

Robust Adaptive Beamforming via Worst-Case SINR Maximization With Nonconvex Uncertainty Sets

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Abstract—This paper considers a formulation of the robust adaptive beamforming (RAB) problem based on worst-case signal-to-interference-plus-noise ratio (SINR) maximization with a nonconvex uncertainty set for the steering vectors. The uncertainty set consists of a similarity constraint and a (nonconvex) double-sided ball constraint. The worst-case SINR maximization problem is turned into a quadratic matrix inequality (QMI) problem using the strong duality of semidefinite programming. Then a linear matrix inequality (LMI) relaxation for the QMI problem is proposed, with an additional valid linear constraint. Necessary and sufficient conditions for the tightened LMI relaxation problem to have a rank-one solution are established. When the tightened LMI relaxation problem still has a high-rank solution, the LMI relaxation problem is further restricted to become a bilinear matrix inequality (BLMI) problem. We then propose an iterative algorithm to solve the BLMI problem that finds an optimal/suboptimal solution for the original RAB problem by solving the BLMI formulations. To validate our results, simulation examples are presented to demonstrate the improved array output SINR of the proposed robust beamformer.

Index Terms—Worst-case SINR maximization, nonconvex uncertainty set, robust adaptive beamforming, quadratic matrix inequality, bilinear matrix inequality relaxation.

I. INTRODUCTION

ADAPTIVE beamforming has been widely employed in many applications to radar, sonar, communications, microphone array speech/audio processing, medical imaging, radio

astronomy, and others [1]. Since traditional adaptive beamforming approaches have weak immunity against even a small mismatch between the presumed and actual signal steering vectors, many robust adaptive beamforming (RAB) techniques have been developed and substantial progress has been made in this area in recent years (see e.g. [2], [3], [4], [5], [6], [7] and references therein). The mismatch may result from look direction errors, imperfect array calibration, distorted antenna shape, and other reasons [2]. Facilitated by the recent development of convex optimization and robust optimization [8], [9], new powerful RAB methods have been proposed with significantly improved performance.

Among all proposed RAB techniques, the approach based on the concept of worst-case performance optimization has drawn much attention [2], [3], [4], [10], [11], [12], [13]. Particularly in [10], the actual signal steering vector is modelled as a sum of the presumed steering vector and a norm bounded mismatch vector, and the worst-case signal-to-interference-plus-noise ratio (SINR) maximization problem is formulated. It turns out that this problem can be reformulated as minimization of the denominator of the SINR subject to infinitely many nonconvex quadratic constraints over a spherical uncertainty set of the mismatch vectors. The (robust) minimization problem can be further transformed equivalently into a second-order cone programming (SOCP) problem, and consequently solved via an interior-point method (see e.g. [14]). Both uncertainties in the signal steering vector and in the array signal matrix are considered in [11], and RAB technique based on the worst-case performance optimization is obtained via an SOCP problem, which is a reformulation for the worst-case SINR maximization over the two (convex) uncertainty sets. In [13], under the assumption that the steering vector and covariance matrix are mismatched, but known to belong to a general convex compact subset, the problem of finding a beamforming vector that maximizes the worst-case SINR over the uncertainty set is studied. It has been shown that when the uncertainty set can be represented in terms of linear matrix inequalities (LMIs), the worst-case SINR maximization problem can be solved via semidefinite programming (SDP).

In many practical scenarios, it is inadequate to describe the uncertainty sets of steering vectors by only convex constraints. Indeed, some important constraints are nonconvex by nature (e.g., the unit norm constraint). Therefore, it is necessary to study the worst-case SINR maximization problem with nonconvex uncertainty sets, which have practical significance. It can also

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be viewed as a desired and technically advanced and nontrivial generalization of the work in [13].

In this paper,¹ we propose a RAB technique that maximizes the worst-case SINR over a nonconvex uncertainty set of the signal-of-interest (SOI) steering vector, with a similarity constraint and a (nonconvex) double-sided ball constraints. The similarity constraint is a spherical constraint, ensuring the desired steering vectors to be sufficiently close to a presumed steering vector, while the ball constraints are imposed to account for the steering vector gain perturbations caused, for example, by the sensor amplitude and phase errors as well as the sensor position errors (cf. [16, pp. 2408 and 2414]). Since the uncertainty set is nonconvex, the solution methods proposed herein to the new worst-case SINR maximization problem are very different from that in [13]. In particular, the contributions of this paper are the following.

- We reformulate the worst-case SINR maximization RAB design problem over the aforementioned nonconvex uncertainty set as a semi-infinite program, namely, a quadratic minimization problem with infinitely many constraints over the nonconvex uncertainty set. Mathematically, the constraints for the uncertainty set consist of an inhomogeneous quadratic constraint and a double-sided quadratic constraint. In this case, a quadratic optimization over such uncertainty set is hidden convex, and its SDP relaxation is tight [17]. Then the constraint set of the semi-infinite program can be represented by a single quadratic matrix inequality (QMI) constraint by applying the strong duality theory of SDP. The quadratic minimization problem with such QMI constraint is called a QMI problem. Thus, our worst-case SINR maximization problem is also shown to be a QMI problem.
- For the QMI problem, we establish an LMI relaxation problem. Often there is a gap between an LMI relaxation and a QMI problem. In order to narrow or close the gap, in our case, a valid linear constraint is added to the LMI problem, which means that the relaxation problem is further tightened. Some necessary and sufficient conditions for the tightened LMI problem to have a rank-one solution is presented. If the tightened LMI relaxation problem admits a rank-one solution (output by an interior-point method), then the solution is globally optimal for the QMI problem, and thus, for the original worst-case SINR maximization problem.
- When the tightened LMI relaxation problem however has a high-rank solution, we impose to the problem an additional rank-one condition, which is characterized by the constraint requiring second largest eigenvalue of a positive semidefinite (PSD) beamforming matrix to be nonpositive. This is a novel approach, and such characterization is universal, thus, such constraint can be added to any problem where one seeks a nonzero rank-one solution. In this way, the tightened LMI relaxation problem can be further restricted. Moreover, the tightened LMI relaxation problem

after the restriction is shown to be equivalent to a bilinear matrix inequality (BLMI) problem, which follows from an SDP representation for the sum of K largest eigenvalues of a Hermitian matrix. Such BLMI problem is nonconvex in general. Nevertheless, we propose an iterative algorithm to solve it, which finds an optimal/suboptimal solution for the original QMI problem associated with the worst-case SINR maximization problem.

- In addition to the aforementioned uncertainty set, we consider another nonconvex uncertainty set of practical significance, which is comprised of a double-sided ball constraint and a quadratic constraint that prevents the SOI's direction-of-arrival (DOA) from converging to the set of DOAs characterized by all linear combinations of the interference steering vectors (cf. [18], [19]). Then, the worst-case SINR maximization problem over the new uncertainty set is formulated in order to compute a RAB weight vector, and an approach similar to the previous one is developed to find an optimal/suboptimal solution of the problem. This approach includes the steps of QMI problem reformulation, solving a tightened LMI relaxation problem, BLMI problem reformulation, and developing an iterative algorithm for solving the BLMI problem.

The remainder of this paper is organized as follows. Section II is devoted to signal models and problem formulation. In Section III, we consider the scenario with the uncertainty set consisting of a similarity constraint and a double-sided ball constraint. In Section IV, we then handle the case with the uncertainty set comprised of a quadratic constraint and the double-sided ball constraint. Numerical examples are presented in Section V. Finally, conclusions are drawn in Section VI.

Notation: We adopt the notation of using boldface for vectors \mathbf{a} (lower case), and matrices \mathbf{A} (upper case). The transpose operator and the conjugate transpose operator are denoted by the superscripts $(\cdot)^T$ and $(\cdot)^H$, respectively. The notation $\text{tr}(\cdot)$ stands for the trace of a square matrix argument; \mathbf{I} and $\mathbf{0}$ denote respectively the identity matrix and the matrix (or the row vector or the column vector) with zero entries (their size is determined from the context). The letter j represents the imaginary unit (i.e., $j = \sqrt{-1}$), while the letter i often serves as an index. For any complex number x , we use $\Re(x)$ and $\Im(x)$ to denote respectively the real and imaginary parts of x , $|x|$ and $\arg(x)$ represent the modulus and the argument of x , and x^* (\mathbf{x}^* or \mathbf{X}^*) stands for the component-wise conjugate of x (\mathbf{x} or \mathbf{X}). The Euclidean norm of vector \mathbf{x} is denoted by $\|\mathbf{x}\|$, and the Frobenius norm (the spectral norm) of matrix \mathbf{X} by $\|\mathbf{X}\|_F$ ($\|\mathbf{X}\|_2$). The curled inequality symbol \succeq (and its strict form \succ) is used to denote generalized inequality: $\mathbf{A} \succeq \mathbf{B}$ meaning that $\mathbf{A} - \mathbf{B}$ is a Hermitian positive semidefinite matrix ($\mathbf{A} \succ \mathbf{B}$ for positive definiteness). The space of Hermitian $N \times N$ matrices (the space of real-valued symmetric $N \times N$ matrices) is denoted by \mathcal{H}^N (\mathcal{S}^N), and the set of all positive semidefinite matrices in \mathcal{H}^N (\mathcal{S}^N) by \mathcal{H}_+^N (\mathcal{S}_+^N). The notation $\mathbb{E}[\cdot]$ represents the statistical expectation. The notation $\text{Rank}(\mathbf{X})$ stands for the rank of a matrix. All eigenvalues $\lambda_n(\mathbf{X})$, $n = 1, \dots, N$ of Hermitian matrix \mathbf{X} are placed in a descending order, namely,

¹Some preliminary results without detailed proofs along this line of research have been initially presented in [15].

$\lambda_1(\mathbf{X}) \geq \lambda_2(\mathbf{X}) \geq \dots \geq \lambda_N(\mathbf{X})$. Finally, $v^*(\cdot)$ represents the optimal value of problem (·).

II. SIGNAL MODELS AND PROBLEM FORMULATION

Let the narrowband signal received by an N -sensor array be given by

$$\mathbf{y}(t) = \mathbf{s}(t) + \mathbf{i}(t) + \mathbf{n}(t),$$

where $\mathbf{s}(t)$, $\mathbf{i}(t)$, and $\mathbf{n}(t)$ are statistically independent vectors corresponding to the SOI, interference, and noise, respectively. In the point signal source case, $\mathbf{s}(t)$ is expressed as

$$\mathbf{s}(t) = s(t)\mathbf{a}$$

where $s(t)$ is the SOI waveform and \mathbf{a} is its steering vector (the actual array response or spatial signature of the SOI). The receive beamformer outputs the signal

$$x(t) = \mathbf{w}^H \mathbf{y}(t)$$

where \mathbf{w} is the $N \times 1$ vector of beamformer complex weight coefficients (called a beamvector). The beamforming problem is to find an optimal beamvector \mathbf{w} maximizing the beamformer output SINR

$$\text{SINR} = \frac{\sigma_s^2 |\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H \mathbf{R}_{i+n} \mathbf{w}} \quad (1)$$

where σ_s^2 is the SOI power and \mathbf{R}_{i+n} is the interference-plus-noise covariance matrix. In practical applications, the true covariance matrix \mathbf{R}_{i+n} is unavailable and, consequently, the data sample estimate

$$\bar{\mathbf{R}} = \frac{1}{T} \sum_{t=1}^T \mathbf{y}(t) \mathbf{y}(t)^H \quad (2)$$

is used instead, with T being the number of available snapshots and $\mathbf{y}(t)$ being the beamformer training data. The SINR maximization problem is equivalent to the following convex problem

$$\underset{\mathbf{w}}{\text{minimize}} \quad \mathbf{w}^H \bar{\mathbf{R}} \mathbf{w} \quad \text{subject to} \quad |\mathbf{w}^H \mathbf{a}| = 1, \quad (3)$$

and its solution is

$$\mathbf{w}^* = \frac{\bar{\mathbf{R}}^{-1} \mathbf{a}}{\mathbf{a}^H \bar{\mathbf{R}}^{-1} \mathbf{a}}. \quad (4)$$

This is the well-known minimum variance distortionless response (MVDR) beamformer (see e.g. [12]) or Capon beamformer (see e.g. [16]).

In practical scenarios (e.g., multi-antenna wireless communications and passive source localization), the beamformer suffers from dramatic performance degradation due to mismatch between the actual steering vector \mathbf{a} and the presumed steering vector $\hat{\mathbf{a}}$. To mitigate the degradation, many robust adaptive beamforming techniques have been proposed in the past two decades (see e.g. [2], [10], [12], [13], [16], [18], [19] and references therein).

