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A Prelude to Finite Elements: The Fruitful Problem of the Brachistochrone

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When did finite elements begin? It is difficult to trace the origins of finite-element methods because of a basic problem in defining precisely what constitutes a "finite-element method." To most mathematicians, it is a method of piecewise polynomial approximation. Its origins are therefore frequently traced to the appendix of a paper by Courant [1], in which piecewise-linear approximations of the Dirichlet problem over a network of triangles was discussed. The "Interpretation of Finite Differences" by Polya [2] is regarded as embodying piecewise-polynomial approximation aspects of finite elements.

On the other hand, the approximation of variational problems on a mesh of triangles goes back much further: 92 years. In 1851, Schellbach [3] proposed a finite-element-like solution to Plateau's problem of determining the surface, S, of minimum area enclosed by a given closed curve. Schellbach used an approximation, S_h, of S by a mesh of triangles over which the surface was represented by piecewise-linear functions. He then obtained an approximation of the solution to Plateau's problem by minimizing S_h with respect to the coordinates of hexagons formed by six elements (see [4]). This was not quite the conventional finite-element approach, but certainly as much a finite-element technique as that of Courant.

Some say that there is even an earlier work that uses some of the ideas underlying finite-element methods: Leibniz himself employed a piecewise-linear approximation of the Brachistochrone problem, proposed by Bernoulli in 1696 (see the historical volume [5]). With the help of his newly developed calculus tools, Leibniz derived the governing differential equation for the problem, the solution of which is a cycloid. However, most would agree that to credit this work as a finite-element approximation is somewhat stretching the point. Leibniz had no intention of approximating a differential equation; rather, his purpose was to derive one. Two and one-half centuries later, it was realized that useful approximations of differential equations could be determined by not necessarily taking infinitesimal elements, as in the calculus, but by keeping the elements finite in size. This idea is, in fact, the basis of the term "finite elements" [6, p. 153].

Figure 1a. The façade of the *Basilica di Santa Croce* in Florence.

Figure 1b. The tomb of Vittorio Fossombroni [Arezzo, Italy, September 15, 1754 – Florence, Italy, April 15, 1844] in *Santa Croce*.

This long text is an excerpt from a historical contribution on finite elements by J. T. Oden [6] with which many of the specialists in Finite Elements might be familiar. It traced the history of Finite Elements up to its maturity in the 1980s, and was mostly concerned with structural and mechanic engineering. Other details can be found in [7] and, more specifically on electromagnetics, in a recently reprinted historical paper by P. P. Silvester [8].

These paragraphs come back to us in the *Basilica di Santa Croce*, in Florence (Figure 1a), where there are the tombs, cenotaphs, or commemorative plaques of the greatest Italians. The most known at an international level might be Dante Alighieri, Michelangelo Bonarroti, Leonardo da Vinci, Lorenzo Ghiberti, Niccolò Machiavelli, and Gioacchino Rossini. Among the scientists, there are Eugenio Barsanti, Galileo Galilei, Enrico Fermi, Guglielmo Marconi, and Girolamo Segato.

SAGGIO
DI UN DILETTANTE DI MATEMATICA
SULLE
EQUAZIONI DI CONDIZIONE
E SOPRA
L' INVENZIONE
DELLA BRACHISTOCRONA

1791

Figure 1c. The first page of Fossombroni's essay on the brachistochrone [9].

There is also the tomb of Vittorio Fossombroni [Arezzo, Italy, September 15, 1754 – Florence, Italy, April 15, 1844], a statesman, mathematician, economist, and engineer (Figure 1b). In 1794, he directed the work for the drainage of the marshy valley of the Arno River. In 1814, he was made President of the legislative commission, and was appointed Prime Minister of the Great Duchy of Tuscany.

Among Fossombroni's writings was a booklet dated 1791 on the problem of the Brachistochrone [9], a copy of which is in the personal library of one of the authors (Figure 1c). This problem is fundamental, since it contributed to the birth to the calculus of variations, and eventually to modern numerical techniques such as the Finite Element Method, as Oden said [6]. A step back by a full century since Fossombroni's essay, and more than three centuries since Oden's paper, is now necessary if we want to remember this landmark in mathematics.

