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Stabilization for a Class of Positive Bilinear Systems

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***Abstract*—In this letter, we consider stabilizing control design for a class of positive bilinear systems. Positivity allows us to employ a max-separable function for Lyapunov analysis, enabling to estimate a region of attraction as a box, which is particularly useful for heat exchangers. We take the summation of max-separable functions with respect to the state and input as a Lyapunov candidate and design a dynamic stabilizing controller. Moreover, employing a quadratic function with respect to the input and integral action of a performance output, we propose another dynamic stabilizing controller which can additionally regulate the output.**

***Index Terms*—Bilinear systems, positive systems, max-separable functions, stabilization, output regulation.**

I. Introduction

A SYSTEM is called *positive* if its trajectory stays in the positive orthant. For instance, positivity naturally appears in chemical kinetics and population dynamics. In the linear case, exponential stability of positive systems can be verified by a max-separable Lyapunov function in a necessary and sufficient manner [1], [2], enabling scalable analysis and also to estimate a region of attraction as a box rather than an ellipse. In the nonlinear case, there are two properties corresponding to positivity: positivity and cooperativity (or more generally, monotonicity) [3], [4], where cooperativity is stronger than positivity. Cooperativity is used to proceed with max-separable type analysis; see, e.g., [5], [6], [7], [8], [9]. In contrast, it is known that positivity is too weak to develop such analysis for general nonlinear systems.

Motivated by temperature control of a counter-current heat exchanger [10], we focus on stabilizing control design for a class of *positive bilinear* systems. Although there are numerous results on stabilization of bilinear systems, e.g., [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], positivity has not been incorporated except for a special case without additive terms [21], [22]. It is worth emphasizing that none of [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22] employs a max-separable Lyapunov function for stabilizing control design. In this letter, explicitly utilizing positivity and max-separable Lyapunov functions, we develop two Lyapunov-based stabilizing controllers for bilinear systems. For the first controller, we construct the Lyapunov candidate for the closed-loop system as the summation of a max-separable function with respect to the state and a max-separable function with respect to the input. For the second controller, instead of a max-separable function with respect to the input, we use a quadratic one. Moreover, in the quadratic case, one can further include an integral action of a performance output to achieve output regulation. Because of the construction of each Lyapunov candidate, a region of attraction with respect to the state can be estimated as a box.

Notation: The sets of real numbers and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, respectively. For matrices $A, B \in \mathbb{R}^{n\times m}$, we write $A \leq B$ ($A < B$) if and only if $A_{i,j} \leq B_{i,j}$ ($A_{i,j} < B_{i,j}$) for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, where $A_{i,j}$ denotes the (i,j)-th component of A. Similarly, for $c \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times m}$, $c \leq A$ ($c < A$) means $c \leq A_{i,j}$ ($c < A_{i,j}$) for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The $n \times n$ identity matrix is denoted by I_n. The $n\times m$ matrix whose components are all zero is denoted by $0_{n\times m}$. The sign function is denoted by $\mathrm{sgn}(\cdot)$.

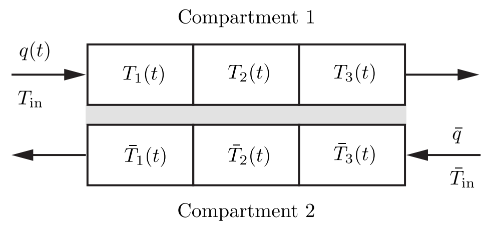

Fig. 1. A counter-current heat exchanger with three cells.

II. Motivating Example

Consider a counter-current heat exchanger with three cells in two layers [10] as in Fig. 1, which can be represented as

$$\dot{x} = Ax + (Bx + b)u + G$$

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$$A = \frac{1}{\beta}\left[\begin{array}{c|ccc} -aI_3 & & aI_3 & \\ \hline aI_3 & -\bar{q}-a & \bar{q} & 0 \\ & 0 & -\bar{q}-a & \bar{q} \\ & 0 & 0 & -\bar{q}-a \end{array}\right], \quad b = \frac{1}{\beta}\begin{bmatrix} T_{\rm in} \\ 0_{5\times 1}\end{bmatrix}$$
$$B = \frac{1}{\beta}\left[\begin{array}{ccc|c} -1 & 0 & 0 & \\ 1 & -1 & 0 & 0_{3\times 3} \\ 0 & 1 & -1 & \\ \hline & 0_{3\times 3} & & 0_{3\times 3} \end{array}\right], \quad G = \frac{1}{\beta}\begin{bmatrix} 0_{5\times 1} \\ \bar{q}\,\bar{T}_{\rm in}\end{bmatrix}$$
$$a = \lambda/c_p > 0, \quad \beta = \rho v > 0 \tag{1}$$