A popular robust beamforming design is the so-called MVDR robust adaptive beamforming (RAB), based on the optimal estimate of the actual SOI's steering vector \mathbf{a} in accordance to

some criterion (for instance, maximizing the beamformer output power subject to some constraints on \mathbf{a} , see e.g. [18], [19], [20] and references therein). Note that the beamformer output power can be expressed as

$$\begin{aligned} \text{E}[|x(t)|^2] &= \text{E}[\mathbf{w}^H \mathbf{y}(t) \mathbf{y}(t)^H \mathbf{w}] \\ &= \mathbf{w}^H \text{E}[\mathbf{y}(t) \mathbf{y}(t)^H] \mathbf{w} \\ &\approx \mathbf{w}^H \bar{\mathbf{R}} \mathbf{w}. \end{aligned}$$

Using the MVDR beamformer formula (4), the array output power at \mathbf{w}^* can be found to be

$$\mathbf{w}^{*H} \bar{\mathbf{R}} \mathbf{w}^* = \frac{1}{\mathbf{a}^H \bar{\mathbf{R}}^{-1} \mathbf{a}}.$$

In order to maximize the array output power, we just can minimize the denominator subject to a set of constraints on the actual steering vector. In other words, the following optimal steering vector estimation problem needs to be solved

$$\underset{\mathbf{a} \in \mathcal{A}}{\text{minimize}} \quad \mathbf{a}^H \bar{\mathbf{R}}^{-1} \mathbf{a}, \quad (5)$$

where \mathcal{A} is a constraint set for the actual steering vector. Once an optimal solution \mathbf{a}^* is obtained, we get an optimal MVDR RAB vector \mathbf{w}^* by substituting \mathbf{a}^* into (4). The uncertainty set \mathcal{A} here can be a nonconvex, for example, (cf. [19])

$$\mathcal{A}_1 = \{\mathbf{a} \mid \|\mathbf{a} - \hat{\mathbf{a}}\|^2 \leq \epsilon, N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2\}. \quad (6)$$

Note that the MVDR RAB vector \mathbf{w}^* (cf. [19]) can be also interpreted as a solution for the minimax problem:

$$\underset{\mathbf{a} \in \mathcal{A}}{\text{minimize}} \quad \underset{\mathbf{w} \neq \mathbf{0}}{\text{maximize}} \quad \frac{|\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H \bar{\mathbf{R}} \mathbf{w}} = \underset{\mathbf{a} \in \mathcal{A}}{\text{minimize}} \quad \mathbf{a}^H \bar{\mathbf{R}}^{-1} \mathbf{a}. \quad (7)$$

In the literature, another interesting robust adaptive beamformer is based on the worst-case SINR maximization design. Specifically, the maximization problem of the worst-case SINR is formulated as

$$\underset{\mathbf{w} \neq \mathbf{0}}{\text{maximize}} \quad \underset{\mathbf{a} \in \mathcal{A}, \mathbf{\Delta} \in \mathcal{B}}{\text{minimize}} \quad \frac{|\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H (\bar{\mathbf{R}} + \mathbf{\Delta}) \mathbf{w}}, \quad (8)$$

where \mathcal{B} is the uncertainty set for matrix error term $\mathbf{\Delta}$. The robust beamforming problem can be equivalently transformed into

$$\underset{\mathbf{w} \neq \mathbf{0}}{\text{maximize}} \quad \frac{\underset{\mathbf{a} \in \mathcal{A}}{\text{minimize}} |\mathbf{w}^H \mathbf{a}|^2}{\underset{\mathbf{\Delta} \in \mathcal{B}}{\text{maximize}} \mathbf{w}^H (\bar{\mathbf{R}} + \mathbf{\Delta}) \mathbf{w}}, \quad (9)$$

which can be further reexpressed as

$$\min_{\mathbf{w} \neq \mathbf{0}} \max_{\mathbf{\Delta} \in \mathcal{B}} \mathbf{w}^H (\bar{\mathbf{R}} + \mathbf{\Delta}) \mathbf{w} \quad \text{subject to} \quad \min_{\mathbf{a} \in \mathcal{A}} |\mathbf{w}^H \mathbf{a}|^2 = 1. \quad (10)$$

In other words, if a vector \mathbf{w}^* is optimal for (10), then it is also optimal for (9), and thus, for (8). Therefore, the worst-case performance optimization design takes the following formulation

$$\min_{\mathbf{w} \neq \mathbf{0}} \max_{\mathbf{\Delta} \in \mathcal{B}} \mathbf{w}^H (\bar{\mathbf{R}} + \mathbf{\Delta}) \mathbf{w} \quad \text{subject to} \quad |\mathbf{w}^H \mathbf{a}|^2 \geq 1, \forall \mathbf{a} \in \mathcal{A}, \quad (11)$$

since both problems (11) and (10) are tantamount to each other.

In the seminal paper [10], the following robust adaptive beamforming problem has been studied

$$\underset{\mathbf{w}}{\text{minimize}} \mathbf{w}^H \bar{\mathbf{R}} \mathbf{w} \text{ subject to } |\mathbf{w}^H \mathbf{a}|^2 \geq 1, \forall \mathbf{a} : \|\mathbf{a} - \hat{\mathbf{a}}\|^2 \leq \epsilon. \quad (12)$$

In words, the set $\mathcal{A} = \{\mathbf{a} \mid \|\mathbf{a} - \hat{\mathbf{a}}\| \leq \sqrt{\epsilon}\}$ is convex and it is a ball centered at a presumed steering vector $\hat{\mathbf{a}}$ (with $\|\hat{\mathbf{a}}\|^2 = N$). It turned out that the robust optimization problem (12) can be equivalently transformed into an SOCP of the form

$$\underset{\mathbf{w}}{\text{minimize}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \text{ subject to } \Re(\mathbf{w}^H \hat{\mathbf{a}}) \geq \sqrt{\epsilon} \|\mathbf{w}\| + 1. \quad (13)$$

In this paper, we focus on the robust adaptive beamforming problem (11) with a nonconvex set of possible steering vectors and a convex set of perturbations of the interference-plus-noise covariance matrix, which generalizes the works in [10], [13]. Specifically, the uncertainty set \mathcal{B} is defined as

$$\mathcal{B}_1 = \{\Delta \mid \|\Delta\|_F^2 \leq \gamma, \bar{\mathbf{R}} + \Delta \succeq \mathbf{0}\}. \quad (14)$$

Therefore, problem (11) is rewritten as

$$\min_{\mathbf{w} \neq \mathbf{0}} \mathbf{w}^H (\bar{\mathbf{R}} + \sqrt{\gamma} \mathbf{I}) \mathbf{w} \text{ subject to } |\mathbf{w}^H \mathbf{a}|^2 \geq 1, \forall \mathbf{a} \in \mathcal{A}, \quad (15)$$

since

$$\mathbf{w}^H \Delta \mathbf{w} = \text{tr}(\Delta \mathbf{w} \mathbf{w}^H) \leq \|\Delta\|_F \|\mathbf{w} \mathbf{w}^H\|_F \leq \sqrt{\gamma} \|\mathbf{w}\|^2, \quad (16)$$

and the equalities hold as long as $\Delta = \sqrt{\gamma} \mathbf{w} \mathbf{w}^H / \|\mathbf{w}\|^2 \in \mathcal{B}_1$. For notational convenience, we let

$$\hat{\mathbf{R}} := \bar{\mathbf{R}} + \sqrt{\gamma} \mathbf{I}, \quad (17)$$

then the robust adaptive beamforming problem (15) can be recast into

$$\underset{\mathbf{w} \neq \mathbf{0}}{\text{minimize}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \text{ subject to } |\mathbf{w}^H \mathbf{a}|^2 \geq 1, \forall \mathbf{a} \in \mathcal{A}. \quad (18)$$

As for \mathcal{A} in (18), we first assume that it is specified to

$$\mathcal{A}_1 = \{\mathbf{a} \mid \|\mathbf{a} - \hat{\mathbf{a}}\|^2 \leq \epsilon, N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2\}, \quad (19)$$

namely, (6) is considered, where user parameters η_1 and η_2 control the norm bounds of perturbation. In particular, when $\eta_1 = \eta_2 = 0$, the set of the desired steering vectors reduces to

$$\mathcal{A}'_1 = \{\mathbf{a} \mid \|\mathbf{a} - \hat{\mathbf{a}}\|^2 \leq \epsilon, \|\mathbf{a}\|^2 = N\}. \quad (20)$$

Also, we assume subsequently that $\|\hat{\mathbf{a}}\|^2 = N > \epsilon$ (for $\epsilon \in [N, 2N]$, the discussion is similar).

Second, we assume (cf. [18], [19]) that

$$\mathcal{A}_2 = \{\mathbf{a} \mid \mathbf{a}^H \bar{\mathbf{C}} \mathbf{a} \leq \Delta_0, N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2\}, \quad (21)$$

and

$$\mathcal{A}_3 = \{\mathbf{a} \mid \mathbf{a}^H \mathbf{C} \mathbf{a} \geq \Delta_1, N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2\}. \quad (22)$$

Here, the matrix parameter in (21) $\bar{\mathbf{C}} = \int_{\bar{\Theta}} \mathbf{d}(\theta) \mathbf{d}^H(\theta) d\theta$, $\mathbf{d}(\theta)$ is the steering vector associated with direction θ that has the structure defined by the antenna array geometry, and $\bar{\Theta}$ is the complement of an angular sector Θ , where the desired signal is located. Also, parameter Δ_0 is equal to $\max_{\theta \in \Theta} \mathbf{d}^H(\theta) \bar{\mathbf{C}} \mathbf{d}(\theta)$. Similarly, matrix parameter $\mathbf{C} = \int_{\Theta} \mathbf{d}(\theta) \mathbf{d}^H(\theta) d\theta$, and the

threshold value $\Delta_1 = \min_{\theta \in \Theta} \mathbf{d}^H(\theta) \mathbf{C} \mathbf{d}(\theta)$. Both the first constraint in (21) and that in (22) are used to avoid the convergence of the steering vector \mathbf{a} to any linear combinations of the interference steering vectors (see e.g. [19, Fig. 2]). For more motivations and interpretations of the uncertainty sets in (21) and (22), we refer respectively to [18] and [19], where MVDR RAB designs have been studied. In this paper, we will consider the maximization of the worst-case SINR over nonconvex uncertainty sets (21) and (22).

We remark that a trivial upper bound for problem (8) is the optimal value for the following minimax problem (due to the weak duality theorem):

$$\underset{\mathbf{a} \in \mathcal{A}, \Delta \in \mathcal{B}}{\text{minimize}} \quad \underset{\mathbf{w} \neq \mathbf{0}}{\text{maximize}} \quad \frac{|\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H (\bar{\mathbf{R}} + \Delta) \mathbf{w}}. \quad (23)$$

It follows from [13] that if both \mathcal{A} and \mathcal{B} are compact and convex sets, then the two problems (8) and (23) are equivalent to each other, and can be solved via convex optimization. However, when \mathcal{A} and \mathcal{B} are specified to \mathcal{A}_1 (or \mathcal{A}_2 or \mathcal{A}_3) and \mathcal{B}_1 , respectively, (8) and (23) are not equivalent any more since the uncertainty sets \mathcal{A}_i , $i = 1, 2, 3$, are nonconvex.

III. SOLVING ROBUST OPTIMIZATION PROBLEM (18) WITH UNCERTAINTY SET \mathcal{A}_1 IN (19)

In this section, we show how to convert robust adaptive beamforming design problem (18) into a QMI problem, and present a method for finding a solution for a tightened LMI relaxation for such QMI problem.

A. QMI Reformulation for Optimization Problem (18) With Uncertainty Set \mathcal{A}_1

Since the original worst-case SINR maximization problem (8) is equivalent to problem (18), we need only to focus on solving (18) with a specific uncertainty set \mathcal{A} .

To begin with, let us look into the minimization problem for the feasible set in (18) with $\mathcal{A} = \mathcal{A}_1$, that is,

$$\underset{\mathbf{a} \in \mathbb{C}^N}{\text{minimize}} \quad \mathbf{a}^H \mathbf{w} \mathbf{w}^H \mathbf{a} \\ \text{subject to} \quad \|\mathbf{a} - \hat{\mathbf{a}}\|^2 \leq \epsilon, \\ N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2. \quad (24)$$

The optimization problem (24) amounts to the following homogeneous quadratically constrained quadratic program (QCQP) (see, e.g., [17, eq. (32)])

$$\underset{\mathbf{x} \in \mathbb{C}^{N+1}}{\text{minimize}} \quad \mathbf{x}^H \mathbf{A}_0 \mathbf{x} \\ \text{subject to} \quad \mathbf{x}^H \mathbf{A}_1 \mathbf{x} \leq 0, \\ N - \eta_1 \leq \mathbf{x}^H \mathbf{A}_2 \mathbf{x} \leq N + \eta_2, \\ \mathbf{x}^H \mathbf{A}_3 \mathbf{x} = 1, \quad (25)$$

with the matrix parameters being

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{w} \mathbf{w}^H & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} \mathbf{I} & -\hat{\mathbf{a}} \\ -\hat{\mathbf{a}}^H & \|\hat{\mathbf{a}}\|^2 - \epsilon \end{bmatrix},$$

and

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}.$$

The equivalence between (24) and (25) is in the sense that they share the same optimal value and if $\mathbf{x}^* = [\mathbf{w}^{*T}, t^*]^T \in \mathbb{C}^{N+1}$ solves (25), then \mathbf{w}^*/t^* solves (24).

