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Optimal Decision Strategies for the Generalized *Cuckoo* Card Game

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***Abstract*—*Cuckoo* is a popular card game, which originated in France during the 15th century and then spread throughout Europe, where it is currently well-known under distinct names and with different variants. Cuckoo is an imperfect information game-of-chance, which makes the research regarding its optimal strategies determination interesting. The rules are simple: each player receives a covered card from the dealer; starting from the player at the dealer's left, each player looks at its own card and decides whether to exchange it with the player to their left, or keep it; the dealer plays at last and, if it decides to exchange card, it draws a random one from the remaining deck; the player(s) with the lowest valued card lose(s) the round. We formulate the gameplay mathematically and provide an analysis of the optimal decision policies. Different card decks can be used for this game, e.g., the standard 52-card deck or the Italian 40-card deck. We generalize the decision model for an arbitrary number decks' cards, suites, and players. Lastly, through numerical simulations, we compare the determined optimal decision strategy against different benchmarks, showing that the strategy outperforms the random and naive policies and approaches the performance of the ideal oracle.**

***Index Terms*—Card games, games of chance, optimization, optimal decision strategy.**

I. Introduction

Cuckoo (also known as *Coucou*, *As Qui Court*, and *Hère*) is a historical French card game (CG), and is currently very popular throughout Europe. The game was mentioned as early as 1490 in France, where it was known as *Mécontent* [1] and played with the standard 52-card deck, while it was referenced in Italy, for the first time, in 1547 as *Malcontento* [2]. It appears with the name *Hère* [3] in 1690 and *Coucou* [4] in 1721. In the early 18th century, dedicated decks started to be produced, comprising 38 cards [5]. The earliest deck including the rules of the game was produced in Bologna, Italy, in 1717 by Borzaghi [6]. The game, then, spread through north Europe, where both the deck and the name changed [7]: in Bavaria (Germany) it was called *Hexenspiel* or *Vogelspiel*. It was known in Denmark as *Gniao*, and its deck had 42 cards. This name was then translated to *Gnav* when it reached Norway, and it was played with a set of dedicated wooden tokens, instead of cards. The token values were printed and glued on its bottom, as showed in a mid-17th century Venetian print by Stefano Scolari [8]. It was also brought to the Netherlands, where it was known as *Slabberjan*. Finally, from 1881, the game was known in England with the names *Chase-the-Ace* and *Ranter-Go-Round*, and it was presumably first played in Cornwall [9].

Despite the existence of all the aforementioned variants, the general rules of Cuckoo are simple: given a covered card from the dealer, players looks at it and decide whether to exchange it with the player to the left or keep it; the player(s) with the lowest card value lose the round. Nonetheless, the design and definition of optimal winning strategies a challenging problem; indeed, the decision space of possible strategies is large due to the high number of cards and action combinations. However, to the best of our knowledge, no studies have address the modeling and analysis of Cuckoo under a comprehensive theoretical and simulation-based framework. For the sake of filling such a gap, in this paper, after describing and mathematically formulating the gameplay, we provide an analysis of the optimal decision policies, which maximize the winning likelihood. Different card decks can be used for this game, e.g., the standard 52-card deck or the Italian 40-card deck. For this reason, we generalize the decision model for decks with an arbitrary number of cards and suites as well as players. Lastly, through numerical game simulations, we show that the determined optimal decision strategy outperforms the random and naive policies and approaches the performance of the ideal oracle.

The remainder of the paper is structured as follows. In Section II, related works on determining the optimal strategies of card games are discussed, highlighting the paper contributions. Section III describes the game rules. After introducing the basic definitions for the generalized card game, Section IV formalizes the optimal strategy. Section V reports the game results obtained from numerical experiments, and discusses the outcomes of benchmarking analysis. Conclusions are drawn in Section VI.

II. Literature Review and Paper Positioning

A. Related Works

The existing literature has prominently discussed the topic of CGs both from its historical and cultural perspective [11], being strongly linked to the traditions of a geographical area and to the roots of its inhabitants. However, in the last decades, the subject has been studied also from the strategical point of view, with the aim of finding algorithms that would yield the

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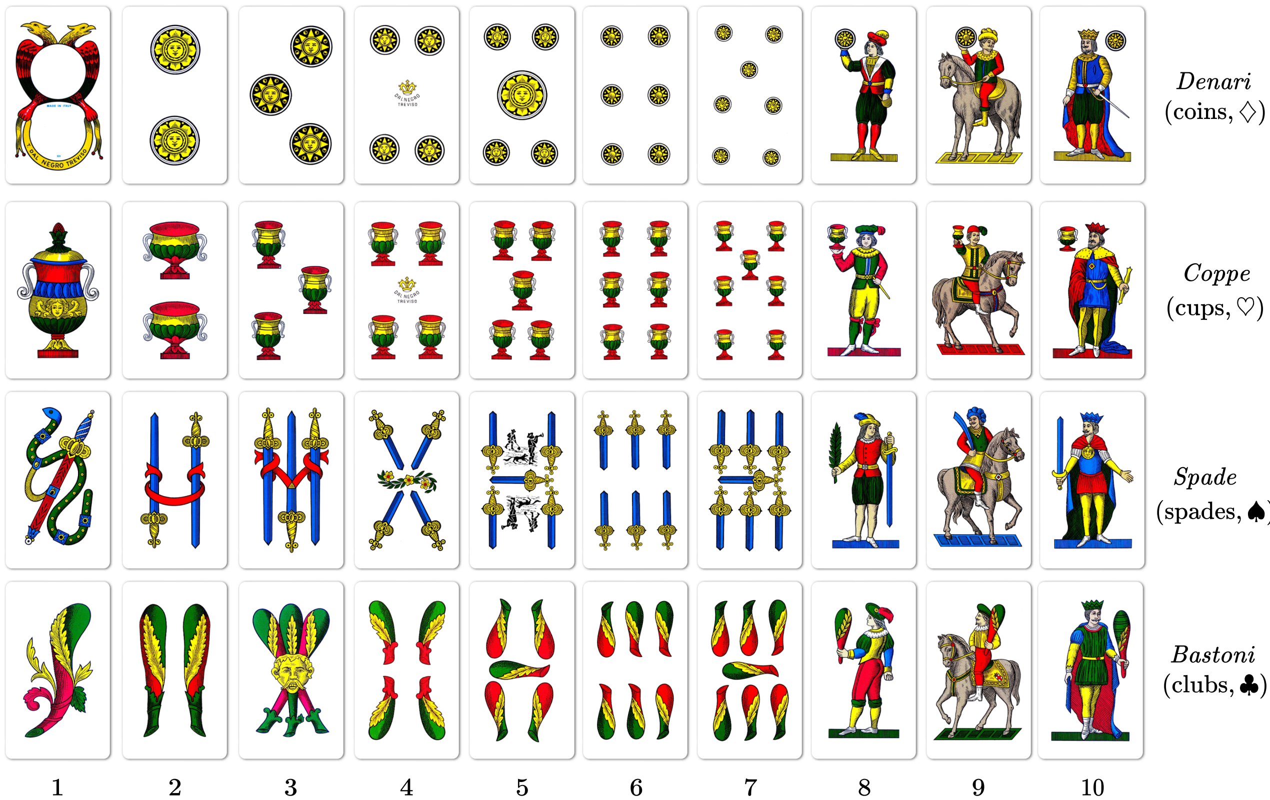

Fig. 1. *Italian 40-card deck* with *Neapolitan* suites, manufactured by Dal Negro [10]. It is the most popular and commonly used deck, especially in Southern Italy. Columns denotes the card value, while rows indicate the specific suit, with the corresponding one of the standard 52-cards deck. Some cards have a proper name, in addition to their value: the card with the lowest value (i.e., equal to 1) is called *asso* (ace), while cards with value from 8 to 10 are called *donna* (woman, queen), *cavallo* (horse, knight), and *re* (king), respectively

optimal game policy. For this reason, the analysis of CGs is investigated not only for recreational purposes, but also for its close proximity with computer and decision sciences.