with the state $x(t) = [T_1, T_2, T_3, \bar{T}_1, \bar{T}_2, \bar{T}_3]^\top(t)$ and input $u(t) = q(t)$, where $T_i(t)$ and $\bar{T}_i(t)$ denote the temperature of the ith cell in the compartments 1 and 2, respectively, and $q(t)$ denotes the mass flow rate in the compartment 1. The positive constants $T_{\rm in}$ and $\bar{T}_{\rm in}$ denote the stream temperatures at the inlet of the compartments 1 and 2, respectively, and the positive constant $\bar{q}$ denotes the mass flow rate in the compartment 2. The constants λ, c_p, ρ, and v denote the heat transfer coefficient, heat capacity, mass density of the fluid, and volume of the fluid in the heat exchanger, respectively.

The control objective is to stabilize a desired equilibrium point (x^*, u^*) while the state and input fulfill physical constraints, e.g., $x_i(t) \in [\bar{T}_{\rm in}, T_{\rm in}] \subset \mathbb{R}_{\geq 0}$, $i = 1, \dots, 6$, and $u(t) \in [u_{\min}, u_{\max}] \subset \mathbb{R}_{\geq 0}$ for all $t \geq 0$. In [19], by extending [10], a stabilizing dynamic controller is proposed based on a quadratic Lyapunov function, and a set of initial states whose corresponding trajectories satisfy physical constraints is estimated as a level set of the quadratic Lyapounov function. However, this estimation can be conservative because the level set is an ellipse contained in a box $[\bar{T}_{\rm in}, T_{\rm in}]^6$.

As detailed below, a heat exchanger (1) is a positive system. Motivated by this, in this letter we develop a new control scheme for positive bilinear systems. By virtue of positivity, one can use a max-separable Lyapunov function [1], [2] for control design, which is suitable for heat exchangers since its level set is a box.

III. Problem Formulation

Consider a bilinear system, described by

$$\dot{x} = f(x, u) := Ax + \sum_{i=1}^{m}(B_i x + b_i)u_i + G, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) := [u_1, \dots, u_m]^\top(t) \in \mathbb{R}^m$ denote the state and input, respectively. Motivated by the structure of the heat exchanger (1), $A, B_i \in \mathbb{R}^{n\times n}$, $i = 1, \dots, m$ are assumed to be Metzler, i.e., their off-diagonal elements are non-negative. Also, we assume $b_i, G \in \mathbb{R}^n_{\geq 0}$, $i = 1, \dots, m$. Then, the bilinear system (2) is positive in the following sense.

Proposition 1: For a system (2), it follows that

$$x(0) \in \mathbb{R}^n_{\geq 0} \text{ and } u(t) \in \mathbb{R}^m_{\geq 0},\ \forall t \geq 0 \implies x(t) \in \mathbb{R}^n_{\geq 0}$$

as long as $x(t)$ exits.

Proof: The system can also be represented as

$$\dot{x} = \left(A + \sum_{i=1}^{m} B_i u_i\right)x + \sum_{i=1}^{m} b_i u_i + G. \tag{3}$$

For any $u_i \geq 0$, $i = 1, \dots, m$, $A + \sum_{i=1}^m B_i u_i$ is Metzler. In addition, $b_i, G \geq 0$, $i = 1, \dots, m$. Thus, the statement follows from theory for positive linear systems, e.g., [23]. ∎

For stabilization of a bilinear system, it is standard in the literature to assume some open-loop stability (see, e.g., [10], [11], [13], [14], [15], [17], [19], [20]). In the case of the heat exchanger (1), $A + Bu^*$ is Hurwitz for all $u^* > 0$ as confirmed in Section V. Thus, we impose the same assumption as [10], [11], [17], [19], [20], i.e., there exists $u^* \geq 0$ such that $A + \sum_{i=1}^m B_i u_i^*$ is Hurwitz.[1] Then, for fixed u^*, the system (3) has the following unique equilibrium point:

$$x^* = -\left(A + \sum_{i=1}^{m} B_i u_i^*\right)^{-1}\left(\sum_{i=1}^{m} b_i u_i^* + G\right). \tag{4}$$

Note that $x^* \geq 0$ because $A + \sum_{i=1}^m B_i u_i^*$ being Metzler and Hurwitz implies $-(A + \sum_{i=1}^m B_i u_i^*)^{-1} \geq 0$; see, e.g., [23, Proposition 2]. Moreover, by [23, Proposition 2], for sufficiently small $\lambda > 0$, there exist $w \in \mathbb{R}^n$ such that

$$\begin{cases} w > 0 \\ (A + \sum_{i=1}^m B_i u_i^*)w \leq -\lambda w. \end{cases} \tag{5}$$

Fixed $\lambda > 0$, solving (5) with respect to w is a linear programming problem, which can be easily solved numerically.