Accordingly, the conventional SDP relaxation of (24) is

$$\begin{aligned} & \underset{\mathbf{X} \in \mathcal{H}^{N+1}}{\text{minimize}} && \text{tr}(\mathbf{A}_0 \mathbf{X}) \\ & \text{subject to} && \text{tr}(\mathbf{A}_1 \mathbf{X}) \leq 0, \\ & && N - \eta_1 \leq \text{tr}(\mathbf{A}_2 \mathbf{X}) \leq N + \eta_2, \\ & && \text{tr}(\mathbf{A}_3 \mathbf{X}) = 1, \\ & && \mathbf{X} \succeq \mathbf{0}. \end{aligned} \quad (26)$$

Moreover, the dual problem is the following SDP:

$$\begin{aligned} & \underset{\{y_i\}_{i=1}^4}{\text{maximize}} && (N - \eta_1)y_2 + (N + \eta_2)y_3 + y_4 \\ & \text{subject to} && \mathbf{A}_0 - y_1 \mathbf{A}_1 - (y_2 + y_3) \mathbf{A}_2 - y_4 \mathbf{A}_3 \succeq \mathbf{0}, \\ & && y_1 \leq 0, y_2 \geq 0, y_3 \leq 0, y_4 \in \mathbb{R}. \end{aligned} \quad (27)$$

Given the primal and dual SDPs, we claim strict feasibility of both SDPs in the following lemma.

Lemma III.1: It holds that the primal SDP (26) and the dual SDP (27) are strictly feasible.

Proof: See Appendix A.

It follows then from the strong duality theorem (see, e.g., [21, Th. 1.4.2]) that both SDPs are solvable.² Furthermore, one can always find a rank-one solution for (26) (see e.g. [17]). In other words, all the optimal values are equal, that is,

$$v^*((24)) = v^*((25)) = v^*((26)) = v^*((27)).$$

Accordingly, the robust adaptive beamforming problem (18) with $\mathcal{A} = \mathcal{A}_1$ is recast into the following QMI problem

$$\underset{\mathbf{w}, \{y_i\}_{i=1}^4}{\text{minimize}} \quad \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad (28a)$$

$$\text{s.t.} \quad (N - \eta_1)y_2 + (N + \eta_2)y_3 + y_4 = 1, \quad (28b)$$

$$\begin{bmatrix} \mathbf{w} \mathbf{w}^H - (y_1 + y_2 + y_3) \mathbf{I} & y_1 \hat{\mathbf{a}} \\ y_1 \hat{\mathbf{a}}^H & -y_4 - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) \end{bmatrix} \succeq \mathbf{0}, \quad (28c)$$

$$\mathbf{w} \in \mathbb{C}^N, y_1 \leq 0, y_2 \geq 0, y_3 \leq 0, y_4 \in \mathbb{R}. \quad (28d)$$

The conventional LMI relaxation problem can be formulated as

$$\underset{\mathbf{W}, \{y_i\}_{i=1}^4}{\text{minimize}} \quad \text{tr}(\hat{\mathbf{R}} \mathbf{W}) \quad (29a)$$

$$\text{s.t.} \quad (N - \eta_1)y_2 + (N + \eta_2)y_3 + y_4 = 1, \quad (29b)$$

$$\begin{bmatrix} \mathbf{W} - (y_1 + y_2 + y_3) \mathbf{I} & y_1 \hat{\mathbf{a}} \\ y_1 \hat{\mathbf{a}}^H & -y_4 - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) \end{bmatrix} \succeq \mathbf{0}, \quad (29c)$$

$$\mathbf{W} \succeq \mathbf{0}, y_1 \leq 0, y_2 \geq 0, y_3 \leq 0, y_4 \in \mathbb{R}. \quad (29d)$$

Suppose that problem (29) is solvable, and that $(\mathbf{W}^*, \{y_i^*\}_{i=1}^4)$ is an optimal solution. If \mathbf{W}^* is of rank one, then we claim that robust adaptive beamforming problem (18) is solved and the optimal beamforming vector is \mathbf{w}^* such

that $\mathbf{W}^* = \mathbf{w}^* \mathbf{w}^{*H}$. If rank of \mathbf{W}^* is larger than one, then we have to find another rank-one solution for (29).

B. Tightened LMI Relaxation by Adding One More Valid Linear Constraint

In this subsection, we study how to impose one more constraint to (29), such that the feasible region of the LMI relaxation problem is reduced and it is more likely that the optimal solution is of rank one. First, observe the lemma below.

Lemma III.2: The second constraint of (28) implies that

$$y_1 + y_2 + y_3 \leq 0. \quad (30)$$

The proof is immediate, and thus, is omitted.

In other words, the set of optimal solutions for (28) will not be altered if one enforces the linear constraint (30) to the problem, that is,

$$\begin{aligned} & \underset{\mathbf{w}, \{y_i\}_{i=1}^4}{\text{minimize}} && \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \\ & \text{subject to} && (28b), (28c), (28d) \text{ satisfied,} \\ & && y_1 + y_2 + y_3 \leq 0. \end{aligned} \quad (31)$$

This problem shares the same optimal value with problem (28).

The LMI relaxation problem for (31) can be written as

$$\begin{aligned} & \underset{\mathbf{W}, \{y_i\}_{i=1}^4}{\text{minimize}} && \text{tr}(\hat{\mathbf{R}} \mathbf{W}) \\ & \text{subject to} && (29b), (29c), (29d) \text{ satisfied,} \\ & && y_1 + y_2 + y_3 \leq 0. \end{aligned} \quad (32)$$

Clearly, the difference between (32) and (29) lies in the additional linear constraint. Therefore, LMI relaxation (32) appears tighter than (29). More importantly, if a rank-one solution $\mathbf{W}^* = \mathbf{w}^* \mathbf{w}^{*H}$ is optimal for (32), then \mathbf{w}^* is optimal for (31), and thus, also for (28). Therefore, let us focus hereafter on tightened LMI relaxation problem (32).

It is not hard to derive the dual problem of (32) as

$$\begin{aligned} & \underset{\mathbf{Z}, \mathbf{z}_1, z_0, x}{\text{maximize}} && z_0 \\ & \text{subject to} && \hat{\mathbf{R}} - \mathbf{Z} \succeq \mathbf{0}, \\ & && \text{tr} \mathbf{Z} - 2\Re(\hat{\mathbf{a}}^H \mathbf{z}_1) + z_0(\|\hat{\mathbf{a}}\|^2 - \epsilon) \leq x, \\ & && \text{tr} \mathbf{Z} - (N - \eta_1)z_0 \geq x, \\ & && \text{tr} \mathbf{Z} - (N + \eta_2)z_0 \leq x, \\ & && \begin{bmatrix} \mathbf{Z} & \mathbf{z}_1 \\ \mathbf{z}_1^H & z_0 \end{bmatrix} \succeq \mathbf{0}, x \leq 0. \end{aligned} \quad (33)$$

Suppose that the primal and dual SDPs are solvable, and possess the same optimal value. Thus, the complementary conditions include

$$\text{tr}((\hat{\mathbf{R}} - \mathbf{Z}) \mathbf{W}) = 0 \quad (34)$$

$$y_1(\text{tr} \mathbf{Z} - 2\Re(\hat{\mathbf{a}}^H \mathbf{z}_1) + z_0(\|\hat{\mathbf{a}}\|^2 - \epsilon) - x) = 0 \quad (35)$$

$$y_2(\text{tr} \mathbf{Z} - (N - \eta_1)z_0 - x) = 0 \quad (36)$$

$$y_3(\text{tr} \mathbf{Z} - (N + \eta_2)z_0 - x) = 0 \quad (37)$$

$$(y_1 + y_2 + y_3)x = 0 \quad (38)$$

$$\text{tr} \left(\mathbf{Q} \begin{bmatrix} \mathbf{Z} & \mathbf{z}_1 \\ \mathbf{z}_1^H & z_0 \end{bmatrix} \right) = 0, \quad (39)$$

²Here by ‘‘solvable,’’ we mean that the minimization (or maximization) problem is feasible, bounded below (above) and the optimal value is attained [21, p. 13].

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{W} - (y_1 + y_2 + y_3)\mathbf{I} & y_1 \hat{\mathbf{a}} \\ y_1 \hat{\mathbf{a}}^H & -y_4 - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) \end{bmatrix}. \quad (40)$$

It follows from (34)–(39) that

$$\text{tr}(\hat{\mathbf{R}}\mathbf{W}) = z_0, \quad (41)$$

i.e., the primal optimal value is equal to the dual optimal value.

Suppose that $(\mathbf{W}^*, y_1^*, y_2^*, y_3^*, y_4^*)$ and $(\mathbf{Z}^*, z_1^*, z_0^*, x^*)$ are the primal and the dual SDP solutions, respectively. They clearly comply with the complementary conditions (34)–(39). In order to make the notations simpler and clearer, we drop the superscript $*$ for the optimal solutions, without leading to confusion. The following propositions are in order.

Proposition III.3: Suppose that $(\mathbf{W}, y_1, y_2, y_3, y_4)$ and $(\mathbf{Z}, z_1, z_0, x)$ are the solutions for primal SDP (32) and dual SDP (33), respectively. Then, it holds that

- 1) $y_1 < 0$;
- 2) $y_2 y_3 = 0$;
- 3) if $y_1 + y_2 + y_3 = 0$, then $y_1 = -y_2$ and $y_3 = 0$.

Proof: See Appendix B.

Proposition III.4: Suppose that $(\mathbf{W}, y_1, y_2, y_3, y_4)$ and $(\mathbf{Z}, z_1, z_0, x)$ are the solutions for primal SDP (32) and dual SDP (33), respectively. Then, it holds that

- 1) $z_0 = \frac{2\Re(\hat{\mathbf{a}}^H \mathbf{z}_1) - (\text{tr} \mathbf{Z} - x)}{\|\hat{\mathbf{a}}\|^2 - \epsilon}$ or $z_0 = \frac{y_1 x + (y_2 + y_3) \text{tr} \mathbf{Z}}{1 - y_4}$ for $y_4 \neq 1$;
- 2) $-y_4 - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) > 0$;
- 3) if $y_1 + y_2 + y_3 = 0$, then $y_2 > \frac{1}{2N - \epsilon - \eta_1}$.

Proof: See Appendix C.

Note that the second constraint (the LMI constraint) in (32) can be recast into

$$\mathbf{W} - (y_1 + y_2 + y_3)\mathbf{I} \succeq \bar{\mathbf{a}}\bar{\mathbf{a}}^H, \quad (42)$$

where

$$\bar{\mathbf{a}} = \frac{y_1 \hat{\mathbf{a}}}{\sqrt{-y_4 - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon)}}. \quad (43)$$

Accordingly, a necessary condition for the existence of rank-one solution for problem (32) can be established as follows.

Theorem III.5: Suppose that $(\mathbf{w}\mathbf{w}^H, y_1, y_2, y_3, y_4)$ and $(\mathbf{Z}, z_1, z_0, x)$ are the solutions for primal SDP (32) and dual SDP (33), respectively. Then, it holds that

- 1) $(\hat{\mathbf{R}} - \mathbf{Z})\mathbf{w} = \mathbf{0}$;
- 2) $y_1 + y_2 + y_3 \leq \frac{\|\bar{\mathbf{a}}\|^2 - \|\mathbf{w}\|^2 + \sqrt{(\|\bar{\mathbf{a}}\|^2 + \|\mathbf{w}\|^2)^2 - 4\|\bar{\mathbf{a}}^H \mathbf{w}\|^2}}{2} < 0$, if $\bar{\mathbf{a}}$ and \mathbf{w} are linearly independent.

Proof: See Appendix D.

In what follows, we present sufficient conditions for the existence of rank-one solution for problem (32). To facilitate our analysis, we cite a special rank-one matrix decomposition lemma as follows.

Lemma III.6 (Theorem 2.1 in [22]): Suppose that \mathbf{X} is an $N \times N$ complex Hermitian PSD matrix of rank R , and \mathbf{A} and \mathbf{B} are given $N \times N$ Hermitian matrices. Then, there is a rank-one decomposition $\mathbf{X} = \sum_{r=1}^R \mathbf{x}_r \mathbf{x}_r^H$ such that

$$\mathbf{x}_r^H \mathbf{A} \mathbf{x}_r = \frac{\text{tr}(\mathbf{A}\mathbf{X})}{R} \quad \text{and} \quad \mathbf{x}_r^H \mathbf{B} \mathbf{x}_r = \frac{\text{tr}(\mathbf{B}\mathbf{X})}{R}, \quad r = 1, \dots, R.$$

The rank-one decomposition synthetically is denoted as $\mathbf{x} = \mathcal{D}_1(\mathbf{X}, \mathbf{A}, \mathbf{B})$.

Suppose that $y_1 + y_2 + y_3 < 0$. We have the following theorem.

Theorem III.7: Suppose that $(\mathbf{W}, y_1, y_2, y_3, y_4)$ and $(\mathbf{Z}, z_1, z_0, x)$ are the solutions for primal SDP (32) and dual SDP (33), respectively. Suppose that $\text{Rank}(\mathbf{W})$ is higher than one and $y = y_1 + y_2 + y_3 < 0$. Suppose also that

$$y^2 + y(\|\bar{\mathbf{a}}\|^2 - \text{tr} \mathbf{W}) + \bar{\mathbf{a}}^H \mathbf{W} \bar{\mathbf{a}} - \|\bar{\mathbf{a}}\|^2 \text{tr} \mathbf{W} \geq 0. \quad (44)$$

Then there is a rank-one optimal solution for SDP problem (32).