1. The Brachistochrone Problem

In the June, 1696, issue of the *Acta Eruditorum*, the young mathematician Johann Bernoulli [Basel, Switzerland, July 27, 1667 – Basel, Switzerland, January 1, 1748] (Figure 2a) posed a challenge to all mathematicians ([10] and Figures 3 and 4):

> Given in a vertical plane two points A and B, assign to the moving body M, the path AMB, by means of which, descending by its own weight from point A, it would arrive at the other point B in the shortest time.

Seldom in the history of science did a challenge lead to so fruitful results.

*The challenge was aimed mainly at Isaac Newton [*Woolsthorpe, England, January 4, 1643 – Kensington, England, March 31, 1727*] (Figure 2b). This was because*

Figure 2a. The protagonists of the Brachistochrone challenge: Johann Bernoulli [Basel, Switzerland, July 27, 1667 – Basel, Switzerland, January 1, 1748].

Figure 2b. Isaac Newton /Woolsthorpe, England, January 4, 1643 – Kensington, England, March 31, 1727*/.*

Figure 2c. Jakob Bernoulli [Basel, Switzerland, December 27, 1654 - Basel, Switzerland, August 16, 1705].

Figure 2d. Gottfried Wilhelm Leibniz [Leipzig, Saxony July 1, 1646 – Hanover, November 14, 1716].

Figure 2e. Ehrenfried Walther von Tschirnhaus [Kieslingswalde, Saxony, April 10, 1651 – Dresden, Saxony, October, 11, 1708].

Figure 2f. Guillaume de l'Hôpital [Paris, France, 1661 – Paris, France, February 2, 1704].

Bernoulli was a follower of Gottfried Wilhelm Leibniz [Leipzig, Germany, July 1, 1646 – Hanover, Germany, November 14, 1716] (Figure 2d), and sided with him in the famous Newton-Leibniz controversy about the invention of the calculus.

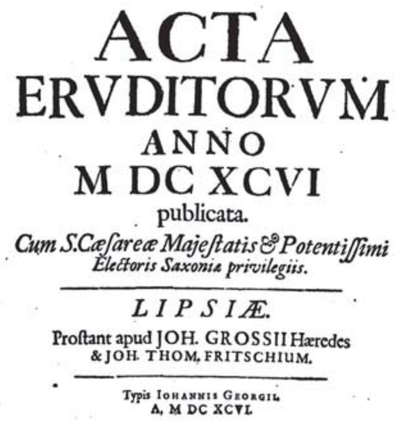
Bernoulli initially allowed six months for the solutions, but none was received during this period. He then allowed an extension. Isaac Newton become aware of the challenge on January 29, 1697. According to records, Newton solved the problem that same night, and mailed the solution anonymously by the next post. Upon reading the solution, Bernoulli immediately recognized its author, stating that *tanquam ex ungue leone* [one recognizes a lion from his claws].

In the end, five mathematicians responded with solutions: Newton, Jakob Bernoulli (Johann's brother, Figure 2c), the already mentioned Leibniz, Ehrenfried Walther von Tschirnhaus (Figure 2e), and Guillaume de l'Hôpital (Figure 2f).

2. Other Contemporary Problems

The Brachistochrone was not an isolated case. Several similar problems were proposed in those years. In his own reply to [10], Jakob Bernoulli posed a second challenge: a complex problem of surface maximization given the boundary perimeter, which was even harder than the brachistochrone challenge (Figure 5):

> Of all isoperimetric curves on a given axis BN, we seek the one that, like BFN, does not contain the greatest surface, but which maximizes another one contained by the curve BZN, after having extended FP in such a way that PZ is any ratio multiplied or divided by PF or the arc BF, that is to say that PZ is any proportion of a given A and of the distance PF or the arc BF [11, p. 214].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LIPSIÆ.
Proſtant apud JOH. GROSSII Hæredes
& JOH. THOM. FRITSCHIUM.

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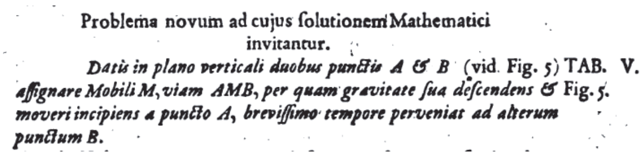
Figure 3. The first page of the *Acta Eroditorum* year 1696 [10], hosting Bernoulli's challenge.


Problema novum ad cujus ſolutionem Mathematici
invitantur.

Datis in plano verticali duobus punctis A & B (vid. Fig. 5) TAB. V.
aſſignare Mobili M, viam AMB, per quam gravitate ſua deſcendens & Fig. 5.
moveri incipiens a puncto A, breviſſimo tempore perveniat ad alterum punctum B.