The most notable CG which has attracted active research is probably Poker [12]. In [13] authors review algorithms and methods in the area of computer Poker, starting from the earliest attempts to create strong computerised Poker players, up to more modern computational game-theoretical approaches. The most recent Poker agents, based on techniques derived from artificial intelligence, have reached superhuman capabilities: in [14] an algorithm called "Pluribus" is presented, shown to be stronger than top human professionals in six-player no-limit Texas hold'em Poker. The game has also been studied under different lenses, other than strategical optimality: in [15] an emotion recognition model is described, which aims at assisting the human player by providing advice based on the current game situations and human player's recognized emotions. Further examples of well-known CGs for which optimal strategies have been derived are Blackjack and *Baccarat* [16], both sharing a strong gambling component. In [17] the optimal strategy for Blackjack is discussed, where a mathematical expression is derived, providing a general solution to the player's problem of drawing additional cards, with a given hand, or not. The game is also used in [18] as a baseline for examining the advantages that quantum strategies allow in communication-limited games. Regarding *Baccarat*, it has been shown in [19] that, differently from Blackjack, the "card counting" technique does not provide significant favorable strategies, both for the *Trente-et-Quarante* and "Nevada" variants. A game-theoretic discussion of the possible strategies for the game is provided in [20], where the variant known as *Chemin de fer* is also taken into consideration. A brief history of the evolution of the game and its variants is provided in [21]. Another popularly studied CG is Bridge [22], which shares an international diffusion with the ones mentioned above, but is not prone towards gambling. In [23] the existence of equilibrium points of a best-defence model, for games with imperfect information, is devised and used for analyzing search architectures that have been proposed for Bridge. In [24], instead, a recursive Monte Carlo (MC) approach is employed, in place of the traditional depth-one MC Tree Search (MCTS), showing better results, at the cost of a higher computational effort.

So far, we mentioned CGs which have a worldwide diffusion. If we lower the granularity to the regional level, the number of existing CGs roughly is estimated to be around 10,000 [25]. One of these, which belongs to the Italian cultural heritage, is *Scopone*, whose strategy has been studied in [26]. Authors compare rule-based players using the most established strategies against players using MCTS and Information Set MCTS, with different reward functions and simulation strategies.

On the top of this large list, there exists a further wide

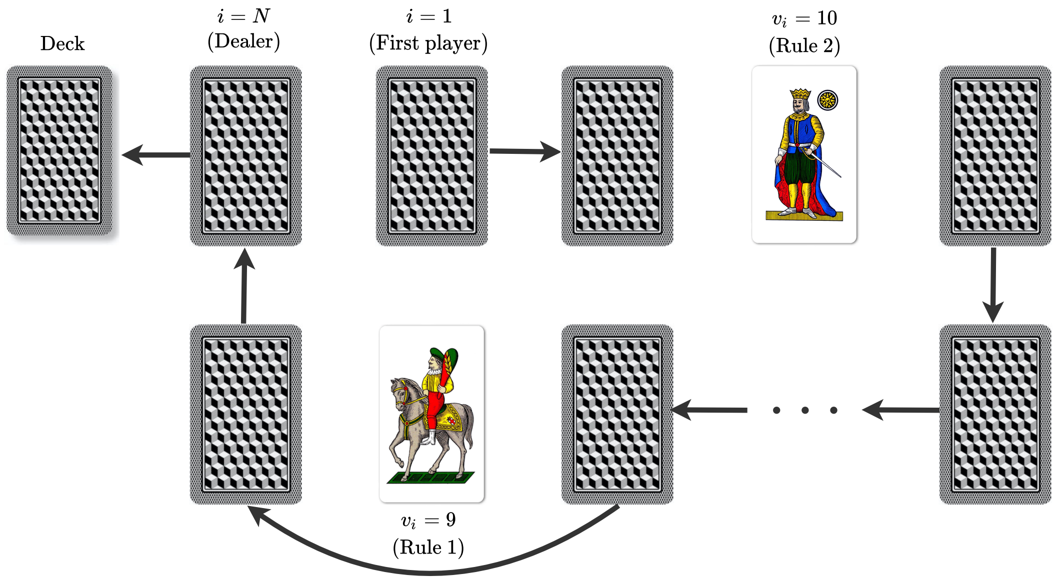

Fig. 2. Example of a game round with an *Italian 40-card* deck (characterized by $S = 4$ *Neapolitan suites*, each having $C = 10$ cards). Arrows indicate possible card exchanges, whose decision is eventually made by players with the arrow tails. The dealer has the remaining deck from which a random card is eventually extracted to fulfill its card exchange. The player having a card valued $C = 10$ is isolated from the adjacent players (Rule 2), while the player with a card valued $C - 1 = 9$ makes the player on its right eventually "jump", if the latter desires to change card (Rule 1).

category of CGs, denoted as *collectible* CG (CCG) [27], such as *Magic: The Gathering* and *Hearthstone*. They are all characterized by the possibility for players to customize their own deck, deciding which cards to use for a specific match. This feature adds a further layer of complexity to their analysis, the since the in-game strategy strongly depends on the optimal mix of cards constituting the deck [28], [29].

B. Paper Contributions

As many CGs, Cuckoo is a game of chance with imperfect information, which makes the discussion regarding its optimal strategy determination interesting. To the best of the authors' knowledge, this is the first study regarding the decision modeling and analysis of the Cuckoo game. To fill this gap, in the following, after describing and mathematically formulating the gameplay, we provide an analysis of the optimal game strategies, which maximize the winning likelihood. The game can be played with a variety of decks, from the standard 52-card deck, up to national and regional ones, such as the Italian 40-card deck (Fig. 1). It can also be played with dedicated tokens. For this reason, we generalize the decision model for decks with an arbitrary number of cards and suites as well as players. Lastly, we analyze the winning results achieved by the optimal decision policies through numerical game simulations, showing that their performance is closed to those of the ideal oracle.

III. Game Rules

The game is played with a S-suites deck, each containing cards from 1 to C, for a total of CS cards. A match consists of several rounds, which are played in accordance with the rules described in the following. We refer to Fig. 2 as an illustrative example of a typical game turn, specifically played with the Italian 40-card deck, for which $S = 4$, $C = 10$.