Proof: See Appendix E.

Suppose $y_1 + y_2 + y_3 = 0$. Eq. (42) reduces to

$$\mathbf{W} \succeq \bar{\mathbf{a}}\bar{\mathbf{a}}^H.$$

Therefore, we have $\text{tr}(\hat{\mathbf{R}}\mathbf{W}) \geq \bar{\mathbf{a}}^H \hat{\mathbf{R}} \bar{\mathbf{a}}$. It is known that if \mathbf{w} satisfies

$$\mathbf{w}\mathbf{w}^H \succeq \bar{\mathbf{a}}\bar{\mathbf{a}}^H,$$

then there is $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ such that $\mathbf{w} = \lambda \bar{\mathbf{a}}$, and vice versa. Hence, the optimal value is equal to $|\lambda|^2 \bar{\mathbf{a}}^H \hat{\mathbf{R}} \bar{\mathbf{a}}$, provided that \mathbf{w} is an optimal solution. Observe that our goal is to minimize $\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w}$, and therefore, the optimal value is $\bar{\mathbf{a}}^H \hat{\mathbf{R}} \bar{\mathbf{a}}$ when $|\lambda| = 1$. Then we obtain the following proposition.

Proposition III.8: Suppose that $(\mathbf{W}, y_1, y_2, y_3, y_4)$ and $(\mathbf{Z}, z_1, z_0, x)$ are the solutions for primal SDP (32) and dual SDP (33), respectively. Suppose that $\text{Rank}(\mathbf{W})$ is higher than one and $y_1 + y_2 + y_3 = 0$. Then $\mathbf{w} = \bar{\mathbf{a}}$ is optimal for (31), and thus, for (28).

C. BLMI Approximation Approach

Note that if the optimal solution \mathbf{W}^* of (32) is of rank one, then we claim that the solution \mathbf{w}^* with $\mathbf{W}^* = \mathbf{w}^* \mathbf{w}^{*H}$ is optimal for (31), and thus, for (28). If the solution \mathbf{W}^* is of higher rank, we need to build another method to solve (31).

In this subsection, we present a BLMI reformulation for QMI problem (31) and propose an approximation algorithm for such BLMI problem.

Note that for a nonzero $\mathbf{W} \succeq \mathbf{0}$, the statement that \mathbf{W} is of rank one is equivalent to that $\lambda_2(\mathbf{W}) \leq 0$ (recall that throughout the paper, we place the eigenvalues of a matrix in a descending order, e.g., $\lambda_1(\mathbf{W}) \geq \lambda_2(\mathbf{W}) \geq \dots \geq \lambda_N(\mathbf{W})$). Note that an optimal solution for the relaxation problem (32) cannot be zero. In fact, if an optimal solution for (32) is the zero matrix, then $\mathbf{w} = \mathbf{0}$ is optimal for (31), and thus, for (28), which is a contradiction since we maximize the worst-case SINR (it is always assumed to be positive and finite).

Clearly, we have that for the nonzero PSD matrix \mathbf{W} , the following statement is true.

$$\text{Rank}(\mathbf{W}) = 1 \Leftrightarrow \lambda_2(\mathbf{W}) \leq 0 \Leftrightarrow \lambda_1(\mathbf{W}) + \lambda_2(\mathbf{W}) \leq \lambda_1(\mathbf{W}).$$

Therefore, we can put the new rank-one constraint $\lambda_1(\mathbf{W}) + \lambda_2(\mathbf{W}) \leq \lambda_1(\mathbf{W})$ into (32), and obtain a reformulation for (31),

that is,

$$\begin{aligned} & \text{minimize} && \text{tr}(\hat{\mathbf{R}}\mathbf{W}) \\ & \mathbf{W}, \{y_i\}_{i=1}^4 && \\ & \text{subject to} && (29\text{b}), (29\text{c}), (29\text{d}) \text{ satisfied,} \\ & && y_1 + y_2 + y_3 \leq 0, \\ & && \lambda_1(\mathbf{W}) + \lambda_2(\mathbf{W}) \leq \lambda_1(\mathbf{W}). \end{aligned} \quad (45)$$

It is worth highlighting that this method is universal, namely, the constraint $\lambda_1(\mathbf{W}) + \lambda_2(\mathbf{W}) \leq \lambda_1(\mathbf{W})$ can be added to an LMI relaxation problem for which one seeks a nonzero rank-one solution. Note that this approach based on adding the constraint $\lambda_1(\mathbf{W}) + \lambda_2(\mathbf{W}) \leq \lambda_1(\mathbf{W})$ to an LMI relaxation problem is different from another approach in [23, pp. 11, Algorithm 2.1], where the rank reduction algorithm for SDPs is developed based on the observation that updated solution with reduced rank can be obtained by multiplying on the middle the factorized higher rank solution by a non-negative definite and singular matrix $\mathbf{I} + \alpha\mathbf{\Delta}$. Here $\mathbf{\Delta}$ and α are unknowns, which need to be found so that the non-negative definiteness and singularity of $\mathbf{I} + \alpha\mathbf{\Delta}$ are guaranteed. Thus, the approach in [23] is also based on factorization of the general rank solution, but the rank reduction is achieved in a different manner from that of solving (45).

In order to represent the constraint $\lambda_1(\mathbf{W}) + \lambda_2(\mathbf{W}) \leq \lambda_1(\mathbf{W})$, we invoke a result in [21, Sec. 4.2, Example 18c], which states the equivalence between the sum of K largest eigenvalues of a real symmetric matrix and a finite LMI representation. For the completeness, we cite the result as follows.

Lemma III.9: Suppose that $S_K(\mathbf{W})$ is the sum of K largest eigenvalues of the Hermitian matrix \mathbf{W} ($K \leq N$). Then the epigraph $\{(\mathbf{W}, t) \mid S_K(\mathbf{W}) \leq t\}$ of the function admits the LMI representation

$$t - Ks - \text{tr}(\mathbf{Z}) \geq 0 \quad (46)$$

$$\mathbf{Z} - \mathbf{W} + s\mathbf{I} \succeq \mathbf{0} \quad (47)$$

$$\mathbf{Z} \succeq \mathbf{0}, \quad (48)$$

where \mathbf{Z} is an $N \times N$ Hermitian matrix and s is a real number.

We note that the original result [21] holds for a symmetric matrix \mathbf{W} , but it can be extended to a Hermitian matrix straightforwardly, as already stated in the previous lemma.

Therefore, it follows from Lemma III.9 that the constraint $\lambda_1(\mathbf{W}) + \lambda_2(\mathbf{W}) \leq \lambda_1(\mathbf{W})$ has the following LMI representation.

$$\lambda_1(\mathbf{W}) - 2s - \text{tr}(\mathbf{Z}) \geq 0, \mathbf{Z} - \mathbf{W} + s\mathbf{I} \succeq \mathbf{0}, \text{ and } \mathbf{Z} \succeq \mathbf{0}, \quad (49)$$

where \mathbf{Z} and s are additional variables. Considering that

$$\lambda_1(\mathbf{W}) = \text{maximize } \text{tr}(\mathbf{W}\mathbf{X}) \text{ subject to } \text{tr } \mathbf{X} = 1, \mathbf{X} \succeq \mathbf{0},$$

we can reexpress QMI problem (45) into the following problem

$$\begin{aligned} & \text{minimize} && \text{tr}(\hat{\mathbf{R}}\mathbf{W}) \\ & \mathbf{W}, \mathbf{X}, \mathbf{Z}, \{y_i\}_{i=1}^4, s && \\ & \text{subject to} && (29\text{b}), (29\text{c}), (29\text{d}) \text{ satisfied,} \\ & && y_1 + y_2 + y_3 \leq 0, \\ & && \text{tr}(\mathbf{W}\mathbf{X}) - 2s - \text{tr}(\mathbf{Z}) \geq 0, \\ & && \mathbf{Z} - \mathbf{W} + s\mathbf{I} \succeq \mathbf{0}, \\ & && \text{tr } \mathbf{X} = 1, \\ & && \mathbf{Z} \succeq \mathbf{0}, \mathbf{X} \succeq \mathbf{0}, s \in \mathbb{R}. \end{aligned} \quad (50)$$

Algorithm 1: Procedure for Solving Problem (28).

Input: $\hat{\mathbf{R}}, \hat{\mathbf{a}}, \eta_1, \eta_2, \epsilon, \xi$;

Output: A solution \mathbf{w}^* for problem (28);

- 1: solve (32), returning \mathbf{W}^* ;
 - 2: **if** \mathbf{W}^* is of rank one **then**
 - 3: output \mathbf{w}^* with $\mathbf{W}^* = \mathbf{w}^*\mathbf{w}^{*H}$, and terminate;
 - 4: **end if**
 - 5: let $k = 0$; let \mathbf{W}_0 be the optimal solution \mathbf{W}^* for (32);
 - 6: **repeat**
 - 7: solve (50) with the fourth constraint changed to $\text{tr}(\mathbf{W}_k\mathbf{X}) - 2s - \text{tr}(\mathbf{Z}) \geq 0$, obtaining the solution \mathbf{W}_{k+1} ;
 - 8: $k := k + 1$;
 - 9: **until** $\|\mathbf{W}_k - \mathbf{W}_{k-1}\|_2 \leq \xi$
 - 10: output $\mathbf{w}^* = \sqrt{\lambda_1}\mathbf{w}_1$, where λ_1 is the largest eigenvalue of \mathbf{W}_k and \mathbf{w}_1 is a corresponding eigenvector.
-

Suppose that \mathbf{W}^* is the optimal solution of (50). It is noticed that \mathbf{W}^* must be a rank-one since $\lambda_2(\mathbf{W}^*) \leq 0$ (or $\lambda_1(\mathbf{W}^*) + \lambda_2(\mathbf{W}^*) \leq \lambda_1(\mathbf{W}^*)$) has been fulfilled automatically in (50). Therefore, the vector \mathbf{w}^* with $\mathbf{W}^* = \mathbf{w}^*\mathbf{w}^{*H}$ is optimal for (28), and thus, for original RAB problem (10) with $\mathcal{A} = \mathcal{A}_1$.

We observe that the only difficulty in (50) is the bilinear term $\text{tr}(\mathbf{W}\mathbf{X})$ in the fourth constraint (for this reason, the problem is called a BLMI problem). To overcome the aforementioned difficulty with (50), we take the high-rank solution \mathbf{W}^* for (32) as an initial point \mathbf{W}_0 , and solve LMI problem (50) with the fourth constraint changed to $\text{tr}(\mathbf{W}_0\mathbf{X}) - 2s - \text{tr}(\mathbf{Z}) \geq 0$, obtaining \mathbf{W}_1 . Set $\mathbf{W}_0 = \mathbf{W}_1$ and repeat the step of solving (50) with the fourth constraint changed, obtaining \mathbf{W}_2 . Repeat the previous two steps, until a stopping criteria (for example, $\|\mathbf{W}_k - \mathbf{W}_{k-1}\|_2 \leq \xi$, or $|\lambda_1(\mathbf{W}_k) - \lambda_1(\mathbf{W}_{k-1})| \leq \xi$) is satisfied. We summarize the procedure for solving (28) in Algorithm 1.

If the rank of \mathbf{W}_k in Step 10 of Algorithm 1 is one, then \mathbf{w}^* with $\mathbf{W}^* = \mathbf{w}^*\mathbf{w}^{*H}$ is the solution for (28). Otherwise, the solution is $\mathbf{w}^* = \sqrt{\lambda_1}\mathbf{w}_1$ as stated in Step 10. The computational cost of Algorithm 1 is dominated by solving the SDP problem in Step 7 of each iteration.

The complexity of Algorithm 1, if \mathbf{W}^* has rank 1, is equivalent to the complexity of solving SDP problem (32) with $N(N+1)/2 + 3$ optimization variables. Then the worst-case complexity of solving (32) by standard interior-point methods is of $\mathcal{O}((0.5N^2)^{4.5})$ [14], [24]. If \mathbf{W}^* has a higher rank, SDP problem (50) with $3N(N+1)/2 + 5$ optimization variables needs to be solved additionally. The worst-case complexity of solving (50) is of $\mathcal{O}((1.5N^2)^{4.5})$. Thus, the order of complexity is comparable to that of the MVDR robust adaptive beamformer of [19] where the constraints are convex, but also SDPs with $N(N+1)/2$ optimization variables need to be solved. There is only a constant factor difference of complexities in different cases.

Note that it is possible that LMI problem (50) with the fourth constraint changed to $\text{tr}(\mathbf{W}_k\mathbf{X}) - 2s - \text{tr}(\mathbf{Z}) \geq 0$ leads to

infeasibility due to several constraints added. In this case, we set $\mathbf{W}_k := 2\mathbf{W}_k$ and repeat Step 7. By doing so, we wish to provide a way to enlarge the feasible set of (50). Our simulations show that a rank-one solution $\mathbf{W}^* := \mathbf{W}_k$ in Step 10 can always be obtained.