The set of inequalities (5) ensures that

$$V_x(x) := \max_{i=1,\dots,n}\left\{\frac{|x_i - x_i^*|}{w_i}\right\} \tag{6}$$

is a Lyapunov function for

$$\dot{x} = \left(A + \sum_{i=1}^{m} B_i u_i^*\right)(x - x^*).$$

Namely, we have

$$D^+ V_x(x) \leq -\lambda V_x(x), \tag{7}$$

where $D^+ V_x(x)$ denotes the upper-right Dini derivative [25] of $V_x(x)$:

$$D^+ V_x(x) := \limsup_{h\to 0^+} \frac{V_x(x + hf(x, u)) - V_x(x)}{h}.$$

Denoting the set of indices

$$\mathcal{I}_x(x) := \{i \in \{1, \dots, n\} : V_x(x) = |x_i - x_i^*|/w_i\}, \tag{8}$$

the Dini derivative can be computed by

$$D^+ V_x(x) = \max_{j_x \in \mathcal{I}_x(x)}\left\{\frac{1}{w_{j_x}}\frac{x_{j_x} - x_{j_x}^*}{|x_{j_x} - x_{j_x}^*|} f_{j_x}(x, u)\right\}.$$

This is the maximum value of the directional derivative of $|x_i - x_i^*|/w_i$, $i \in \mathcal{I}(x)$ along $f(x, u)$. The function (6) is called a max-separable Lyapunov function [1], [2].

Now, we are ready to state the main problem of this letter.

Problem 1: Consider a positive bilinear system (2). Given a compact $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m_{\geq 0}$, assume that there exists $u^* \geq 0$ such that $A + \sum_{i=1}^m B_i u_i^*$ is Hurwitz and $(x^*, u^*) \in \mathcal{X} \times \mathcal{U}$ for x^* satisfying (4). Then, design a controller such that the closed-loop system is asymptotically stable at (x^*, u^*), and estimate a region of attraction contained in $\mathcal{X} \times \mathcal{U}$.

[1] Another system satisfying this assumption is the DC-DC boost converter [24].

A heat exchange (1) satisfies the assumption for u^* in Problem 1. One notices that $u(t) = u^*$ is a trivial stabilizing controller, but open-loop controllers are usually vulnerable against uncertainties. To increase robustness, our objective is to design a state feedback controller. Differently from [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], we take advantage of positivity and max-separable functions for control design. In [21], [22], stabilization of positive bilinear systems by static controllers are considered only when $b_i = 0$, $i = 1, \dots, m$. We study dynamic control design without this assumption, i.e., each b_i, $i = 1, \dots, m$, can be either zero or non-zero.

IV. Main Results

In this section, we design dynamic state feedback controllers based on the max-separable Lyapunov function (6) with respect to the state. With respect to the input, we first select a max-separable function and then a quadratic one. In the latter case, we further consider (robust) output regulation.

A. Max-Separable Functions With Respect to Inputs

In this subsection, we design dynamic state feedback controllers based on the following max-separable Lyapunov candidate:

$$\begin{aligned} V(x,u) &:= V_x(x) + V_u(u) \\ V_u(u) &:= \max_{i=1,\dots,m}\left\{\frac{|u_i - u_i^*|}{v_i}\right\}, \end{aligned} \tag{9}$$

where $V_x(x)$ is defined by (6) and $v_i > 0$, $i = 1, \dots, m$. To describe a controller equation, we define the following set similarly to $\mathcal{I}_x(x)$ in (8):

$$\mathcal{I}_u(u) \ := \{i \in \{1, \dots, m\} : V_u(u) = |u_i - u_i^*|/v_i\}.$$

Now, we are ready to provide the following dynamic controller:

$$\dot{u}_i = -\lambda_u\big(u_i - u_i^*\big) - v_i \operatorname{sgn}\big(u_i - u_i^*\big) S(x,u), \tag{10a}$$

with $\lambda_u > 0$ and

$$S(x,u) := \max_{j_x \in \mathcal{I}_x(x)}\left\{\frac{\operatorname{sgn}\Big(x_{j_x} - x_{j_x}^*\Big)}{w_{j_x}} \sum_{i=1}^m (B_i x + b_i)_{j_x}\big(u_i - u_i^*\big)\right\}, \tag{10b}$$

where $(\cdot)_j$ denotes the jth element of vector $(\cdot)$.