At the end of each round, the objective of the player(s) is not to be the one(s) with the lowest card value. At the beginning of each round, the dealer gives to each player a card, which has to be kept covered. Then, starting from the players on the left of the dealer, each player looks at its own card and decides whether to keep it or exchange it with the player immediately on the left. This process is repeated for each player in clockwise order, until the player on the right of the dealer decides to change the card with the dealer or not. Then, the dealer, being the last player of the round, may choose to exchange its own card with a random card drawn from the deck. Finally, all players uncover their cards. The player(s) with the lowest card value lose the round. Usually, each player has an initial number of *lives*: each lost round corresponds to a lost life. A player who loses all lives must exit the game. Rounds are repeated upon dealer role shifting over players and cards reshuffling and redistribution, until there is only a player left, i.e., the winner of the match. In addition, there are two further rules that define specific conditions and modalities of card exchange:

Rule 1. *If a player has a $(C-1)$-valued card, the player on the right has to "jump" the player with the $(C-1)$-valued card, i.e., it can exchange its own card with the first player on its left not having $(C-1)$-valued card, if desired. The player with the $(C-1)$-valued card cannot change card at all.*

Rule 2. *If a player has a C-valued card, the player on the right cannot change its own card at all. The player with the C-valued card cannot change card as well.*

We further assume that all players with either a $(C-1)$ or a C-valued card have to uncover their card at the beginning

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of the round, showing the corresponding values to the other players. The last player allowed to exchange its card, but having no available players on its left, can randomly extract a card for the deck: for instance, this is the case addressed by player $N-1$ (that does not have a $(C-1)$ or a C-valued card) when the dealer (i.e., the player N) has a $(C-1)$-valued card. Several variants of the game exist, which convey more and different rules. In order not to render the discussion too cumbersome, we adopt only the two previously mentioned rules. Finally, in the case of a draw – i.e., all the players have the same valued card – we adopt the convention such that all players lose a life.

IV. The Decision Strategies

Notation: $\mathbb{N}$ and $\mathbb{Z}$ denotes the set of natural and integer numbers, respectively; given $a, b \in \mathbb{N}$, with $a \leq b$, $[a,b]_{\mathbb{N}}$ indicates $[a,b] \cap \mathbb{N}$; given $n, k \in \mathbb{N}$, $(n)_k := n(n-1)\cdots(n-k+1)$ is the Pochhammer notation for the falling factorial; given $n, k \in \mathbb{N}$, $\binom{n}{k} := n!/(k!(n-k)!)$ is the notation for the binomial coefficient; symbols $\vee$ and $\wedge$ represent logical operators OR and AND.

A. Preliminaries

The generic game with N players and a deck having S suites and C valued cards for each suite, is denoted as $\mathcal{G}(S, C, N)$. To ensure fairness, the number of players cannot be equal to the number of cards, i.e., $N \leq CS - 1$, otherwise the last player would not have the possibility to change its card by randomly drawing a card from the remaining deck. The N players are indicated with the identifiers $\mathcal{I} = [\mathrm{Id}_1, \cdots, \mathrm{Id}_N]_{\mathbb{N}}$; however, due to the dealer role shifting, in each round players are identified by the indices in the set $\mathcal{N} = [1, \cdots, N]_{\mathbb{N}}$: without loss of generality, we assume that player $i+1$ (for each $i \leq N-1$) is on the left of player i and player $i = N$ indicates the dealer. Clearly, these indices shift clockwise over players round by round. The value of the card hold by the i-th player is denoted as $v_i \in [1, C]_{\mathbb{N}}$.

We define the set of successors of player i, $\mathcal{S}_i$, as the set of all players on its left, whose card is not $(C-1)$-valued, i.e.:

$$\mathcal{S}_i := \{j \in [i+1, N]_{\mathbb{N}} \mid v_j \neq C-1\}. \tag{1}$$

Afterwards, we define the “subsequent player” of player i as s_i:

$$s_i := \begin{cases} \min(\mathcal{S}_i), & \text{if } \mathcal{S}_i \neq \emptyset \\ N+1, & \text{if } i = \min\{j \in [1, N]_{\mathbb{N}} \mid \mathcal{S}_i = \mathcal{S}_j = \emptyset\} \\ \nexists, & \text{otherwise} \end{cases} \tag{2}$$

where $s_i = N+1$ conventionally indicates the remaining deck. In other terms, if $\mathcal{S}_i \neq \emptyset$, then s_i is the first successor on the left of i. Note that player i might not have any successor (i.e., $\mathcal{S}_i = \emptyset$), e.g., when player i is the dealer (i.e., $i = N$). If there are no successors, player i might extract a random card from the deck, to eventually fulfill its card exchange, if it is the first player without successors. For example, consider the case where $v_i \neq C-1$, $v_{i+1} = \cdots = v_N = C-1$, and $\mathcal{S}_1 = \cdots = \mathcal{S}_{i-1} \neq \emptyset$: i is the first player without successors and might extract a random card from the deck; contrarily, players from $i+1$ to N would not have any subsequent player.

Similarly to (1), we define the set of predecessors of player i, $\mathcal{P}_i$, as the set of all players on its right, whose card is not $(C-1)$-valued, i.e.:

$$\mathcal{P}_i = \{j \in [1, i-1]_{\mathbb{N}} \mid v_j \neq C-1\}. \tag{3}$$

Subsequently, we define the “preceding player” of player i as p_i:

$$p_i := \begin{cases} \max(\mathcal{P}_i), & \text{if } \mathcal{P}_i \neq \emptyset \\ 0, & \text{if } i = \max\{j \in [1, N]_{\mathbb{N}} \mid \mathcal{P}_i = \mathcal{P}_j = \emptyset\} \\ \nexists, & \text{otherwise} \end{cases} \tag{4}$$

where $p_i = 0$ conventionally indicates that i is the first player. In other terms, if $\mathcal{P}_i \neq \emptyset$, then p_i is the first predecessor on the right of i. Note that player i might not have any predecessor (i.e., $\mathcal{P}_i = \emptyset$), e.g., when player i is the first player (i.e., $i = 1$). A player $i > 1$ may not have any predecessors, e.g., if $v_1 = \cdots = v_i = C-1$. If there are no predecessors, player i does not undergo any request of card exchange.

Given that player i has a card of value v_i, in the whole deck there are $v_i^- := S \cdot (v_i - 1)$ cards whose value is lower than v_i. Similarly, in the whole deck there are $v_i^+ := S \cdot (C - v_i)$ cards whose value is higher than v_i. Clearly, the number of cards equal to v_i is always S, including the v_i-valued card of player i. Hence, the following equality holds:

$$v_i^- + v_i^+ + S = CS. \tag{5}$$

For convenience, we also define $\tilde{v}_i^+ := v_i^+ + S$ and $\tilde{v}_i^- := v_i^- + S$, i.e, the number of cards whose value is higher or equal to v_i, and lower or equal to v_i, respectively. The objective of each player, for each round, is not to be the one with the lowest value card. For player i, the taken action is denoted as a_i, whilst the set $\mathcal{A}_i$ collects all the potential actions as follows:

$$\mathcal{A}_i := \{v_i \rightleftarrows v_{s_i}, \varnothing\} \tag{6}$$

where the notation $v_i \rightleftarrows v_{s_i}$ means that player i exchanges its card with player s_i, while the symbol $\varnothing$ represents no action. For $s_i = N+1$, $v_i \rightleftarrows v_{N+1}$ means that player i exchanges its card with a random one from the remaining deck.