IV. SOLVING ROBUST OPTIMIZATION PROBLEM (18) WITH UNCERTAINTY SET \mathcal{A}_2 IN (21)

In this section, we study robust adaptive beamforming problem via worst-case SINR maximization with nonconvex uncertainty set \mathcal{A}_2 and its extensions.

A. Robust Optimization Problem (18) With Uncertainty Set \mathcal{A}_2

Let us first look into the minimization problem in the constraint of robust optimization problem (18), when $\mathcal{A} = \mathcal{A}_2$. The problem then can be written as

$$\begin{aligned} & \underset{\mathbf{a} \in \mathbb{C}^N}{\text{minimize}} && \mathbf{a}^H \mathbf{w} \mathbf{w}^H \mathbf{a} \\ & \text{subject to} && \mathbf{a}^H \bar{\mathbf{C}} \mathbf{a} \leq \Delta_0, \\ & && N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2. \end{aligned} \quad (51)$$

From the feasibility of (51), we have

$$\lambda_N(\bar{\mathbf{C}}) \leq \frac{\mathbf{a}^H \bar{\mathbf{C}} \mathbf{a}}{\|\mathbf{a}\|^2} \leq \frac{\Delta_0}{N - \eta_1};$$

in other words,

$$\lambda_N(\bar{\mathbf{C}})(N - \eta_1) \leq \Delta_0. \quad (52)$$

On the other hand, if Δ_0 is sufficiently large, then the constraint $\mathbf{a}^H \bar{\mathbf{C}} \mathbf{a} \leq \Delta_0$ vanishes in the feasible set since it is fulfilled always. We claim that if $\Delta_0 \geq \lambda_1(\bar{\mathbf{C}})(N + \eta_2)$, then it follows that $\mathbf{a}^H \bar{\mathbf{C}} \mathbf{a} \leq \Delta_0$ always holds. In fact, $\Delta_0 \geq \lambda_1(\bar{\mathbf{C}})(N + \eta_2) \geq \lambda_1(\bar{\mathbf{C}})\|\mathbf{a}\|^2$, which means that $\Delta_0/\|\mathbf{a}\|^2 \geq \lambda_1(\bar{\mathbf{C}}) \geq \mathbf{a}^H \bar{\mathbf{C}} \mathbf{a}/\|\mathbf{a}\|^2$. In other words, $\Delta_0 \geq \mathbf{a}^H \bar{\mathbf{C}} \mathbf{a}$. Therefore, in order to make the first inequality constraint in (51) hold, it should be satisfied that

$$\lambda_N(\bar{\mathbf{C}})(N - \eta_1) \leq \Delta_0 \leq \lambda_1(\bar{\mathbf{C}})(N + \eta_2), \quad (53)$$

considering (52).

Suppose that there is a vector \mathbf{a}_0 such that $\mathbf{a}_0^H \bar{\mathbf{C}} \mathbf{a}_0 < \Delta_0$ and $\|\mathbf{a}_0\|^2 = N$. In other words, problem (51) is strictly feasible.

The SDP relaxation for (51) is

$$\begin{aligned} & \underset{\mathbf{X} \in \mathcal{H}^N}{\text{minimize}} && \text{tr}(\mathbf{w} \mathbf{w}^H \mathbf{X}) \\ & \text{subject to} && \text{tr}(\bar{\mathbf{C}} \mathbf{X}) \leq \Delta_0, \\ & && N - \eta_1 \leq \text{tr} \mathbf{X} \leq N + \eta_2, \\ & && \mathbf{X} \succeq \mathbf{0} \end{aligned} \quad (54)$$

and its dual problem can be written as

$$\begin{aligned} & \underset{x, y_1, y_2}{\text{maximize}} && \Delta_0 x + (N - \eta_1)y_1 + (N + \eta_2)y_2 \\ & \text{subject to} && \mathbf{w} \mathbf{w}^H - x\bar{\mathbf{C}} - (y_1 + y_2)\mathbf{I} \succeq \mathbf{0}, \\ & && x \leq 0, y_1 \geq 0, y_2 \leq 0. \end{aligned} \quad (55)$$

It can be seen easily that the dual SDP is strictly feasible. The primal SDP is also strictly feasible since the point $\epsilon \mathbf{I} + (1 - \epsilon)\mathbf{a}_0 \mathbf{a}_0^H$ is a strictly feasible point for a sufficiently small $\epsilon > 0$. Therefore, it follows again from the strong duality theorem

that problems (54) and (55) are solvable and there is zero gap between them. In addition, there are only two constraints in the primal SDP, and then a rank-one solution for it can be always constructed efficiently (see, e.g., [22]). Thereby, SDP relaxation (54) is tight, and we have $v^*((51)) = v^*((54)) = v^*((55))$.

Therefore, robust beamforming problem (18) with \mathcal{A}_2 can be recast into the following QMI problem:

$$\underset{\mathbf{w}, x, y_1, y_2}{\text{minimize}} \quad \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad (56a)$$

$$\text{subject to} \quad \Delta_0 x + (N - \eta_1)y_1 + (N + \eta_2)y_2 = 1, \quad (56b)$$

$$\mathbf{w} \mathbf{w}^H - x\bar{\mathbf{C}} - (y_1 + y_2)\mathbf{I} \succeq \mathbf{0}, \quad (56c)$$

$$x \leq 0, y_1 \geq 0, y_2 \leq 0. \quad (56d)$$

Its LMI relaxation can be written as

$$\underset{\mathbf{W}, x, y_1, y_2}{\text{minimize}} \quad \text{tr}(\hat{\mathbf{R}} \mathbf{W}) \quad (57a)$$

$$\text{subject to} \quad \Delta_0 x + (N - \eta_1)y_1 + (N + \eta_2)y_2 = 1, \quad (57b)$$

$$\mathbf{W} - x\bar{\mathbf{C}} - (y_1 + y_2)\mathbf{I} \succeq \mathbf{0}, \quad (57c)$$

$$\mathbf{W} \succeq \mathbf{0}, x \leq 0, y_1 \geq 0, y_2 \leq 0. \quad (57d)$$

If the solution of LMI problem (57) is of rank one, then problem (56) is solved. Otherwise, we consider a tightened LMI relaxation by adding one more valid linear constraint, applying the similar idea demonstrated in the previous section.

B. Tightened LMI Relaxation

The second constraint of (56) implies

$$x\lambda_{N-1}(\bar{\mathbf{C}}) + (y_1 + y_2) \leq 0, \quad (58)$$

due to $x \leq 0$. Therefore, putting the additional linear constraint (58) into (56) does not change the set of optimal solutions. Namely, the following problem

$$\begin{aligned} & \underset{\mathbf{w}, x, y_1, y_2}{\text{minimize}} && \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \\ & \text{subject to} && (56b), (56c), (56d) \text{ satisfied}, \\ & && x\lambda_{N-1}(\bar{\mathbf{C}}) + y_1 + y_2 \leq 0, \end{aligned} \quad (59)$$

shares the same optimal solution set and the same optimal value with problem (56). Thus, we obtain the following restricted LMI relaxation problem

$$\begin{aligned} & \underset{\mathbf{W}, x, y_1, y_2}{\text{minimize}} && \text{tr}(\hat{\mathbf{R}} \mathbf{W}) \\ & \text{subject to} && (57b), (57c), (57d) \text{ satisfied}, \\ & && x\lambda_{N-1}(\bar{\mathbf{C}}) + y_1 + y_2 \leq 0. \end{aligned} \quad (60)$$

Hence, if rank-one solution $\mathbf{w}^* \mathbf{w}^{*H}$ is optimal for (60), then the vector \mathbf{w}^* is optimal for (59), and thus, for (56).

Clearly, the dual of LMI problem (60) is

$$\begin{aligned} & \underset{\mathbf{Z}, z_1, z_2}{\text{maximize}} && z_1 \\ & \text{subject to} && \hat{\mathbf{R}} - \mathbf{Z} \succeq \mathbf{0}, \\ & && \text{tr}(\bar{\mathbf{C}} \mathbf{Z}) - \Delta_0 z_1 + \lambda_{N-1}(\bar{\mathbf{C}})z_2 \leq 0, \\ & && \text{tr} \mathbf{Z} + z_2 \leq z_1(N + \eta_2), \\ & && \text{tr} \mathbf{Z} + z_2 \geq z_1(N - \eta_1), \\ & && \mathbf{Z} \succeq \mathbf{0}, z_1 \in \mathbb{R}, z_2 \geq 0. \end{aligned} \quad (61)$$

Suppose that the primal and dual SDPs are solvable, and possess the same optimal value (assumed to be positive and finite). Thus, the complementary conditions include

$$\text{tr}((\hat{\mathbf{R}} - \mathbf{Z})\mathbf{W}) = 0 \quad (62)$$

$$x(\text{tr}(\bar{\mathbf{C}}\mathbf{Z}) - \Delta_0 z_1 + \lambda_{N-1}(\bar{\mathbf{C}})z_2) = 0 \quad (63)$$

$$y_1(\text{tr}(\mathbf{Z} + z_2 - (N - \eta_1)z_1)) = 0 \quad (64)$$

$$y_2(\text{tr}(\mathbf{Z} + z_2 - (N + \eta_2)z_1)) = 0 \quad (65)$$

$$z_2(x\lambda_{N-1}(\bar{\mathbf{C}}) + y_1 + y_2) = 0 \quad (66)$$

$$\text{tr}(\mathbf{Z}(\mathbf{W} - x\bar{\mathbf{C}} - (y_1 + y_2)\mathbf{I})) = 0. \quad (67)$$

It follows that $\text{tr}(\hat{\mathbf{R}}\mathbf{W}) = z_1$ (i.e., the duality gap is zero). Based on the complementary conditions, we claim some necessary optimality conditions for (60) and (61).

Proposition IV.1: Suppose that $(\mathbf{W}, x, y_1, y_2)$ and (\mathbf{Z}, z_1, z_2) are the solutions for primal SDP (60) and dual SDP (61), respectively. If $x\lambda_{N-1}(\bar{\mathbf{C}}) + y_1 + y_2 < 0$, then it holds that

- 1) $x < 0$;
- 2) $z_1 = \frac{\text{tr}(\bar{\mathbf{C}}\mathbf{Z})}{\Delta_0}$ and $z_2 = 0$;
- 3) $y_1 + y_2 > 0$.

Proof: See Appendix F.

Proposition IV.2: Suppose that $(\mathbf{W}, x, y_1, y_2)$ and (\mathbf{Z}, z_1, z_2) are the solutions for primal SDP (60) and dual SDP (61), respectively. If $x\lambda_{N-1}(\bar{\mathbf{C}}) + y_1 + y_2 = 0$, then the following statements are true:

- 1) $x < 0$;
- 2) $z_1 = \frac{\text{tr}(\bar{\mathbf{C}}\mathbf{Z}) + z_2\lambda_{N-1}(\bar{\mathbf{C}})}{\Delta_0}$;
- 3) $y_1 + y_2 > 0$;
- 4) $\lambda_1(\mathbf{W}) \geq -x(\lambda_{N-1}(\bar{\mathbf{C}}) - \lambda_N(\bar{\mathbf{C}}))$;
- 5) if $\mathbf{W} = \mathbf{w}\mathbf{w}^H$, then the optimal value $\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \geq -x(\lambda_{N-1}(\bar{\mathbf{C}})\lambda_N(\hat{\mathbf{R}}) - \lambda_N(\hat{\mathbf{R}}\bar{\mathbf{C}}))$.

Proof: See Appendix G.

In particular, when the solution $\mathbf{W} = \mathbf{w}\mathbf{w}^H$ is of rank-one, the fourth statement of Proposition IV.2 implies that $\|\mathbf{w}\|^2$ has the lower bound $-x(\lambda_{N-1}(\bar{\mathbf{C}}) - \lambda_N(\bar{\mathbf{C}}))$, and when $\hat{\mathbf{R}} = \mathbf{I}$, the fifth statement means the same lower bound for $\|\mathbf{w}\|^2$.

To establish a sufficient condition for (60) to have a rank-one solution, we give the following theorem.

Theorem IV.3: Suppose that $(\mathbf{W}, x, y_1, y_2)$ and (\mathbf{Z}, z_1, z_2) are the solutions for primal SDP (60) and dual SDP (61), respectively. If $\text{tr}(\hat{\mathbf{R}}^{-1}\mathbf{Z}) \leq 1$, then \mathbf{W} must be of rank one.

Proof: See Appendix H.

C. BLMI Approximation Approach

Suppose that the solution \mathbf{W} for (60) has rank higher than one. Therefore, we can apply the BLMI approximation method to obtain a solution for (59), i.e., also (56). Evidently, QMI problem (59) can be rewritten as

$$\begin{aligned} & \underset{\mathbf{W}, x, y_1, y_2}{\text{minimize}} && \text{tr}(\hat{\mathbf{R}}\mathbf{W}) \\ & \text{subject to} && (57b), (57c), (57d) \text{ satisfied,} \\ & && x\lambda_{N-1}(\bar{\mathbf{C}}) + y_1 + y_2 \leq 0, \\ & && \lambda_1(\mathbf{W}) + \lambda_2(\mathbf{W}) \leq \lambda_1(\mathbf{W}). \end{aligned} \quad (68)$$

Algorithm 2: Procedure for Solving Problem (56).