In fact, (10) is a stabilizing controller, stated below.[2]

Theorem 1: For a positive bilinear system (2), let $u^* \geq 0$ be such that $A + \sum_{i=1}^m B_i u_i^*$ is Hurwitz, and let x^* be given by (4). Then, the closed-loop system with a dynamic feedback controller (10) is globally exponentially stable at (x^*, u^*).

Proof: We first consider the set $(\mathbb{R}^n \setminus \{x^*\}) \times (\mathbb{R}^m \setminus \{u^*\})$. The upper-right Dini derivative of $V(x,u)$ in (9), denoted by $D^+V(x,u)$, satisfies

$$\begin{aligned} D^+V(x,u) &= D^+V_x(x) + D^+V_u(u) \\ &= \max_{j_x \in \mathcal{I}_x(x)}\left\{\frac{1}{w_{j_x}}\frac{x_{j_x} - x_{j_x}^*}{|x_{j_x} - x_{j_x}^*|}\dot{x}_{j_x}\right\} \\ &\quad + \max_{j_u \in \mathcal{I}_u(u)}\left\{\frac{1}{v_{j_u}}\frac{u_{j_u} - u_{j_u}^*}{|u_{j_u} - u_{j_u}^*|}\dot{u}_{j_u}\right\}. \end{aligned}$$

We proceed with the computation of the first term. At (x^*, u^*), it follows from (3) that

$$\left(A + \sum_{i=1}^m B_i u_i^*\right)x^* + \sum_{i=1}^m b_i u_i^* + G = 0,$$

yielding

$$\dot{x} = \left(A + \sum_{i=1}^m B_i u_i^*\right)(x - x^*) + \sum_{i=1}^m (B_i x + b_i)\big(u_i - u_i^*\big). \tag{11}$$

From (7) and (11), we have

$$\begin{aligned} D^+V_x(x) \leq -\lambda V_x(x) + \max_{j_x \in \mathcal{I}_x(x)}\Bigg\{&\frac{1}{w_{j_x}}\frac{x_{j_x} - x_{j_x}^*}{|x_{j_x} - x_{j_x}^*|} \\ &\sum_{i=1}^m (B_i x + b_i)_{j_x}\big(u_i - u_i^*\big)\Bigg\}, \end{aligned}$$

and consequently

$$\begin{aligned} D^+V(x,u) &\leq -\lambda V_x(x) + \max_{j_x \in \mathcal{I}_x(x)}\Bigg\{\frac{1}{w_{j_x}}\frac{x_{j_x} - x_{j_x}^*}{|x_{j_x} - x_{j_x}^*|} \\ &\qquad \sum_{i=1}^m (B_i x + b_i)_{j_x}\big(u_i - u_i^*\big)\Bigg\} \\ &\quad + \max_{j_u \in \mathcal{I}_u(u)}\left\{\frac{1}{v_{j_u}}\frac{u_{j_u} - u_{j_u}^*}{|u_{j_u} - u_{j_u}^*|}\dot{u}_{j_u}\right\} \\ &\leq -\lambda V_x(x) + S(x,u) \ + \max_{j_u \in \mathcal{I}_u(u)}\left\{\frac{1}{v_{j_u}}\frac{u_{j_u} - u_{j_u}^*}{|u_{j_u} - u_{j_u}^*|}\dot{u}_{j_u}\right\}, \end{aligned}$$

where (10b) is used. Thus, it follows from (10a) that

$$\begin{aligned} D^+V(x,u) &\leq -\lambda V_x(x) - \lambda_u V_u(u) \\ &\leq -\min\{\lambda, \lambda_u\} V(x,u). \end{aligned} \tag{12}$$

Similarly, (12) holds on $(\mathbb{R}^n \setminus \{x^*\}) \times \{u^*\}$. Finally, we consider the set $\{x^*\} \times (\mathbb{R}^m \setminus \{u^*\})$. Since $\dot{x} \notin \{0\}$, we have $D^+V(x,u) = \max \emptyset = -\infty$. According to [26, Th. 3], (x^*, u^*) is strong asymptotically stable. By (9), $V(x,u)$ is positive definite at (x^*, u^*) and radially unbounded. Also, the convergence speed is exponential. Therefore, (x^*, u^*) is globally exponentially stable. ■

Remark 1: According to the proof of Theorem 1, the closed-loop stability can also be shown if the control law (10a) is implemented for $i \in \mathcal{I}_u(u)$, and the other inputs u_j, $j \notin \mathcal{I}_u(u)$ are arbitrary.