Figure 4a. Bernoulli's challenge in [10] (in Latin).

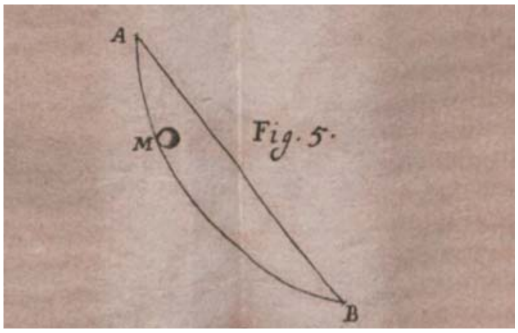


Figure 4b. The figure accompanying Bernoulli's challenge.

The excitement was not limited to these two problems. Immediately afterwards, a third arose (Figure 6). In a letter to Leibniz, Johann Bernoulli wrote:

> I haven't seen what my brother gave you about the shortest line between two points of the same surface, yet; I doubt that it might hold in general. Your method, which is actually a basis for some method, is legitimate and first came to my mind when the problem arose for me and noticed with ease (see Figure 4) that the shortest line, relatively to two intersecting planes, from the point R to point S is the one that makes with the common intersection NM of the dihedron two equal angles, so to say opposite, RTM, STN. But, up to now it seems not useful to the construction of the requested lines on the curved surface. Hence, I found another method, which is very general and founded on it, in order to solve the question, according to which a plane through three close generic points on the requested line must be perpendicular to the tangent plane to the curved surface in any of those points. From this, I realized an equation for any surface that can be easily constructed for any conoid or spheroid of any degree [12, Vol. I, Epist. LXXVI, p. 393].

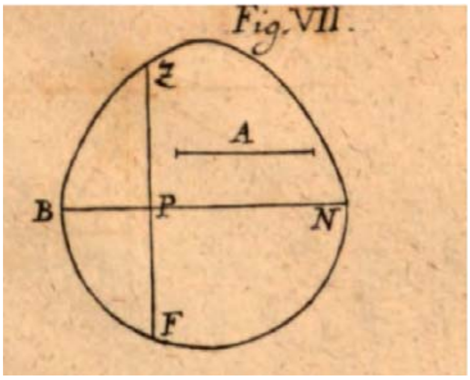


Figure 5. The accompanying figure to Jakob Bernoulli's counter-challenge in [11].

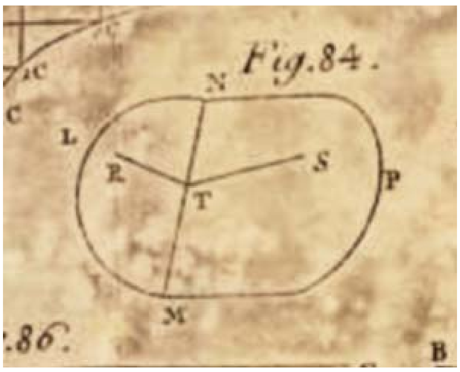


Figure 6. The figure associated with Johann Bernoulli's geodetic problem to Leibniz [12].

This was to become one of the hottest topics in mathematics, the problem of the geodesic: the shortest line between two points on a surface.

It is indeed true that Newton worked on a problem, which we would now call variational, about one decade earlier, in 1685, publishing it in his 1687 book [13, Book 2, prop XXXIV]. This was about the shape of a body of revolution with minimal resistance in a fluid. However, it was only with Bernoulli's challenge that the topic became one of the hottest in mathematics.

3. Conclusions

What really matters is not the problem itself and indeed not the relative, particular, solution proposed, but how the problem was solved. The new method was to be elaborated by Lenard Euler [Basel, Switzerland, April 15, 1707 – Saint Petersburg, Russia, September 18, 1783].

He worked on the geodesic problem in 1732 [14], and significantly improved it by an intuition by Joseph-Louis Lagrange [Turin, Italy, January 25, 1736 – Paris, France, April 10, 1813], communicated to Euler in a letter dated August 12, 1755 [15]. Finally, in 1756, Euler himself gave it its current name: *Calculus of Variations* [16].

Of course, a full history of the Calculus of Variations is out of the scope of a paper, but a matter for a whole book. The interested reader can refer to [17], but it is interesting for our community to remember how far in the past stretch the roots of our modern numerical methods, such as those of Finite Elements.

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