness, "On the accuracy of the ADI-FDTD method," *IEEE Antennas Wireless Propag. Lett.*, vol. 1, no. 1, pp. 31–34, 2002.

[6] S. Wang, F. L. Teixeira, and J. Chen, "An iterative ADI-FDTD with reduced splitting error," *IEEE Microw. Wireless Compon. Lett.*, vol. 15, no. 2, pp. 92–94, Feb. 2005.

[7] G. Sun and C. W. Trueman, "Unconditionally stable Crank-Nicolson scheme for solving the two-dimensional Maxwell's equations," *Electron. Lett.*, vol. 39, no. 7, pp. 595–597, Apr. 2003.

[8] G. Sun and C. W. Trueman, "Approximate Crank-Nicolson schemes for the 2-D finite-difference time-domain method for $\mathrm{TE_z}$ waves," *IEEE Trans. Antennas Propag.*, vol. 52, no. 11, pp. 2963–2972, Nov. 2004.

[9] Y. S. Chung, T. K. Sarkar, B. H. Jung, and M. Salazar-Palma, "An unconditionally stable scheme for the finite-difference time-domain method," *IEEE Trans. Microw. Theory Tech.*, vol. 51, no. 3, pp. 697–704, Mar. 2003.

[10] Z. Y. Cai, B. Chen, Q. Yin, and R. Xiong, "The WLP-FDTD method for periodic structures with oblique incident wave," *IEEE Trans. Antennas Propag.*, vol. 59, no. 10, pp. 3780–3785, Oct. 2011.

[11] Y. T. Duan, B. Chen, D. G. Fang, and B. H. Zhou, "Efficient implementation for 3-D Laguerre-based finite-difference time-domain method," *IEEE Trans. Microw. Theory Tech.*, vol. 59, no. 1, pp. 56–64, Jan. 2011.

[12] Z. Chen, Y. T. Duan, Y. R. Zhang, and Y. Yi, "A new efficient algorithm for the unconditionally stable 2-D WLP-FDTD method," *IEEE Trans. Antennas Propag.*, vol. 61, no. 7, pp. 3712–3720, July 2013.

[13] L. R. L. Conte, R. Merletti, and G. V. Sandri, "Hermite expansions of compact support waveforms: Applications to myoelectric signals," *IEEE Trans. Biomed. Eng.*, vol. 41, no. 12, pp. 1147–1159, Dec. 1994.

[14] S. Saboktakin and B. Kordi, "Time-domain distortion analysis of wideband electromagnetic-field sensors using hermite-gauss orthogonal functions," *IEEE Trans. EMC*, vol. 54, no. 3, pp. 511–521, Jun. 2012.

[15] Y. K. Alp and O. Arıkan, "Time-frequency analysis of signals using support adaptive Hermite-Gaussian expansions," *Digital Signal Processing*, vol. 22, no. 6, pp. 1010–1023, Dec. 2012.

[16] M. M. Rao, T. K. Sarkar, and T. Anjali, "Simultaneous extrapolation in time and frequency domains using hermite expansions," *IEEE Trans. Antennas Propag.*, vol. 47, no. 06, pp. 1108–1115, Jun. 1999.

[17] M. M. Rao and T. K. Sarkar, "Extrapolation of electromagnetic responses from conducting objects in time and frequency domains," *IEEE Trans. Microw. Theory Tech.*, vol. 47, no. 10, pp. 1964–1974, Oct. 1999.

[18] J. L. Hu, C. H. Chan, and T. K. Sarkar, "Generation of broadband response for the ground bounce," in *Proc. IEEE Conf. on Electrical Performance of Electronic Packaging*, 2000, pp. 47–50.

[19] J. L. Hu, C. H. Chan, and T. K. Sarkar, "Optimal simultaneous interpolation/extrapolation algorithm of electromagnetic responses in time and frequency domains," *IEEE Trans. Microw. Theory Tech.*, vol. 49, no. 10, pp. 1725–1732, Oct. 2001.

[20] Y. Mengtao, K. Jinhwan, and T. K. Sarkar, "A comparison of performance of three orthogonal polynomials in extraction of wide-band response using early time and low frequency data," *IEEE Trans. Antennas Propag.*, vol. 53, no. 02, pp. 785–792, Feb. 2005.

[21] R. S. Ran and T. Z. Huang, "An inversion algorithm for a banded matrix," *Comput. Math. Applicat.*, vol. 58, pp. 1699–1710, Jul. 2009.

[22] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*. New York, NY, USA: Dover, 1965.

[23] Y. Mengtao and T. K. Sarkar, "Use of discrete Laguerre sequences to extrapolate wide-band response from early-time and low-frequency data," *IEEE Trans. Microw. Theory Tech.*, vol. 52, no. 07, pp. 1740–1750, Jul. 2004.

[24] Y. Mengtao, A. De, and T. K. Sarkar, "Conditions for generation of stable and accurate hybrid TD-FD MoM solutions," *IEEE Trans. Microw. Theory Tech.*, vol. 54, no. 6, pp. 2552–2563, Jun. 2006.

[25] Z. Bi, K. Wu, C. Wu, and J. Litva, "A dispersive boundary condition for micro strip component analysis using the FD-TD method," *IEEE Trans. Microw. Theory Tech.*, vol. 40, pp. 774–777, Apr. 1992.