A region of attraction contained in $\mathcal{X} \times \mathcal{U}$ can be estimated from the Lyapunov function $V(x,u)$. Since its level set

$$\Omega_c := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m : V(x,u) \ \leq c\}$$

[2] Throughout this letter, we consider Filippov solutions of the corresponding differential inclusions to differential equations with sign functions.

is positively invariant, it suffices to find the largest $c > 0$ such that $\{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : V(x, u) = c\} \subset \mathcal{X} \times \mathcal{U}$.

Utilizing the structure of the Lyapunov function $V(x, u)$ in (9), we proceed with further analysis. Let $\pi_x(\Omega_c) \subset \mathbb{R}^n$ denote the projection of Ω_c onto the x-space. From (9), one notices that for the same $c > 0$,

$$\pi_x(\Omega_c) \subset \bar{\mathcal{X}}_c := \{x \in \mathbb{R}^n : V_x(x) \le c\}. \tag{13}$$

Then, for any trajectory starting from Ω_c, the corresponding state trajectory stays in $\bar{\mathcal{X}}_c$. Thus, one can use the largest $c_{\max} > 0$ satisfying $\bar{\mathcal{X}}_{c_{\max}} \subset \mathcal{X}$ to estimate a region of attraction $\bar{\mathcal{X}}_{c_{\max}}$ with respect to the state as a box. By a similar reasoning, the corresponding input trajectory stays in $\mathcal{U}$ if

$$\bar{\mathcal{U}}_v := \{u \in \mathbb{R}^m : V_u(u) \le c_{\max}\} \subset \mathcal{U},$$

where $\bar{\mathcal{U}}_v$ is also a box. Then, we can select v in (9) to satisfy $\bar{\mathcal{U}}_v \subset \mathcal{U}$, and a region of attraction contained in $\mathcal{X} \times \mathcal{U}$ can be estimated as $\bar{\mathcal{X}}_{c_{\max}} \times \bar{\mathcal{U}}_v$.

The controller (10a) has tuning parameters v and λ_u. The first one can be determined to satisfy the input constraints as mentioned above, while λ_u can be designed to balance the importance of input regulation and regulation of $S(x, u)$.

When implementing the dynamic feedback controller (10), we can estimate the set of initial states such that state constraints are fulfilled and tune the controller gains such that input constraints are satisfied. If one focuses on stabilizing the equilibrium (x^*, u^*) without taking these constraints into account, one can implement the following static feedback controller:

$$u_i = u_i^* - \gamma \max_{j_x \in \mathcal{I}_x(x)} \left\{ \frac{\operatorname{sgn}\left(x_{j_x} - x_{j_x}^*\right)}{w_{j_x}} (B_i x + b_i)_{j_x} \right\}, \tag{14}$$

where $\gamma > 0$. This is stated as follows.

Corollary 1: Under the assumptions of Theorem 1, the closed-loop system with a static feedback controller (14) is globally exponentially stable at x^*.

Proof: The proof can be shown by utilizing $V_x(x)$ in (6) as a Lyapunov candidate. ∎

B. Quadratic Functions With Respect to Inputs

In this subsection, we utilize a quadratic Lyapunov candidate with respect to the input, which allows us to additionally consider regulating a performance output $y(t) = Cx(t) \in \mathbb{R}^p$ to some reference y^*, where $C \ge 0$. This improves robustness of the regulation property, i.e., $\lim_{t\to\infty} y(t) = y^*$ against parameter uncertainties [27].

To this end, we employ for each $i = 1, \dots, m$ the following controller:

$$\begin{aligned} q_i \dot{u}_i = &-\lambda_u q_i\left(u_i - u_i^*\right) \\ &- \max_{j_x \in \mathcal{I}_x(x)} \left\{ \frac{\operatorname{sgn}\left(x_{j_x} - x_{j_x}^*\right)}{w_{j_x}} (B_i x + b_i)_{j_x} \right\} \\ &- \max_{j_z \in \mathcal{I}_z(x,z)} \left\{ -\frac{\operatorname{sgn}\left(z_{j_z} - z_{j_z}^*(x)\right)}{\alpha_{j_z}} (MB_i x + Mb_i)_{j_z} \right\} \end{aligned} \tag{15a}$$

$$\dot{z} = y - y^* = C\left(x - x^*\right), \tag{15b}$$

where $\lambda_u > 0$, $q_i > 0$, $i = 1, \dots, m$, $\alpha_i > 0$, $i = 1, \dots, p$,

$$z^*(x) := M\left(x - x^*\right), \quad M := C\left(A + \sum_{i=1}^m B_i u_i^*\right)^{-1} \tag{16}$$

and

$$\begin{aligned} \mathcal{I}_z(x, z) &:= \{i \in \{1, \dots, p\} : V_z(x, z) = |z_i - z_i^*(x)|/\alpha_i\} \\ V_z(x, z) &:= \max_{i=1,\dots,p} \left\{ \frac{|z_i - z_i^*(x)|}{\alpha_i} \right\}. \end{aligned} \tag{17}$$