We remark that a player i has knowledge of some card value different from its own if any of Rules 1 or 2 occurs (public knowledge), or if player p_i exchanges its card with player i (private knowledge). We will refer to *knowledge* as the totality of private and public knowledge [30]. We introduce $k_i(h) : [1, C]_{\mathbb{N}} \to [0, S]_{\mathbb{N}}$ as the function defining the number of h-valued cards known by player i. In particular, for player i we indicate with K_i^+ and K_i^- the number of known cards whose value is higher and lower than v_i, respectively:

$$K_i^+ := \sum_{h=v_i+1}^{C} k_i(h), \quad K_i^- := \sum_{h=1}^{v_i-1} k_i(h) \tag{7}$$

For convenience, for player i we also define $\tilde{K}_i^+ := K_i^+ + k_i(v_i)$ and $\tilde{K}_i^- := K_i^- + k_i(v_i)$ as the number of known cards whose value is higher or equal and lower or equal to v_i,

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respectively. Notice that $k_i(v_i) \geq 1$, since card v_i is always known by player i. Clearly, $K_i^- \leq v_i^-$ and $K_i^+ \leq v_i^+$.

B. Optimal Strategy

In the following, we establish the optimal strategy to be adopted by the player i. Indicating with $\mathcal{E}_i$ the set of possible events that player i deals with, we introduce the optimal decision policy of player i as the function:

$$\sigma_i : 2^{\mathcal{E}_i} \to \mathcal{A}_i. \tag{8}$$

Let us provide the formulation of $\sigma_i(\cdot)$ for all the game occurrences, classified in the sequel into three categories.

First, there are two possible events that trivially guarantee that player i wins the round:

- *w1)* There is at least one player $j \neq i$ such that $v_j < v_i$ with v_j known by player i, i.e., $K_i^- \geq 1$. This might happen in the case player p_i decided to change card with player i, giving to the latter a card with value $v_{p_i} > v_i$.
- *w2)* Player i has a card whose value is such that $\tilde{v}_i^+ < N-1$, i.e., no matter how many players have a higher value card, there is at least a player with a lower value card.

Let us group all the above described occurrences in the winning event set $\mathcal{W}_i \subset \mathcal{E}_i$, defined as:

$$\mathcal{W}_i := \{(v_{p_i} \rightleftarrows v_i \wedge v_{p_i} > v_i) \ \vee (\tilde{v}_i^+ < N-1)\}. \tag{9}$$

It straightforwardly follows that for player i the best action for all events in $\mathcal{W}_i$ is to not exchange card, i.e.:

$$\sigma_i(\mathcal{W}_i) = \varnothing. \tag{10}$$

Second, there is a unique event that trivially guarantees that player i loses the round:

- *l1)* player i has one of the lowest cards in the deck, either received by the dealer at the beginning of the round or got from player p_i, and simultaneously player s_i on the left has a card with $v_{s_i} = C$, thus preventing any card exchange to player i.

The losing event is indicated with $\mathcal{L}_i \subset \mathcal{E}_i$, which is formally defined as follows:

$$\mathcal{L}_i := \{(v_i = 1 \wedge v_{s_i} = C)\}. \tag{11}$$

In this case the only possible action for player i is to not exchange card, although it always ends up in losing the round, i.e.:

$$\sigma_i(\mathcal{L}_i) = \varnothing. \tag{12}$$

Remark 1. *From* (10) *and* (12)*, it follows that function σ_i is non-invertible, for all $i \in \mathcal{N}$.*

Lastly, the third category of events include all the cases that do not guarantee player i to neither win nor lose the round:

- *u1)* Player p_i does not exchanged card with player i, i.e., $a_{p_i} = \varnothing$.
- *u2)* Player p_i exchanges a card with value $v_{p_i} < v_i$ with player i.
- *u3)* Player i is the player that starts the round, i.e., $p_i = 0$.

Similarly to the previous cases, let us group all the above described occurrences in the set $\mathcal{U}_i \subset \mathcal{E}_i$, defined as:

$$\mathcal{U}_i := \{(a_{p_i} = \varnothing) \ \vee (v_{p_i} \rightleftarrows v_i \wedge v_{p_i} < v_i) \vee (p_i = 0)\}. \tag{13}$$

Note that $\mathcal{W}_i \cup \mathcal{L}_i \cup \mathcal{U}_i \subseteq \mathcal{E}_i$ and $\mathcal{W}_i \cap \mathcal{L}_i = \mathcal{W}_i \cap \mathcal{U}_i = \mathcal{L}_i \cap \mathcal{U}_i = \emptyset$.

In order to establish the optimal policy $\sigma_i(\mathcal{U}_i)$ for player i for all events in the set $\mathcal{U}_i$, we need to characterize its stochastic nature. To this aim, we identify the subset of $\mathcal{U}_i$ containing the winning events for player i, for a given action a_i, which is denoted as Ω_i^- in the sequel.

We preliminarily recall that, for player i to win the round, it is sufficient that there exists at least one player with a card with value lower than the card value of player i. For a given $d < N-1$, the event related to the existence of exactly d players (different from player i) with a card value lower than v_i is denoted as $\Omega_{i,d}^-$ and is defined as follows:

$$\Omega_{i,d}^- = \{\exists \mathcal{D} \subseteq \mathcal{N} \setminus \{i\} \mid v_j < v_i, \ \forall j \in \mathcal{D}, \ |\mathcal{D}| = d\} \tag{14}$$

where $\mathcal{D}$ represents the subset of d players whose card value is lower than v_i. It is apparent that $\mathcal{D}$ is not empty (and consequently the event $\Omega_{i,d}^-$ may occur) if and only if $d \leq M^-$, where M^- is defined as:

$$M^- := \min\,(v_i^-, N - \tilde{K}_i^+). \tag{15}$$

Note that (15) has the following meaning: on the one hand, the number of players with a card value lower than v_i cannot be higher than v_i^- (i.e., the number of cards, in the whole deck, with a value lower than v_i^-); on the other hand, the number of players with a card value lower than v_i is upper bounded by the number of remaining players whose cards are unknown (i.e., $N - \tilde{K}_i^+$). Having defined $\Omega_{i,d}^-$, the event related to the existence of at least one player $j \neq i$ such that $v_j < v_i$ is:

$$\Omega_i^- = \bigcup_{d=1}^{M^-} \Omega_{i,d}^-. \tag{16}$$

Furthermore, for the sake of convenience, we introduce the event $\Omega_{i,b}^+$ related to the existence of exactly b players (different from player i) with a card value higher or equal to v_i. Such an event is defined as follows:

$$\Omega_{i,b}^+ = \{\exists \mathcal{B} \subseteq \mathcal{N} \setminus \{i\} \mid v_j \geq v_i, \ \forall j \in \mathcal{B}, \ |\mathcal{B}| = b\} \tag{17}$$

where $\mathcal{B}$ represents the subset of b players whose card value is higher or equal to v_i. It is apparent that $\mathcal{B}$ is not empty (and consequently the event $\Omega_{i,b}^+$ may occur) if and only if $b \leq M^+$, where M^+ has a meaning similar to M^- in (15) and is defined as:

$$M^+ := \min\,(\tilde{v}_i^+, N - \tilde{K}_i^+). \tag{18}$$

The probability of the occurrence of event Ω_i^- for player i (i.e., its winning probability in the round), depending on $a_i \in \mathcal{A}_i$ and $\tilde{K}_i^+$ is denoted as $\mathbb{P}(\Omega_i^- \mid a_i, v_i, \tilde{K}_i^+)$ and defined as follows.

We know analyze the two cases, $a_i = \varnothing$ and $a_i = v_i \rightleftarrows v_{s_i}$, respectively. The following proposition provides the formulation of the winning probability $\mathbb{P}(\Omega_i^- \mid \varnothing, v_i, \tilde{K}_i^+)$ of player i.