Input: $\hat{\mathbf{R}}, \bar{\mathbf{C}}, \Delta_0, \eta_1, \eta_2, \xi$;

Output: A solution \mathbf{w}^* for problem (56);

- 1: solve (60), returning \mathbf{W}^* ;
 - 2: **if** \mathbf{W}^* is of rank one **then**
 - 3: output \mathbf{w}^* with $\mathbf{W}^* = \mathbf{w}^*\mathbf{w}^{*H}$, and terminate;
 - 4: **end if**
 - 5: let $k = 0$; let \mathbf{W}_k be the optimal solution \mathbf{W}^* for (60);
 - 6: **repeat**
 - 7: solve (69) with the fourth constraint changed to $\text{tr}(\mathbf{W}_k\mathbf{X}) - 2s - \text{tr}(\mathbf{Z}) \geq 0$, obtaining the solution \mathbf{W}_{k+1} ;
 - 8: $k := k + 1$;
 - 9: **until** $\|\mathbf{W}_k - \mathbf{W}_{k-1}\|_2 \leq \xi$
 - 10: output $\mathbf{w}^* = \sqrt{\lambda_1}\mathbf{w}_1$, where λ_1 is the largest eigenvalue of \mathbf{W}_k and \mathbf{w}_1 is a corresponding eigenvector.
-

Moreover, problem (68) can be further recast into

$$\begin{aligned} & \underset{\mathbf{W}, \mathbf{X}, \mathbf{Z}, x, y_1, y_2, s}{\text{minimize}} && \text{tr}(\hat{\mathbf{R}}\mathbf{W}) \\ & \text{subject to} && (57b), (57c), (57d) \text{ satisfied,} \\ & && x\lambda_{N-1}(\bar{\mathbf{C}}) + y_1 + y_2 \leq 0, \\ & && \text{tr}(\mathbf{W}\mathbf{X}) - 2s - \text{tr}(\mathbf{Z}) \geq 0, \\ & && \mathbf{Z} - \mathbf{W} + s\mathbf{I} \succeq \mathbf{0}, \\ & && \text{tr} \mathbf{X} = 1, \\ & && \mathbf{Z} \succeq \mathbf{0}, \mathbf{X} \succeq \mathbf{0}, s \in \mathbb{R}. \end{aligned} \quad (69)$$

This problem can be solved in a similar way as problem (50) is solved. We summarize the procedure in Algorithm 2.

If the solution in Step 10 $\mathbf{W}_k = \mathbf{w}^*\mathbf{w}^{*H}$ is of rank one, then \mathbf{w}^* is a solution for (56). Otherwise, we regard $\sqrt{\lambda_1}\mathbf{w}_1$ as a solution for (56), where λ_1 is the largest eigenvalue and \mathbf{w}_1 is a corresponding eigenvector. If the problem solved in Step 7 is infeasible, we adopt the same strategy as in Subsection III-C to enlarge the feasible set and continue to solve it in an approximate way. Similar to Algorithm 1, the complexity of Algorithm 2, if \mathbf{W}^* has rank 1, is equivalent to the complexity of solving SDP problem (56) only with $N(N+1)/2 + 3$ optimization variables. If \mathbf{W}^* has a higher rank, SDP problem (69) with $3N(N+1)/2 + 4$ optimization variables needs to be solved additionally.

In order to solve robust problem (10) with the uncertainty set \mathcal{A}_3 , we note that

$$\mathcal{A}_3 = \{\mathbf{a} \mid \mathbf{a}^H \mathbf{C} \mathbf{a} \geq \Delta_1, N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2\}$$

is equivalent to

$$\mathcal{A}_3 = \{\mathbf{a} \mid \mathbf{a}^H(-\mathbf{C})\mathbf{a} \leq -\Delta_1, N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2\}. \quad (70)$$

In this case, the valid linear constraint is

$$-x\lambda_2(\mathbf{C}) + y_1 + y_2 \leq 0,$$

which can be added to enhance the LMI relaxation problem. Therefore, Algorithm 2 can be applied to solve the problem as well.

By adding one more similarity constraint, the uncertainty set \mathcal{A}_2 can be generalized to the following set

$$\mathcal{A}_4 = \left\{ \mathbf{a} \mid \mathbf{a}^H \bar{\mathbf{C}} \mathbf{a} \leq \Delta_0, N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2, \|\mathbf{a} - \hat{\mathbf{a}}\|^2 \leq \epsilon \right\}, \quad (71)$$

and the uncertainty set \mathcal{A}_3 can be extended as the following set

$$\mathcal{A}_5 = \left\{ \mathbf{a} \mid \mathbf{a}^H \mathbf{C} \mathbf{a} \geq \Delta_1, N - \eta_1 \leq \|\mathbf{a}\|^2 \leq N + \eta_2, \|\mathbf{a} - \hat{\mathbf{a}}\|^2 \leq \epsilon \right\}. \quad (72)$$

In [19], MVDR RAB problems with \mathcal{A}_4 and \mathcal{A}_5 have been studied. However, robust beamforming problem (10) with either \mathcal{A}_4 or \mathcal{A}_5 can be discussed as well, according to the same vein applied herein. Then, Algorithm 2 modified accordingly can also be employed to solve it.

V. NUMERICAL EXAMPLES

Let us consider a uniform linear array with $N = 12$ omnidirectional sensors spaced half a wavelength apart from each other, and the array noise is a spatially and temporally white Gaussian vector with zero mean and covariance matrix \mathbf{I} , which is an adequate assumption in radar and wireless communications. Two interferers with the same interference-to-noise ratio (INR) of 30 dB are assumed to impinge upon the array from the angles $\theta_1 = -15^\circ$ and $\theta_2 = 15^\circ$ with respect to the array broadside, and the desired signal is always present in the training data cell. The training sample size T is preset to 100 snapshots. The signal of interest impinges upon the array from the direction $\theta = 7^\circ$ while the presumed direction is $\theta_0 = 5^\circ$ (thus $\hat{\mathbf{a}} = \mathbf{d}(\theta_0)$). The norm perturbation parameters η_1 and η_2 are both set to $0.2N$, and $\epsilon = 0.3N$. All results are averaged over 200 simulation runs. Parameter γ in (14) satisfies $\sqrt{\gamma} = 0.1\lambda_N(\hat{\mathbf{R}})$, where the data sample covariance matrix $\hat{\mathbf{R}}$ is different in each run.

In addition to the signal look direction mismatch, we take into account mismatch caused also by wavefront distortion in an inhomogeneous medium [18]. That is, we assume that the signal steering vector is distorted by wave propagation effects in the way that independent-increment phase distortions are accumulated by the components of the steering vector. We assume that the phase increments are independent Gaussian variables each with zero mean and standard deviation 0.02, and they are randomly generated and remain unaltered in each simulation run.

A. Example 1: Uncertainty Set \mathcal{A}_1 (i.e., (19))

In this example, the three problems, i.e., problem (28) (via Algorithm 1), problem (13) (also cf. [10]) and MVDR RAB problem (5) with uncertain set \mathcal{A}_1 (via, e.g., [19, Algorithm 1]), are solved in every simulation run. The beamformers obtained from the aforementioned three methods are termed respectively as ‘‘Proposed QMI beamformer 1,’’ ‘‘SOCP beamformer,’’ and ‘‘MVDR RAB beamformer 1’’ in our figures. The standard CVX toolbox [25] is used for solving the corresponding optimization problems. Since the solver used for solving SOCP and SDP problems in CVX is SeDuMi [26], which is based on interior-point

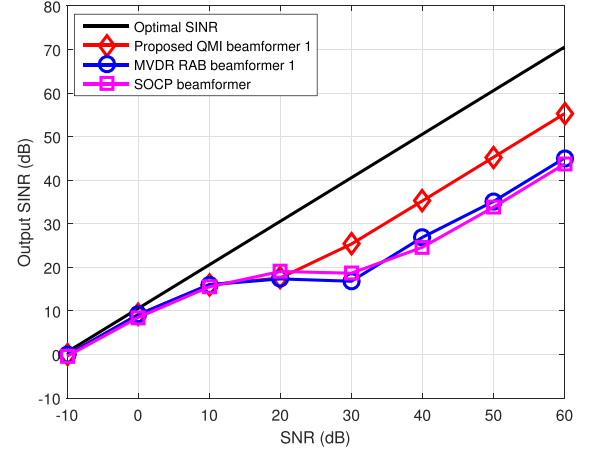


Fig. 1. Average beamformer output SINR versus SNR with $T = 100$. Proposed QMI beamformer 1 corresponds to problem (28) (via Algorithm 1), MVDR RAB beamformer 1 corresponds to problem (5) with uncertainty set \mathcal{A}_1 (via, e.g., [19, Algorithm 1]), and SOCP beamformer corresponds to problem (13) (also cf. [10]).

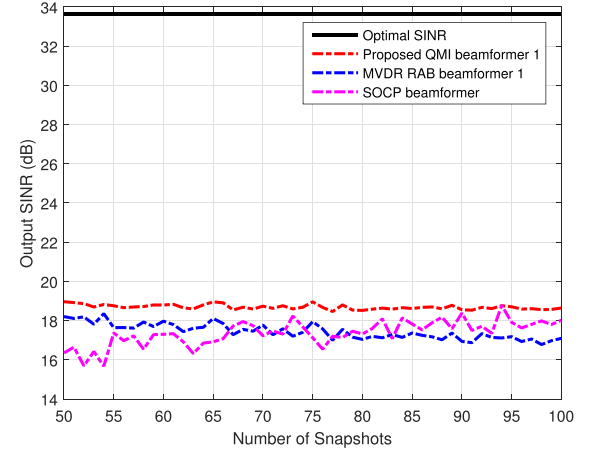


Fig. 2. Average array output power versus number of snapshots with SNR equal to 23 dB. Proposed QMI beamformer 1 corresponds to problem (28) (via Algorithm 1), MVDR RAB beamformer 1 corresponds to problem (5) with uncertainty set \mathcal{A}_1 (via, e.g., [19, Algorithm 1]), and SOCP beamformer corresponds to problem (13) (also cf. [10]).

methods, the worst-case complexities are $\mathcal{O}(N^{3.5})$ for SOCP beamformer and $\mathcal{O}((N^2)^{4.5})$ for MVDR RAB and proposed QMI beamformers. The factor differences of complexities for different instances of MVDR RAB and QMI beamformers are negligible even for $N = 12$ since the number of optimization variables is dominated by $N^2 = 12^2 = 144$ as a result of SDP relaxation for MVDR RAB and QMI beamformers. Thus, the constant factor differences between different instances of these problems account in all cases for less than a small fraction of 1% of all computations required for finding the corresponding solutions.

Fig. 1 demonstrates the beamformer output SINR versus the signal-to-noise ratio (SNR). As we can see, the output SINR obtained through (28) is better than that obtained through the other two beamformers, especially at moderate and high SNRs. Fig. 2 depicts the output SINR versus the number of snapshots with SNR equal to 23 dB ($= 10 \log_{10}(200)$). From the figure, we

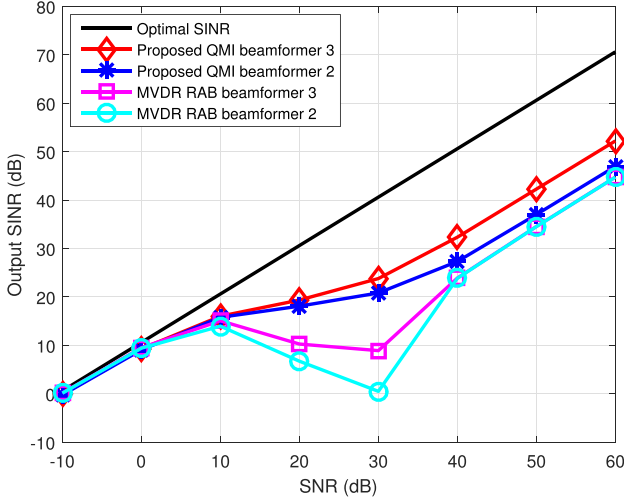


Fig. 3. Average beamformer output SINR versus SNR with $\Theta = [0^\circ, 10^\circ]$ and $T = 100$. Proposed QMI beamformer 2 corresponds to problem (10) with $\mathcal{A} = \mathcal{A}_2$ (i.e., QMI problem (56), via Algorithm 2), Proposed QMI beamformer 3 corresponds to problem (10) with $\mathcal{A} = \mathcal{A}_3$ (i.e., QMI problem (56) with (70), via Algorithm 2), MVDR RAB beamformers 2 and 3 correspond to problem (5) with $\mathcal{A} = \mathcal{A}_2$ and $\mathcal{A} = \mathcal{A}_3$, respectively, solved via [19, Algorithm 1].

observe that our proposed beamformer always provides higher SINR than the other two beamformers.