Note that M in (16) is well defined, since $A + \sum_{i=1}^m B_i u_i^*$ is Hurwitz.

One notices that the first two terms of the u_i-dynamics in (15) are similar to those of (10). In fact, the first two terms are only for stabilization as stated in Corollary 2 below. The additional last term and z-dynamics are for output regulation. Based on the specific application, the importance of each term can be tuned by changing $\lambda_u > 0$, $q_i > 0$, and $\alpha_i > 0$. To achieve output regulation, suppose that

$$\left(MB_i x^* + Mb_i\right)_j \ne 0, \quad \forall j = 1, \dots, p. \tag{18}$$

Given A, B_i, b_i, u_i^*, $i = 1, \dots, m$ and C, we can compute x^* and M by (4) and (16). Then, (18) can be checked.

Now, we are ready to show the closed-loop stability and output regulation as follows.

Theorem 2: Under the assumptions of Theorem 1 and (18), a positive bilinear system (2) in closed-loop with a dynamic controller (15) is globally strongly asymptotically stable at $(x^*, u^*, 0)$.

Proof: As a preliminary step, one can confirm

$$\dot{z} - \dot{z}^*(x) = -M \sum_{i=1}^m (B_i x + b_i)\left(u_i - u_i^*\right) \tag{19}$$

by (11), (15b), and (16).

Consider the following Lyapunov candidate that is quadratic with respect to the input:

$$\begin{aligned} \bar{V}(x, u, z) &:= V_x(x) + \bar{V}_u(u) + V_z(x, z) \\ \bar{V}_u(u) &:= \frac{1}{2} \sum_{i=1}^m q_i\left(u_i - u_i^*\right)^2, \end{aligned} \tag{20}$$

where $V_x(x)$ and $V_z(x, z)$ are defined in (6) and (17), respectively.

We consider the set $(\mathbb{R}^n \setminus \{x^*\}) \times (\mathbb{R}^m \setminus \{u^*\}) \times (\mathbb{R}^m \setminus \{0\})$. According to the proof of Theorem 1 and (19), its upper-right Dini derivative, denoted by $D^+\bar{V}(x, u, z)$, satisfies

$$\begin{aligned} &D^+\bar{V}(x, u, z) \le -\lambda V_x(x) \\ &\quad + \max_{j_x \in \mathcal{I}_x(x)} \left\{ \frac{1}{w_{j_x}} \frac{x_{j_x} - x_{j_x}^*}{|x_{j_x} - x_{j_x}^*|} \sum_{i=1}^m (B_i x + b_i)_{j_x} \left(u_i - u_i^*\right) \right\} \\ &\quad + \sum_{i=1}^m \dot{u}_i q_i\left(u_i - u_i^*\right) \\ &\quad + \max_{j_z \in \mathcal{I}_z(x,z)} \left\{ -\frac{1}{\alpha_{j_z}} \frac{z_{j_z} - z_{j_z}^*(x)}{|z_{j_z} - z_{j_z}^*(x)|} \sum_{i=1}^m (MB_i x + Mb_i)_{j_z} \left(u_i - u_i^*\right) \right\} \\ &\quad = -\lambda V_x(x) \end{aligned}$$

$$+\sum_{i=1}^{m}\Bigg(\dot{u}_i q_i + \max_{j_x\in\mathcal{I}_x(x)}\left\{\frac{1}{w_{j_x}}\frac{x_{j_x}-x^*_{j_x}}{|x_{j_x}-x^*_{j_x}|}(B_i x + b_i)_{j_x}\right\}$$
$$+\max_{j_z\in\mathcal{I}_z(x,z)}\left\{-\frac{1}{\alpha_{j_z}}\frac{z_{j_z}-z^*_{j_z}(x)}{|z_{j_z}-z^*_{j_z}(x)|}(MB_i x + Mb_i)_{j_z}\right\}\Bigg)$$
$$\times\big(u_i-u_i^*\big).$$

Therefore, it follows from (15) that

$$D^+\bar{V}(x,u,z) \ \le -\lambda V_x(x) - \lambda_u \bar{V}_u(u). \tag{21}$$

Invoking the complementary set of $(\mathbb{R}^n\setminus\{x^*\})\times(\mathbb{R}^m\setminus\{u^*\})\times(\mathbb{R}^m\setminus\{0\})$, one can show (21) or $D^+\bar{V}(x,u,z) \le -\infty$. According to [26, Th. 3], $(x^*,u^*,0)$ is a strongly stable equilibrium.