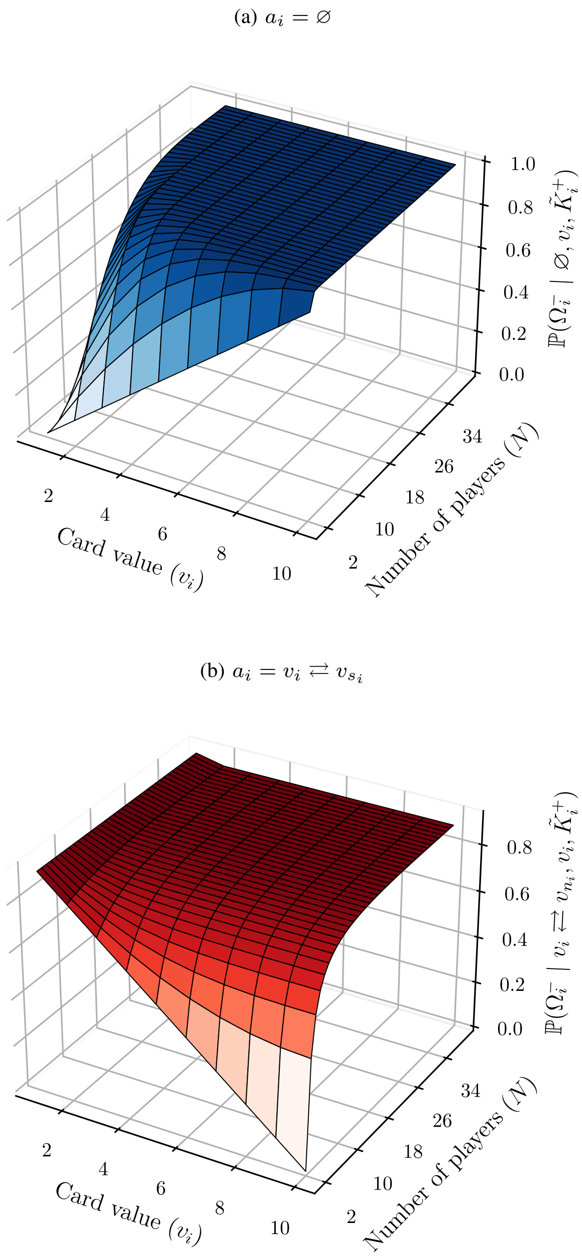

Fig. 3. Probability of the winning event Ω_i^- with $\tilde{K}_i^+ = 1$ for player i in a round of the game $\mathcal{G}(S=4, C=10, N \leq 39)$, when the chosen action is null (a) or card exchange (b).

Proposition 1. *Probability $\mathbb{P}\left(\Omega_i^- \mid \varnothing, v_i, \tilde{K}_i^+\right)$ follows an hypergeometric distribution [31], i.e.,*

$$\mathbb{P}\left(\Omega_i^- \mid \varnothing, v_i, \tilde{K}_i^+\right) = \sum_{d=1}^{M} \frac{\binom{v_i^-}{d}\binom{CS-\tilde{K}_i^+-v_i^-}{N-\tilde{K}_i^+-d}}{\binom{CS-\tilde{K}_i^+}{N-\tilde{K}_i^+}}. \quad (19)$$

Proof. The winning probability $\mathbb{P}(\Omega_i^- \mid a_i, v_i, \tilde{K}_i^+)$ is computed as:

$$\mathbb{P}\left(\Omega_i^- \mid \varnothing, v_i, \tilde{K}_i^+\right) = \sum_{d=1}^{M^-} \mathbb{P}(\Omega_{i,d}^-)\mathbb{P}\left(\Omega_{i,N-\tilde{K}_i^+-d}^+\right). \quad (20)$$

Note that, given d players for which $\Omega_{i,d}^-$ occurs, the remaining players with unknown cards are $N - \tilde{K}_i^+ - K_i^- - d$, although $K_i^- = 0$; otherwise, event $\mathcal{W}_i$ would have occurred. We can explicitly write the term $\mathbb{P}(\Omega_{i,d}^-)$ in (20) as follows:

$$\mathbb{P}(\Omega_{i,d}^-) = \binom{N-\tilde{K}_i^+}{d} \frac{v_i^-}{CS-\tilde{K}_i^+} \cdots \frac{v_i^- - d + 1}{CS-\tilde{K}_i^+-d+1} \quad (21)$$

which represents the probability of d players having a card with a value lower than v_i, for all the possible ways they can be chosen from the $N - \tilde{K}^+$ remaining players with unknown cards. Equation (21) can also be expressed in combinatorial form as:

$$\mathbb{P}(\Omega_{i,d}^-) = \binom{N-\tilde{K}_i^+}{d}\binom{v_i^-}{d}\binom{CS-\tilde{K}_i^+}{d}^{-1}. \quad (22)$$

Similarly, the term $\mathbb{P}\left(\Omega_{N-\tilde{K}_i^+-d}^+\right)$ in (20) can be written as:

$$\mathbb{P}\left(\Omega_{i,N-\tilde{K}_i^+-d}^+\right) = \frac{CS-\tilde{K}_i^+-v_i^-}{(CS-\tilde{K}_i^+)-d} \cdot \ldots \cdot \frac{(CS-\tilde{K}_i^+-v_i^-)-(N-\tilde{K}_i^+-d)+1}{(CS-\tilde{K}_i^+)-(N-\tilde{K}_i^+)+1} \quad (23)$$

which represents the probability of the $N - \tilde{K}_i^+ - d$ remaining players having a card value higher or equal to v_i. Also (23) has an equivalent combinatorial formulation, i.e.:

$$\mathbb{P}\left(\Omega_{i,N-\tilde{K}_i^+-d}^+\right) = \frac{\binom{CS-\tilde{K}_i^+-v_i^-}{N-\tilde{K}_i^+-d}\binom{CS-\tilde{K}_i^+}{d}}{\binom{N-\tilde{K}_i^+}{d}\binom{CS-\tilde{K}_i^+}{N-\tilde{K}_i^+}}. \quad (24)$$

We can now write (20), from (21) and (23), as:

$$\mathbb{P}\left(\Omega_i^- \mid \varnothing, v_i, \tilde{K}_i^+\right) = \sum_{d=1}^{M}\binom{N-\tilde{K}_i^+}{d}\frac{(v_i^-)_d(CS-\tilde{K}_i^+-v_i^-)_{N-\tilde{K}_i^+-d}}{(CS-\tilde{K}_i^+)_{N-\tilde{K}_i^+}} \quad (25)$$

and, equivalently, we can rewrite its combinatorial form, from (22) and (24), as in (19), showing that $\mathbb{P}\left(\Omega_i^- \mid \varnothing, v_i, \tilde{K}_i^+\right)$ follows an hypergeometric distribution. □

A plot of $\mathbb{P}\left(\Omega_i^- \mid \varnothing, v_i, \tilde{K}_i^+\right)$, as computed in (19), is reported in Fig. 3.a, for different values of v_i and N. It can be noticed that the surface becomes flat for games with a large number of players, independently from the card value, as long as $v_i > 1$, which would otherwise result in losing the round if the card does not undergo an exchange.

Let us now focus on the case $a_i = v_i \rightleftarrows v_{s_i}$. The following proposition provides the formulation of the winning probability $\mathbb{P}(\Omega_i^- \mid v_i \rightleftarrows v_{s_i}, v_i, \tilde{K}_i^+)$ of player i.