B. Example 2: Uncertainty Sets \mathcal{A}_2 (i.e., (21)) and \mathcal{A}_3 (i.e., (22))

Suppose that the angular sector Θ of interest is $[0^\circ, 10^\circ]$. Problem (10) with $\mathcal{A} = \mathcal{A}_2$ (i.e., QMI problem (56), via Algorithm 2), and problem (10) with $\mathcal{A} = \mathcal{A}_3$ (i.e., QMI problem (56) with (70), via Algorithm 2) are solved, as well as MVDR RAB problems (5) with $\mathcal{A} = \mathcal{A}_2$ and $\mathcal{A} = \mathcal{A}_3$ are solved via [19, Algorithm 1] in a similar way. The corresponding beamformers from the previous four problems are called respectively, “Proposed QMI beamformer 2,” “Proposed QMI beamformer 3,” “MVDR RAB beamformer 2,” and “MVDR RAB beamformer 3” in our figures. Fig. 3 displays the array output SINR versus the SNR. We can observe from the figure that the output SINR through the worst-case SINR maximization based beamformer (the proposed QMI beamformer 2 or 3) is higher than that obtained through the MVDR RAB beamformer 2 or 3. Also, it can be seen that the beamformer (either the QMI or MVDR RAB beamformer) obtained by corresponding robust problem with \mathcal{A}_3 has better performance than the beamformers obtained by solving the robust problem with \mathcal{A}_2 . Fig. 4 shows the output SINR versus the number of snapshots when SNR=23 dB. From the figure, we can clearly confirm similar behaviors to those in Fig. 3 for the four different beamformers. Moreover, as the number of snapshots increases the performance of the proposed QMI beamformers 2 and 3 also improves by 2–3 dB. Note that by the classic Reed-Mallett-Brennan rule [27], the number of snapshots required for the data covariance matrix estimation which would guaranteed the performance loss of no more than 3 dB is $T = 2N$. In our example, $T = 24$ snapshots is sufficient, and 2–3 dB improvement when the number of snapshots increases to 100 is the possible improvement with

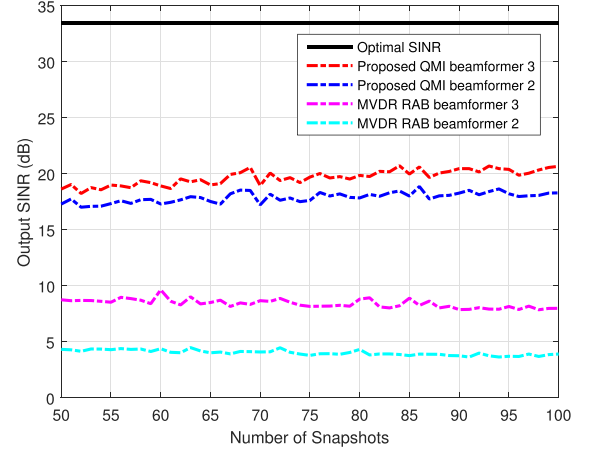


Fig. 4. Average array output power versus number of snapshots with $\Theta = [0^\circ, 10^\circ]$ and SNR equal to 23 dB. Proposed QMI beamformer 2 corresponds to problem (10) with $\mathcal{A} = \mathcal{A}_2$ (i.e., QMI problem (56), via Algorithm 2), Proposed QMI beamformer 3 corresponds to problem (10) with $\mathcal{A} = \mathcal{A}_3$ (i.e., QMI problem (56) with (70), via Algorithm 2), MVDR RAB beamformers 2 and 3 correspond to problem (5) with $\mathcal{A} = \mathcal{A}_2$ and $\mathcal{A} = \mathcal{A}_3$, respectively, solved via [19, Algorithm 1].

respect to the number of snapshots. Only the proposed QMI beamformers 2 and 3 demonstrate this improvement.

VI. CONCLUSION

We have considered the RAB problem by maximizing worst-case SINR with two types of nonconvex uncertainty sets for steering vectors. We have reformulated the SINR maximization problem as QMI problems, and proposed tightened LMI relaxations for them. Some necessary and sufficient conditions for the LMI relaxation problems to admit a rank-one solution have been established. Then the LMI problems with the rank-one constraint have been further recast into corresponding BLMI problems, and an algorithm to solve such BLMI problems has been proposed, returning an optimal/suboptimal solution for the original RAB problem. The improved performance of the proposed robust beamformers has been demonstrated by simulations in terms of the array output SINR.

Future related research topics include a general form of the uncertainty set for steering vectors (with, e.g., three or four double-sided quadratic constraints), general-rank useful signal models (rather than the rank-one signal model considered herein), and other efficient methods to deal with the problem of a rank-one solution for the LMI relaxation problems.

APPENDIX

A. Proof of Lemma III.1

Proof: It can be easily seen that $\hat{\mathbf{a}}$ is an interior point of the uncertainty set \mathcal{A}_1 . Let

$$\mathbf{X}(\lambda) = (1 - \lambda) \begin{bmatrix} \hat{\mathbf{a}}\hat{\mathbf{a}}^H & \hat{\mathbf{a}} \\ \hat{\mathbf{a}}^H & 1 \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \lambda \in (0, 1).$$

It can be seen that $\mathbf{X}(\lambda) \succ \mathbf{0}$ for $0 < \lambda < 1$. It is also easy to check that $\text{tr}(\mathcal{A}_1 \mathbf{X}(\lambda)) = -(1 - \lambda)\epsilon + \lambda(\|\hat{\mathbf{a}}\|^2 - \epsilon + N)$, $\text{tr}(\mathcal{A}_2 \mathbf{X}(\lambda)) = N$, and $\text{tr}(\mathcal{A}_3 \mathbf{X}(\lambda)) = 1$ for any $\lambda \in (0, 1)$.

Therefore, for sufficiently small $\lambda > 0$, we have $\text{tr}(\mathbf{A}_1 \mathbf{X}(\lambda)) < 0$, and the matrix $\mathbf{X}(\lambda) \succ \mathbf{0}$ is strictly feasible for SDP (26).

To construct a strictly feasible point for the dual SDP (27), we note the constraint

$$\begin{bmatrix} \mathbf{w}\mathbf{w}^H - (y_1 + y_2 + y_3)\mathbf{I} & y_1\hat{\mathbf{a}} \\ y_1\hat{\mathbf{a}}^H & -y_4 - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) \end{bmatrix} \succeq \mathbf{0}.$$

Let $y_2 = 1$, $y_3 = -1$, y_1 be any negative number, and y_4 be such that

$$-y_4 > y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) + y_1^2\hat{\mathbf{a}}^H(\mathbf{w}\mathbf{w}^H - y_1\mathbf{I})^{-1}\hat{\mathbf{a}}.$$

Then such a feasible point is strictly feasible for dual SDP (27). Thus, we complete the proof.

B. Proof of Proposition III.3

Proof:

1) Suppose that $y_1 = 0$. Then we have $y_2 + y_3 \leq 0$, and

$$\begin{bmatrix} \mathbf{W} - (y_2 + y_3)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -y_4 \end{bmatrix} \succeq \mathbf{0},$$

which implies that $-y_4 \geq 0$. From the first constraint for the primal SDP, it follows that

$$(N - \eta_1)y_2 + (N + \eta_2)y_3 - 1 = -y_4 \geq 0,$$

which means further that

$$0 \geq \eta_2 y_3 \geq 1 + \eta_1 y_2 - N(y_2 + y_3) \geq 1.$$

This is a contradiction, and thus, we have $y_1 < 0$.

2) The third and the fourth constraints of problem (33) allow for the constraints

$$\frac{\text{tr} \mathbf{Z} - x}{N + \eta_2} \leq z_0 \leq \frac{\text{tr} \mathbf{Z} - x}{N - \eta_1}. \quad (73)$$

Therefore, on one hand, $\text{tr} \mathbf{Z} - x > 0$ (otherwise $z_0 \leq 0$, which clearly is impossible according to (41)) while on the other hand, at least one of the two inequality in (73) is strict. It follows from complementary conditions (36) and (37) that at least one of the two numbers y_2 and y_3 is zero. In other words, $y_2 y_3 = 0$.

3) It follows from the assumption that $y_2 + y_3 = -y_1 > 0$. Considering $y_2 \geq 0$ and $y_3 \leq 0$, we have $y_2 > 0$. Therefore, it holds that $y_3 = 0$, since $y_2 y_3 = 0$.

C. Proof of Proposition III.4

Proof:

1) According to (35) and the established fact that $y_1 < 0$ (the first claim in Proposition III.3), we immediately have $\text{tr} \mathbf{Z} - 2\Re(\hat{\mathbf{a}}^H \mathbf{z}_1) + z_0(\|\hat{\mathbf{a}}\|^2 - \epsilon) = x$, and then obtain

$$z_0 = \frac{2\Re(\hat{\mathbf{a}}^H \mathbf{z}_1) - (\text{tr} \mathbf{Z} - x)}{\|\hat{\mathbf{a}}\|^2 - \epsilon}.$$

In addition, it follows from (34), (35), (39), (40) and (41) that

$$z_0 = \frac{y_1 x + (y_2 + y_3)\text{tr} \mathbf{Z}}{1 - y_4},$$

as long as $y_4 \neq 1$.

2) If

$$-y_4 - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) = 0,$$

then $y_1 = 0$ since \mathbf{Q} in (40) is positive semidefinite. This is a contradiction to the established fact that $y_1 < 0$ (the first claim in Proposition III.3), and therefore, we have

$$-y_4 - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) > 0.$$

3) From the first constraint in problem (32) and the earlier established fact that $y_3 = 0$ (the third claim in Proposition III.3), we have

$$-y_4 = y_2(N - \eta_1) - 1.$$

According to the previous claim, we obtain

$$y_2(N - \eta_1) - y_1(\|\hat{\mathbf{a}}\|^2 - \epsilon) > 1.$$

It follows from the fact that $-y_1 = y_2$ (also the third claim in Proposition III.3) that

$$y_2 > \frac{1}{2N - \epsilon - \eta_1}.$$

D. Proof of Theorem III.5

Proof:

1) It follows from (34) that $\mathbf{w}^H(\hat{\mathbf{R}} - \mathbf{Z})\mathbf{w} = 0$, which immediately implies that $(\hat{\mathbf{R}} - \mathbf{Z})\mathbf{w} = \mathbf{0}$ since $\hat{\mathbf{R}} - \mathbf{Z} \succeq \mathbf{0}$.
2) Recalling (42), we obtain

$$-(y_1 + y_2 + y_3)\mathbf{I} \succeq \bar{\mathbf{a}}\bar{\mathbf{a}}^H - \mathbf{w}\mathbf{w}^H. \quad (74)$$

It can be verified that if $\bar{\mathbf{a}}$ and \mathbf{w} are linearly independent, then $\bar{\mathbf{a}}\bar{\mathbf{a}}^H - \mathbf{w}\mathbf{w}^H$ is a rank-two indefinite matrix with eigenvalues

$$\lambda = \frac{\|\bar{\mathbf{a}}\|^2 - \|\mathbf{w}\|^2 \pm \sqrt{(\|\bar{\mathbf{a}}\|^2 + \|\mathbf{w}\|^2)^2 - 4|\bar{\mathbf{a}}^H \mathbf{w}|^2}}{2}. \quad (75)$$

Therefore, due to (74), we have

$$-(y_1 + y_2 + y_3) \geq \frac{\|\bar{\mathbf{a}}\|^2 - \|\mathbf{w}\|^2 + \sqrt{(\|\bar{\mathbf{a}}\|^2 + \|\mathbf{w}\|^2)^2 - 4|\bar{\mathbf{a}}^H \mathbf{w}|^2}}{2},$$

which is the positive eigenvalue of $\bar{\mathbf{a}}\bar{\mathbf{a}}^H - \mathbf{w}\mathbf{w}^H$, and thus, the proof is complete.

E. Proof of Theorem III.7

Proof: Let $(\text{tr}(\hat{\mathbf{R}}\mathbf{W}), \text{tr}(\mathbf{W}), \text{tr}(\bar{\mathbf{a}}\bar{\mathbf{a}}^H \mathbf{W})) = (z_0, z_1, z_2)$, where in fact z_0 is the optimal value. Observe that

$$\text{tr} \left(\left(\hat{\mathbf{R}} - \frac{z_0}{z_1} \mathbf{I} \right) \mathbf{W} \right) = 0$$

and

$$\text{tr} \left(\left(\bar{\mathbf{a}}\bar{\mathbf{a}}^H - \frac{z_2}{z_1} \mathbf{I} \right) \mathbf{W} \right) = 0.$$

It follows from Lemma III.6 that there exists a rank-one matrix decomposition $\mathbf{W} = \sum_{r=1}^R \mathbf{w}_r \mathbf{w}_r^H$ such that

$$\text{tr} \left(\left(\hat{\mathbf{R}} - \frac{z_0}{z_1} \mathbf{I} \right) \mathbf{w}_r \mathbf{w}_r^H \right) = 0$$

and

$$\text{tr} \left(\left(\bar{\mathbf{a}}\bar{\mathbf{a}}^H - \frac{z_2}{z_1}\mathbf{I} \right) \mathbf{w}_r \mathbf{w}_r^H \right) = 0, \forall r.$$

Let

$$\mathbf{w} = \sqrt{z_1} \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}. \quad (76)$$

Then, it is not hard to verify that

$$\left(\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w}, \|\mathbf{w}\|^2, |\bar{\mathbf{a}}^H \mathbf{w}|^2 \right) = (z_0, z_1, z_2). \quad (77)$$

Therefore, we claim that assumption (44) leads to

$$y^2 + y (\|\bar{\mathbf{a}}\|^2 - \|\mathbf{w}\|^2) - (\|\mathbf{w}\|^2 \|\bar{\mathbf{a}}\|^2 - |\bar{\mathbf{a}}^H \mathbf{w}|^2) \geq 0. \quad (78)$$

Hence, it follows that

$$\bar{\mathbf{a}}^H \left(-(y_1 + y_2 + y_3)\mathbf{I} + \mathbf{w} \mathbf{w}^H \right)^{-1} \bar{\mathbf{a}} \leq 1. \quad (79)$$

In other words,

$$\begin{bmatrix} \mathbf{w} \mathbf{w}^H - (y_1 + y_2 + y_3)\mathbf{I} & y_1 \hat{\mathbf{a}} \\ y_1 \hat{\mathbf{a}}^H & -y_4 - y_1 (\|\hat{\mathbf{a}}\|^2 - \epsilon) \end{bmatrix} \succeq \mathbf{0}. \quad (80)$$

This fact together with (77) implies that $\mathbf{w} \mathbf{w}^H$ (with \mathbf{w} defined by (76)) is optimal for SDP problem (32).