To show closed-loop stability, we apply the invariance principle [26, Th. 2]. The Lyapunov candidate (20) is positive definite at $(x^*,u^*,0)$ and radially unbounded. Thus, its level set is compact and, from (21), strongly positively invariant. Now, we analyze the largest weakly invariant set contained in

$$\begin{aligned}&\overline{\{(x,u,z)\ \in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p : \bar{V}(x,u,z)\ \le c\}}\\ &\cap\{(x,u,z)\ \in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p : \lambda V_x(x)+\lambda_u\bar{V}_u(u)=0\}\\ &=\{(x,u,z)\ \in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p :\\ &\qquad \bar{V}(x,u,z)\ \le c,\ x=x^*,\ u=u^*\},\quad c>0,\end{aligned}$$

where $\overline{E}$ denotes the closure of a set E. From the u-dynamics in (15a) and $z^*(x^*)=0$ by (16), on the largest weakly invariant set, we have

$$\max_{j_x\in\mathcal{I}_x(x)}\left\{\frac{\operatorname{sgn}\big(x_{j_x}-x^*_{j_x}\big)}{w_{j_x}}\big(B_i x^*+b_i\big)_{j_x}\right\}$$
$$+\max_{j_z\in\mathcal{I}_z(x^*,z)}\left\{-\frac{\operatorname{sgn}\big(z_{j_z}\big)}{\alpha_{j_z}}\big(MB_i x^*+Mb_i\big)_{j_z}\right\}=0, \tag{22}$$

where $\mathcal{I}_z(x^*,z) := \{i\in\{1,\dots,p\} : V_z(x^*,z)=|z_i|/\alpha_i\}$ by (16) and (17).[3] From (18), the largest weakly invariant set is $\{(x^*,u^*,0)\}$. According to [26, Th. 2], every solution converges to $\{(x^*,u^*,0)\}$. ■

Similarly to the previous subsection, a region of attraction contained in $\mathcal{X}\times\mathcal{U}$ can be estimated from the Lyapunov function $\bar{V}(x,u,z)$ in (20) as a box with respect to the state.

In the previous section, we design a controller without the integral action of the output. For the comparison, we consider a special case of (15):

$$\begin{aligned}q_i\dot{u}_i = &-\lambda_u q_i\big(u_i-u_i^*\big)\\ &-\max_{j_x\in\mathcal{I}_x(x)}\left\{\frac{\operatorname{sgn}\big(x_{j_x}-x^*_{j_x}\big)}{w_{j_x}}(B_i x+b_i)_{j_x}\right\},\\ &\qquad\qquad i=1,\dots,m.\end{aligned} \tag{23}$$

This is also a stabilizing controller, stated below.

Corollary 2: Under the assumptions of Theorem 1, a positive bilinear system (2) in closed-loop with a dynamic controller (23) is globally exponentially stable at (x^*,u^*).

[3]More precisely, the differential inclusion defined by the left-hand side of (22) contains only 0.

Proof: The statement can be show by using $V_x(x)+\bar{V}_u(u)$ as a Lyapunov candidate. ■

It is also possible to design controllers without the integral action of the input and a static state feedback controller. Due to the space limitation, we omit to show them.

Remark 2: By virtue of the output integral action (15b), we confirm in the next section by simulation that under small parameter perturbations, the system (2) in closed-loop with a dynamic controller (15) is asymptotically stabilized at an equilibrium $(\tilde{x}^*,\tilde{u}^*,\tilde{z}^*)$ of the perturbed system, satisfying $y=y^*$; see [27] for more details on integral action in output feedback control.

V. Revisiting Motivating Example

Now, we revisit a heat exchanger (1) in Section II. First, we confirm that $A+Bu^*$ is Hurwitz for any $u^* > 0$. We construct a max-separable Lyapunov candidate (6) with $w=[1,1,1,1,1,1]^\top$. Although (7) does not hold, LaSalle's invariance principle concludes that $A+Bu^*$ is Hurwitz for any $u^*>0$. In this case, still our results are applicable.