Proposition 2. *Probability $\mathbb{P}(\Omega_i^- \mid v_i \rightleftarrows v_{s_i}, v_i, \tilde{K}_i^+)$ can be expressed as the following compound distribution:*

$$\mathbb{P}(\Omega_i^- \mid v_i \rightleftarrows v_{s_i}, v_i, \tilde{K}_i^+) = \frac{v_i^+ - K_i^+}{CS-\tilde{K}_i^+} + \frac{\tilde{v}_i^-}{CS-\tilde{K}_i^+}\sum_{h=1}^{v_i} k_i(h)\mathbb{P}\left(\Omega_i^- \mid \varnothing, h, \tilde{K}_i^+ + 1\right). \quad (26)$$

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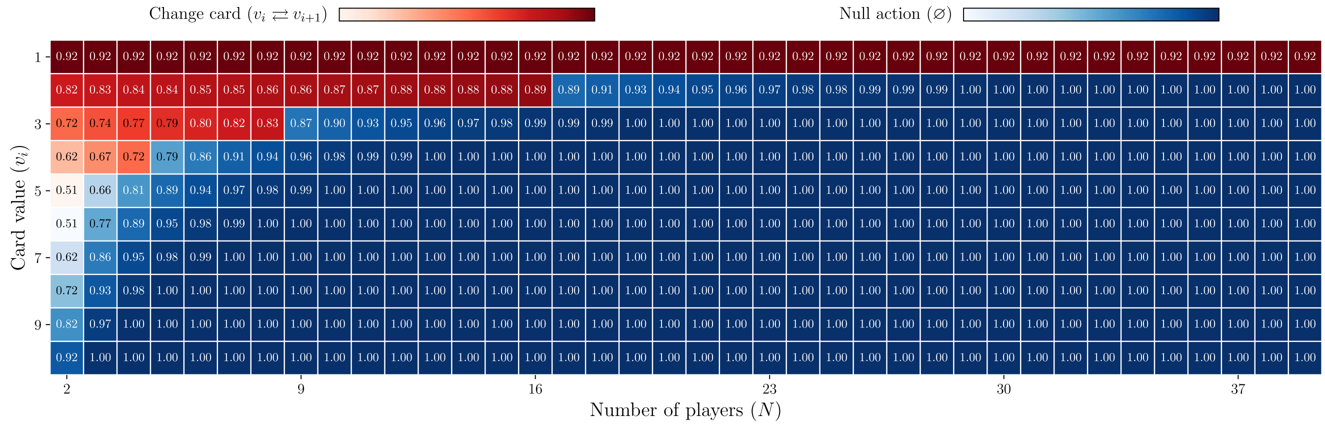

Fig. 4. Heatmap of the optimal decision policy defined in (32) with $\tilde{K}_i^+ = 1$ for the game $\mathcal{G}(S=4, C=10, N \in [2,39]_{\mathbb{N}})$. Given a number of players N and a card valued v_i, shades represent the optimal action a_i, while their intensity indicates the winning probability $\mathbb{P}(\Omega_i^- \mid a_i, v_i, \tilde{K}_i^+)$.

Proof. Differently from the previous case $a_i = \varnothing$, we now have dependency on the card value v_{s_i} of the subsequent player. Therefore, we distinguish two possible sub-cases, i.e.:

$$\mathbb{P}(\Omega_i^- \mid v_i \rightleftarrows v_{s_i}, v_i, \tilde{K}_i^+) =$$

$$= \begin{cases} 1, & \text{if } v_{s_i} > v_i \qquad (27a) \\ \mathbb{P}\left(\Omega_i^- | \varnothing, v_{s_i}, \tilde{K}_i^+ + 1\right), & \text{if } v_{s_i} \leq v_i. \qquad (27b) \end{cases}$$

Sub-case (27a) results in the existence of at least one player with a card value higher than v_i (i.e., $K_i^- \geq 1$), who corresponds to the subsequent player of player i. Conversely, in sub-case (27b), the occurrence probability of event Ω_i^- coincides with (19), evaluated with respect to v_{s_i} and considering that the number of known players with card value higher or equal to v_i has to be increased of one unit (corresponding to the card v_i that player i has exchanged with its subsequent player s_i). The compound distribution of (27) can be written as

$$\mathbb{P}(\Omega_i^- \mid v_i \rightleftarrows v_{s_i}, v_i, \tilde{K}_i^+) =$$
$$= \mathbb{P}(v_{s_i} > v_i) + \mathbb{P}(v_{s_i} \leq v_i)\mathbb{P}\left(\Omega_i^- | \varnothing, v_{s_i}, \tilde{K}_i^+ + 1\right). \quad (28)$$

The occurrence probabilities of the above discussed sub-cases (i.e, events $v_{s_i} > v_i$ and $v_{s_i} \leq v_i$) are computed as:

$$\mathbb{P}(v_{s_i} > v_i) = \sum_{h=v_i+1}^{C} \frac{S - k_i(h)}{CS - \tilde{K}_i^+} \quad (29a)$$

$$\mathbb{P}(v_{s_i} \leq v_i) = \sum_{h=1}^{v_i} \frac{S - k_i(h)}{CS - \tilde{K}_i^+}. \quad (29b)$$

It is apparent that the two events $v_{s_i} > v_i$ and $v_{s_i} \leq v_i$ are complementary, and thus it can be straightforwardly demonstrated that their probabilities add to 1, i.e.:

$$\mathbb{P}(v_{s_i} > v_i) + \mathbb{P}(v_{s_i} \leq v_i) =$$
$$= \sum_{h=1}^{C} \frac{S - k_i(h)}{CS - \tilde{K}_i^+} = \frac{CS - \tilde{K}_i^+ - K_i^-}{CS - \tilde{K}_i^+} = 1 \quad (30)$$

recalling from (7) that $\sum_{h=1}^{C} k_i(h) = K_i^+ + K_i^- + k_i(v_i)$, $\tilde{K}_i^+ = K_i^+ + k_i(v_i)$, and, given that event $\mathcal{U}_i$ occurred, $K_i^- = 0$. Substituting (29a) and (29b) in (28), yields

$$\mathbb{P}(\Omega_i^- \mid v_i \rightleftarrows v_{s_i}, v_i, \tilde{K}_i^+) =$$
$$= \sum_{h=v_i+1}^{C} \frac{S - k_i(h)}{CS - \tilde{K}_i^+} + \sum_{h=1}^{v_i} \frac{S - k_i(h)}{CS - \tilde{K}_i^+} \mathbb{P}\left(\Omega_i^- | \varnothing, h, \tilde{K}_i^+ + 1\right). \quad (31)$$

By definition, $\sum_{h=v_i+1}^{C} S = v_i^+$ and $\sum_{h=1}^{v_i} S = \tilde{v}_i^-$, from which (31) can be further simplified to (26). ☐

A plot of $\mathbb{P}(\Omega_i^- | v_i \rightleftarrows v_{s_i}, v_i, \tilde{K}_i^+)$, as computed in (26), is reported in Fig. 3.b for different values of v_i and N. It can be noticed that the trend is opposed to (19) for low values of N. Also, for v_i close to 1, the probability of Ω_i^- becomes high when player i changes card with its subsequent player. This implies that if $v_i = 1$ and $\mathcal{L}_i$ does not occur, player i is likely not to lose the round.