As for how to get (79) from (78), we calculate it as follows. Using the equality

$$\begin{aligned} & \left(-(y_1 + y_2 + y_3)\mathbf{I} + \mathbf{w} \mathbf{w}^H \right)^{-1} \\ &= \frac{-1}{y_1 + y_2 + y_3} \mathbf{I} - \frac{-1}{-(y_1 + y_2 + y_3) + \mathbf{w}^H \mathbf{w}} \mathbf{w} \mathbf{w}^H, \end{aligned}$$

it is straightforward to see that (79) is equivalent to (78).

F. Proof of Proposition IV.1

Proof:

- 1) Suppose that $x = 0$. Then $y_1 + y_2 < 0$. From (62) and (67), we then can see that

$$(y_1 + y_2) \text{tr} \mathbf{Z} = \text{tr} (\mathbf{W} \mathbf{Z}) = \text{tr} (\hat{\mathbf{R}} \mathbf{W}) > 0,$$

which is a contradiction. Here we use the fact that the optimal values $\text{tr} (\hat{\mathbf{R}} \mathbf{W}) = z_1 > 0$. Therefore, we necessarily have $x < 0$.

- 2) Using the assumption that $x \lambda_{N-1}(\bar{\mathbf{C}}) + y_1 + y_2 < 0$, it follows from (66) that $z_2 = 0$, and it also follows from (63) and the above established fact that $x < 0$ that $z_1 = \frac{\text{tr}(\bar{\mathbf{C}} \mathbf{Z})}{\Delta_0}$.
- 3) From (67), we have

$$\begin{aligned} (y_1 + y_2) \text{tr} (\mathbf{Z}) &= \text{tr} (\mathbf{Z} \mathbf{W}) - x \text{tr} (\bar{\mathbf{C}} \mathbf{Z}) \\ &= z_1 - x \Delta_0 z_1 \\ &= z_1 (1 - x \Delta_0) \\ &> 0, \end{aligned}$$

where we apply $\text{tr} (\mathbf{Z} \mathbf{W}) = \text{tr} (\hat{\mathbf{R}} \mathbf{W}) = z_1$ and $\text{tr} (\bar{\mathbf{C}} \mathbf{Z}) = z_1 \Delta_0$. Hence, $y_1 + y_2 > 0$.

G. Proof of Proposition IV.2

Proof:

- 1) It follows from (62) and (67) that

$$\begin{aligned} 0 < \text{tr} (\hat{\mathbf{R}} \mathbf{W}) &= \text{tr} (\mathbf{Z} \mathbf{W}) = \text{tr} ((x \bar{\mathbf{C}} + (y_1 + y_2)\mathbf{I}) \mathbf{Z}) \\ &= x \text{tr} ((\bar{\mathbf{C}} - \lambda_{N-1}(\bar{\mathbf{C}})\mathbf{I}) \mathbf{Z}). \end{aligned}$$

Since $x \leq 0$, hence we have $x < 0$, and

$$\text{tr} (\bar{\mathbf{C}} \mathbf{Z}) < \lambda_{N-1}(\bar{\mathbf{C}}) \text{tr} \mathbf{Z}. \quad (81)$$

- 2) According to (63), the optimal value is

$$z_1 = \frac{\text{tr}(\bar{\mathbf{C}} \mathbf{Z}) + z_2 \lambda_{N-1}(\bar{\mathbf{C}})}{\Delta_0} > 0.$$

- 3) From the previous statement, we have

$$\begin{aligned} 0 < z_1 &= \frac{\text{tr}(\bar{\mathbf{C}} \mathbf{Z}) + z_2 \lambda_{N-1}(\bar{\mathbf{C}})}{\Delta_0} \\ &< \frac{\lambda_{N-1}(\bar{\mathbf{C}}) \text{tr} \mathbf{Z} + z_2 \lambda_{N-1}(\bar{\mathbf{C}})}{\Delta_0} \\ &= \frac{\lambda_{N-1}(\bar{\mathbf{C}}) (\text{tr} \mathbf{Z} + z_2)}{\Delta_0} \\ &\leq \frac{\lambda_{N-1}(\bar{\mathbf{C}}) z_1 (N + \eta_2)}{\Delta_0}, \end{aligned}$$

where the first inequality is due to (81) and the second inequality is due to the dual feasibility (the third constraint of (61)). Therefore, it follows that

$$0 < \Delta_0 \leq \lambda_{N-1}(\bar{\mathbf{C}}) (N + \eta_2),$$

which means that $\lambda_{N-1}(\bar{\mathbf{C}}) > 0$. Thereby, $y_1 + y_2 = -x \lambda_{N-1}(\bar{\mathbf{C}}) > 0$.

- 4) By the assumption, we have $y_1 + y_2 = -x \lambda_{N-1}(\bar{\mathbf{C}})$. From the second constraint of (60), it follows that

$$\mathbf{W} - x \bar{\mathbf{C}} + x \lambda_{N-1}(\bar{\mathbf{C}}) \mathbf{I} \succeq \mathbf{0},$$

which implies that

$$\mathbf{W} - x \bar{\mathbf{C}} \succeq -x \lambda_{N-1}(\bar{\mathbf{C}}) \mathbf{I}. \quad (82)$$

Therefore, we can write that

$$\lambda_1(\mathbf{W}) - x \lambda_N(\bar{\mathbf{C}}) \geq \lambda_N(\mathbf{W} - x \bar{\mathbf{C}}) \geq -x \lambda_{N-1}(\bar{\mathbf{C}}),$$

where we use the property $\lambda_1(\mathbf{A}) + \lambda_n(\mathbf{B}) \geq \lambda_n(\mathbf{A} + \mathbf{B})$ in the first inequality and apply the fact that $\lambda_n(\mathbf{A}) \geq \lambda_n(\mathbf{B})$ for $\mathbf{A} \succeq \mathbf{B}$ to (82) in the second inequality. In other words,

$$\lambda_1(\mathbf{W}) \geq -x (\lambda_{N-1}(\bar{\mathbf{C}}) - \lambda_N(\bar{\mathbf{C}})).$$

- 5) It follows from (82) that

$$\mathbf{w} \mathbf{w}^H - x \bar{\mathbf{C}} \succeq -x \lambda_{N-1}(\bar{\mathbf{C}}) \mathbf{I}, \quad (83)$$

which implies that

$$\hat{\mathbf{R}}^{1/2} \mathbf{w} \mathbf{w}^H \hat{\mathbf{R}}^{1/2} - x \hat{\mathbf{R}}^{1/2} \bar{\mathbf{C}} \hat{\mathbf{R}}^{1/2} \succeq -x \lambda_{N-1}(\bar{\mathbf{C}}) \hat{\mathbf{R}}. \quad (84)$$

Therefore, we have

$$\begin{aligned} & \lambda_N \left(\hat{\mathbf{R}}^{1/2} \mathbf{w} \mathbf{w}^H \hat{\mathbf{R}}^{1/2} - x \hat{\mathbf{R}}^{1/2} \bar{\mathbf{C}} \hat{\mathbf{R}}^{1/2} \right) \\ & \geq -x \lambda_{N-1}(\bar{\mathbf{C}}) \lambda_N(\hat{\mathbf{R}}). \end{aligned} \quad (85)$$

Note that

$$\begin{aligned} \lambda_1(\hat{\mathbf{R}}^{1/2} \mathbf{w} \mathbf{w}^H \hat{\mathbf{R}}^{1/2}) &= \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w}, \quad \lambda_N(\hat{\mathbf{R}}^{1/2} \bar{\mathbf{C}} \hat{\mathbf{R}}^{1/2}) \\ &= \lambda_N(\hat{\mathbf{R}} \bar{\mathbf{C}}). \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} - x \lambda_N(\hat{\mathbf{R}} \bar{\mathbf{C}}) \\ & \geq \lambda_N \left(\hat{\mathbf{R}}^{1/2} \mathbf{w} \mathbf{w}^H \hat{\mathbf{R}}^{1/2} - x \hat{\mathbf{R}}^{1/2} \bar{\mathbf{C}} \hat{\mathbf{R}}^{1/2} \right) \\ & \geq -x \lambda_{N-1}(\bar{\mathbf{C}}) \lambda_N(\hat{\mathbf{R}}), \end{aligned}$$

which means that

$$\begin{aligned} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} & \geq x \lambda_N(\hat{\mathbf{R}} \bar{\mathbf{C}}) - x \lambda_{N-1}(\bar{\mathbf{C}}) \lambda_N(\hat{\mathbf{R}}) \\ & = -x (\lambda_{N-1}(\bar{\mathbf{C}}) \lambda_N(\hat{\mathbf{R}}) - \lambda_N(\hat{\mathbf{R}} \bar{\mathbf{C}})). \end{aligned}$$

H. Proof of Theorem IV.3

Proof: It can be observed that if \mathbf{Z} is a rank-one matrix, then it follows from (62) that \mathbf{W} must be of rank one (since $\hat{\mathbf{R}}$ is positive definite and $\mathbf{W} \neq \mathbf{0}$). Now, we suppose that the rank of \mathbf{Z} is greater than one. Let $(\text{tr}(\hat{\mathbf{R}}^{-1} \mathbf{Z}), \text{tr}(\bar{\mathbf{C}} \mathbf{Z}), \text{tr}(\mathbf{Z})) = (\delta_1, \delta_2, \delta_3)$. Then, we have

$$\text{tr} \left(\left(\hat{\mathbf{R}}^{-1} - \frac{\delta_1}{\delta_3} \mathbf{I} \right) \mathbf{Z} \right) = 0, \quad \text{tr} \left(\left(\bar{\mathbf{C}} - \frac{\delta_2}{\delta_3} \mathbf{I} \right) \mathbf{Z} \right) = 0.$$

From Lemma III.6, it follows that there is a rank-one matrix decomposition $\mathbf{Z} = \sum_{r=1}^R \mathbf{z}_r \mathbf{z}_r^H$ (where R is the rank of \mathbf{Z}) such that

$$\text{tr} \left(\left(\hat{\mathbf{R}}^{-1} - \frac{\delta_1}{\delta_3} \mathbf{I} \right) \mathbf{z}_r \mathbf{z}_r^H \right) = 0, \quad \text{tr} \left(\left(\bar{\mathbf{C}} - \frac{\delta_2}{\delta_3} \mathbf{I} \right) \mathbf{z}_r \mathbf{z}_r^H \right) = 0,$$

$r = 1, \dots, R$.

Define

$$\mathbf{z} = \sqrt{\delta_3} \frac{\mathbf{z}_1}{\|\mathbf{z}_1\|}.$$

It can be verified that

$$(\mathbf{z}^H \hat{\mathbf{R}}^{-1} \mathbf{z}, \mathbf{z}^H \bar{\mathbf{C}} \mathbf{z}, \mathbf{z}^H \mathbf{z}) = (\text{tr}(\hat{\mathbf{R}}^{-1} \mathbf{Z}), \text{tr}(\bar{\mathbf{C}} \mathbf{Z}), \text{tr}(\mathbf{Z})). \quad (86)$$

Since $\text{tr}(\hat{\mathbf{R}}^{-1} \mathbf{Z}) \leq 1$, hence $\mathbf{z}^H \hat{\mathbf{R}}^{-1} \mathbf{z} \leq 1$, which implies that

$$\begin{bmatrix} \hat{\mathbf{R}} & \mathbf{z} \\ \mathbf{z}^H & 1 \end{bmatrix} \succeq \mathbf{0},$$

which in turn means that $\hat{\mathbf{R}} - \mathbf{z} \mathbf{z}^H \succeq \mathbf{0}$. Therefore, it follows from (86) that $(\mathbf{z} \mathbf{z}^H, z_1, z_2)$ is an optimal solution for problem (61), and thus, the optimal solution $(\mathbf{W}, x, y_1, y_2)$ for primal problem (60) fulfills the complementary conditions (62)–(67). Again from (62), we claim that the rank of \mathbf{W} is one.

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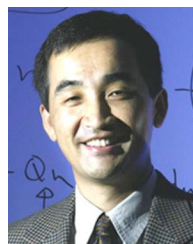
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