Finally, we can define the optimal decision policy for $\mathcal{U}_i$, which consists in choosing the action $\sigma_i(\mathcal{U}_i)$ such that maximizes either (19) or (26), i.e.:

$$\sigma_i(\mathcal{U}_i) = \arg\max_{a_i \in \mathcal{A}_i} \mathbb{P}(\Omega_i^- \mid a_i, v_i, \tilde{K}_i^+). \quad (32)$$

A heatmap for the optimal strategy defined in (32) is reported in Fig. 4, where, for a given number of players N and card value v_i, the color shade and intensity indicate the optimal action and the winning probability of player i, respectively. It can be noticed how, for large values of N, the predominant action is $\varnothing$: intuitively, this is due to the fact that the presence of a low valued card is highly likely. Therefore, only players with a 1-valued card will exchange it. Even if a player receives a 1-valued card, exchanging it with the subsequent player will likely lead to not losing the round, since the probability of the event $v_i = v_{s_i} = 1$ is low.

C. Inference from Previous Players Strategies

The optimal strategy defined in (32) addresses the agent's best strategy without knowing the actions previously taken by the preceding agents. We now take into account how the preceding agents' decisions can affect the game, with the aim of gaining strategic advantage under the occurrence of event $\mathcal{U}$.

Consider player $i \in \mathcal{N}$, whose turn to play has not come, such that $\mathcal{P}_i \neq \emptyset$. Player i and $j \in \mathcal{P}_i$ possess the same amount of knowledge (i.e., apart from v_i and v_j), if the preceding player of j, p_j, did not exchange its card. In fact, players who still have the card received from the dealer, when it is their turn to play, only know the numbers of uncovered C and $C-1$ valued card from the current hand, other than their own card. Formally, given $i \in \mathcal{N}$, we have

$$(j \in \mathcal{P}_i \wedge a_{p_j} = \varnothing) \implies \begin{cases} \tilde{K}_i^+ = \tilde{K}_j^+ \\ K_i^- = K_j^- \end{cases} . \quad (33)$$

Note that (33) does not generally hold in the opposite direction: suppose there are players $k, h \in \mathcal{N}$ such that $a_{p_k} = v_{p_k} \rightleftarrows v_k$, $a_{p_h} = v_{p_h} \rightleftarrows v_h$ and $v_{p_k} < v_k$, $v_{p_h} < v_h$; then, players k and h would share the same knowledge. We know from Remark 1 that σ_i is non-invertible, that is, even if players i and j share the same amount of knowledge, the former can neither determine v_j nor which event of $2^{\mathcal{E}_j}$ occurred. Nonetheless, since we have $a_{p_j} = \varnothing$ from (33), we can restrict the conditions of $\mathcal{W}_j$ and $\mathcal{U}_j$, since both events consider $a_{p_j} = v_{p_j} \rightleftarrows v_j$ as occurrence. Specifically, recalling (9) and (13), we define the restricted events $\tilde{\mathcal{W}}_j$ and $\tilde{\mathcal{U}}_j$ as follows:

$$\forall j \in \mathcal{P}_i \mid a_{p_j} = \varnothing : \begin{cases} \tilde{\mathcal{W}}_j := \{\tilde{v}_j^+ < N-1\} \\ \tilde{\mathcal{U}}_j := \{(a_{p_j} = \varnothing) \vee (p_i = 0)\} \end{cases} \quad (34)$$

Note that event $\mathcal{L}_j$ in (11) does not undergo any variation if (33) occurs. It can be already seen that if $a_j = \varnothing$ occurs, player i can not gain any useful insight since $\sigma_j(\mathcal{L}_i) = \sigma_j(\tilde{\mathcal{W}}_j) = \varnothing$. Also, $a_j = \varnothing$ may result from event $\tilde{\mathcal{U}}_j$ occurring. However, if $a_j = v_j \rightleftarrows v_{s_j}$, player i can infer that event $\tilde{\mathcal{U}}_j$ occurred, since it is the only event that admits a card exchange. If player j decided to exchange card, player i can easily detect the subset of cards that v_j belongs to. We denote it as $\mathcal{V}_j \subset [1, C]_{\mathbb{N}}$, collecting all cards v such that:

$$\forall v \in \mathcal{V}_j : (v_j \rightleftarrows v_{s_j}) = \underset{a_j \in \mathcal{A}_j}{\arg\max}\, \mathbb{P}(\Omega_j^- \mid a_j, v, \tilde{K}_j^+) \quad (35)$$

that is, all cards valued v such that, given $\tilde{K}_j^+$, the best action is to exchange card. Now, player i will not exchange cards, in spite of what (32) suggests, if $v_i > v$, for all $v \in \mathcal{V}_j$. However, this does not crucially improves the optimal strategy so far outlined, since it does not provides information for critical cases: e.g., if $v_i = 2$, and $\mathcal{U}_i$ occurs, $a_i = \sigma_i(\mathcal{U}_i) = v_i \rightleftarrows v_{s_i}$, for $N \leq 16$ (see Fig. 4), then $2 \in \mathcal{V}_j$ as well, which is not enough for making $a_i = \varnothing$ optimal.

V. Numerical Results

The decision strategies presented in Section IV are tested numerically through simulations of the game $\mathcal{G}(S = 4, C = 10, N)$ with an Italian 40-card deck and with the players' number N varying in the range $[2, 39]_{\mathbb{N}}$. Each player has $L = 1$ lives at the beginning of each match, for all game simulations. For each N a total of 800 games have been run. The optimal strategy is compared with three different benchmarking policies, described in the following. The complete simulation source code is available at [32].

A. Benchmarking Strategies

1) Ideal oracle: We define an *ideal oracle*, as the decision policy players would employ if they could see all covered cards, i.e., if $\tilde{K}_i^+ + K_i^- = N - 1$ for all $i \in \mathcal{N}$. We denote it with ψ_i, and define

$$\psi_i : \mathbb{N}^N \times 2^{\mathcal{E}_i} \to \mathcal{A}_i. \quad (36)$$

Clearly, the action taken in case either $\mathcal{W}_i$ or $\mathcal{L}_i$ occurs is the same as in (10) and (12), respectively, i.e.,

$$\psi_i(\boldsymbol{v}, \mathcal{W}_i) = \psi_i(\boldsymbol{v}, \mathcal{L}_i) = \varnothing, \quad \forall i \in \mathcal{N} \quad (37)$$

where $\boldsymbol{v} = (v_1, \cdots, v_N)$ is the vector of cards' values of all the players, for the given hand. In case $\mathcal{U}_i$ occurs, we have instead

$$\forall i \in \mathcal{N}: \ \psi_i(\boldsymbol{v}, \mathcal{U}_i) = \begin{cases} v_i \rightleftarrows v_{s_i}, & \text{if } v_i = \min(\boldsymbol{v}) \\ \varnothing, & \text{otherwise} \end{cases} \quad (38)$$

i.e., the i-th player exchanges its card only if v_i is the lower valued card in the current hand. Note that if $v_i = v_{s_i} = \min(\boldsymbol{v})$, either actions would be losing. The oracle, thus, constitutes an ideal strategy, which bounds the performance of any realistic conceivable policy.

2) Naive strategy: The naive strategy of player i is denoted as:

$$\mu_i : 2^{\mathcal{E}_i} \to \mathcal{A}_i. \quad (39)$$

Player i keeps its card if any of these two events occurs:

- *n1)* The preceeding player p_i exchanges its card with player i, with $v_{p_i} < v_i$, thus implying that $K_i^- \geq 1$.
- *n2)* Player i believes that the subsequent player s_i has a lower valued card, i.e., if $\mathbb{P}(v_{s_i} < v_i) > 1/2$.

Occurrence (n1) is equivalent to the one described in (w1) for the winning event $\mathcal{W}_i$. Moreover, the probability in (n2) can be calculated, from (29b), as:

$$\mathbb{P}(v_{s_i} < v_i) = \sum_{h=1}^{v_i - 1} \frac{S - k_i(h)}{CS - \tilde{K}_i^+} = \frac{v_i^-}{CS - \tilde{K}_i^+}. \quad (40)$$

Conversely, if none of the events (n1) and (n2) occurs, player i exchanges its card. The strategy is considered naive since player i only reasons with respect to the state of its subsequent player, ignoring that the winning condition resides in the occurrences of event Ω_i^-. Therefore, grouping the occurrence (n1) and (n2) in the event set $\mathcal{M}_i \subset \mathcal{E}_i$ we get

$$\mathcal{M}_i = \left\{ (v_{p_i} \rightleftarrows v_i \wedge v_{p_i} < v_i) \vee \mathbb{P}(v_{s_i} < v_i) > \frac{1}{2} \right\}. \quad (41)$$

The naive strategy μ_i can thus be formally expressed as follows:

$$\mu(\Omega) = \begin{cases} \varnothing, & \text{if } \Omega \in \mathcal{M}_i \\ v_i \rightleftarrows v_{s_i}, & \text{otherwise.} \end{cases} \quad (42)$$

3) Random strategy: We define the random strategy of player i as follows:

$$\rho_i : [0,1]_{\mathbb{N}} \to \mathcal{A}_i \tag{43}$$

depending on a binary random variable $x \in [0,1]_{\mathbb{N}}$, which follows a uniform probability distribution, i.e $x \sim U[0,1]$. Therefore, the random strategy ρ_i yields:

$$\rho_i(x) = \begin{cases} \varnothing, & \text{if } x = 0 \\ v_i \rightleftarrows v_{s_i}, & \text{if } x = 1. \end{cases} \tag{44}$$

B. Discussion of the Results

Figure 5 illustrates the distribution of winnings with respect to each strategy, i.e., the fraction of games won using a given strategy. During each game each player is randomly assigned a specific strategy, so that all players play using different policies game by game: strategies are assigned following a uniform distribution, so that all strategies are played roughly the same number of times during the whole simulation. In particular, Fig. 5a reports the overall winning distribution over all the simulated games. As expected, the random and oracle strategy provide the performance lower and upper bounding, respectively: players adopting the random strategy won 1.36% of the time, whilst players playing as an oracle won 43.91% of the time. Playing using the optimal strategy yields the best results among the realistic policies, allowing players to win 31.44% of the time and thus improving the result of the naive strategy, whose winning rate is 23.29%. In our simulations, we coherently observe that the result of the naive strategy is improved by the optimal strategy by 8.15%. The observed gap between the ideal oracle and the optimal strategy is instead 12.47%, also coherently with Fig. 5a. Figure 5b reports the same winning distribution with respect to the number of players in the game. It can be noted how the winning dynamics is unaffected by the number of players, remaining uniform for all N.

Finally, for the sake of highlighting the game dynamics, Fig. 6 reports the average number of turns in which players survive, when all of them adopt the same strategy, i.e., when players share the same degree of rationality. Note that the number of survived turns is an appropriate measure of how well players perform, even when losing the game. From the plots, it is apparent that the oracle and optimal strategy are comparable, suggesting that players "resist" for the same number of turns.

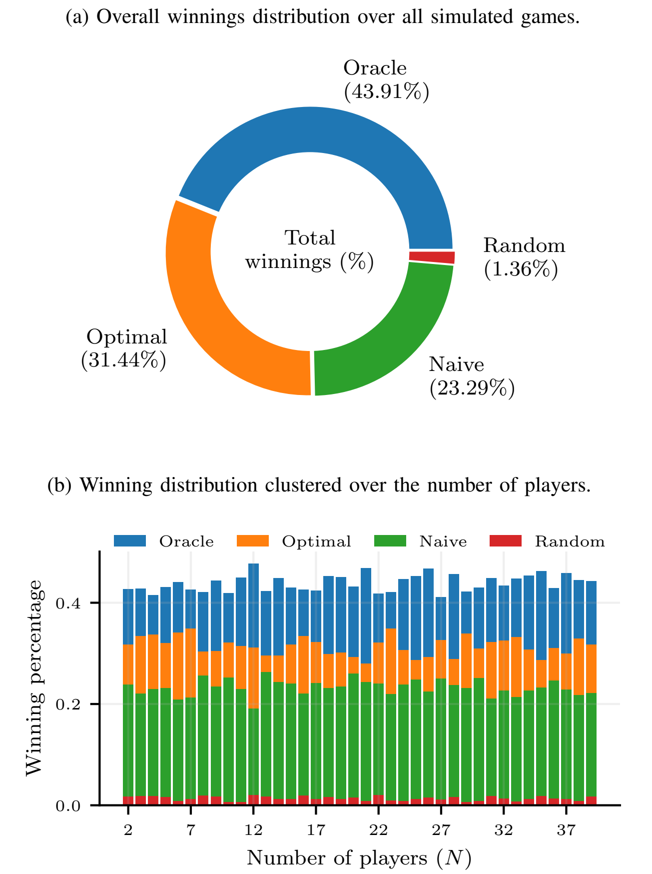

Fig. 5. Composition of total winnings with respect to each strategy: overall (a) and clustered (b) distributions.

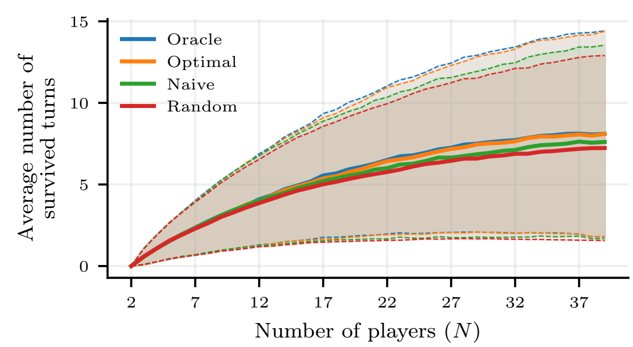

Fig. 6. Average number of survived turns for each strategy over the number of players: colored solid lines denote the average values, while shaded areas represent the standard deviation bounds and are delimited by dashed lines colored with the corresponding colors.

VI. Conclusions

In this paper we studied the optimal strategy for the well-known Cuckoo card game. Since the game can be played with a variety of dedicated decks, including the French-suited 52-cards and the Italian 40-card deck, we analyzed the game for a generic deck, with an arbitrary number of valued cards and suites. The formal characterization of the stochastic nature of the game, as well as the optimal strategies, were modeled and tested numerically, showing that the optimal strategy outperforms the naive and random strategies and approaches the ideal oracle.

Future works will investigate different paths: the game will be analyzed under the hypothesis of non-rational players, and it will be used as a testing ground for machine learning based control models, which will aim at enhancing the effectiveness of the decision policies, for instance, through reinforcement learning.

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