Following [10], [19], for the heat exchanger (1), we select in simulation the system parameters as $\lambda = 10\,\mathrm{J/Ks}$, $\nu = 0.002\,\mathrm{m}^3$, $\rho = 997\,\mathrm{kg/m}^3$, $c_p = 4185\,\mathrm{J/kgK}$, $\bar{q} = 0.01\,\mathrm{kg/s}$, $T_{\mathrm{in}} = 360\,\mathrm{K}$, and $\bar{T}_{\mathrm{in}} = 300\,\mathrm{K}$. Regarding the constraints on the states and control input, let $\mathcal{X} := \{x_{\min,i} \le x_i \le x_{\max,i}\}$, $i=1,\dots,6$, with

$$\begin{aligned}x_{\min} &= [345, 338, 330.5, 314.5, 307, 300]^\top\\ x_{\max} &= [360, 353, 345.5, 329.5, 322, 315]^\top,\end{aligned}$$

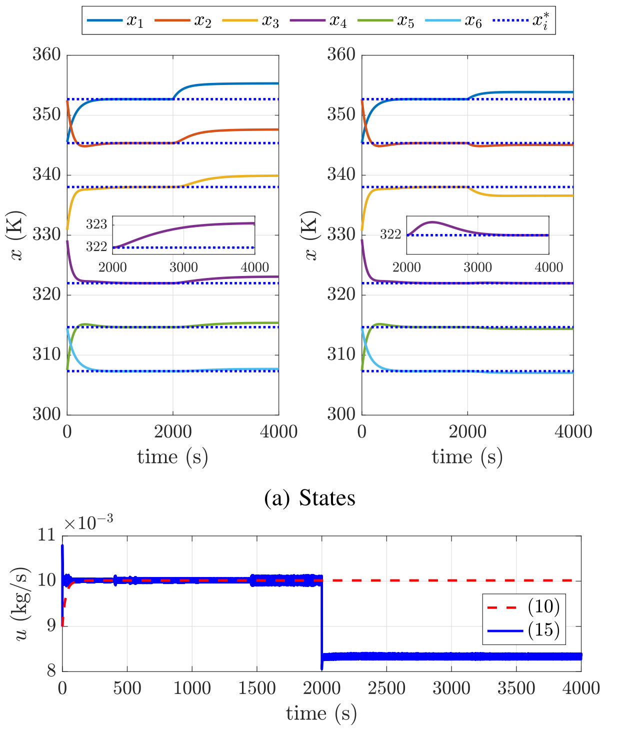
and $\mathcal{U} := \{0 < u \le 0.05\}$. In the following, we test the proposed controllers (10) and (15) to stabilize the heat exchanger (1) at the desired equilibrium point

$$x^* = [352.68, 345.36, 338.04, 322.00, 314.67, 307.34]^\top$$

and $u^*=0.01$. Moreover, to show the robustness properties of the controller (15), at time instant $t=2000\,\mathrm{s}$ we consider a system perturbation by changing the value of T_{in} from 360 K to 363 K.

First, we apply the controller (10) with $\lambda_u = 1\times10^{-3}$. As discussed immediately after Remark 1, the tuning parameter ν is designed based on the set $\mathcal{U}$. More precisely, we have $\mathcal{X}_c := \{x\in\mathbb{R}^n : \max_{i=1,\dots,6}\{|x_i-x_i^*|\}\le c\}$, and the maximum $c_{\max}>0$ satisfying $\mathcal{X}_{c_{\max}}\subset\mathcal{X}$ is given by $c_{\max}=7.32$. Then, for $\nu=0.0055$, it can be verified that also the input constraints are satisfied. Secondly, to robustly regulate $y=x_4$, we apply the controller (15) with $\lambda_u=30$ and $\alpha=1$. We use the same $c_{\max}$ as above and thus, for $q=9.16\times10^3$, also the input constraints are satisfied.

We can observe from Fig. 2(a) that when no perturbation occurs ($t<2000\,\mathrm{s}$), the trajectories converge to the corresponding equilibrium values (dotted lines). Differently, when a perturbation of T_{in} occurs ($t\ge2000\,\mathrm{s}$), only the controller (15) is capable to perform output regulation (see the enlargements in Fig. 2(a)). In both cases, for $t\ge2000\,\mathrm{s}$ the other temperature trajectories are stable but deviate from the equilibrium point. Moreover, we note that when the controller (15) is applied, such deviations are smaller than those obtained by applying



(a) States

(b) Inputs

Fig. 2. **(a)** Heat exchanger states x_i (temperatures) and the corresponding equilibrium x_i^*, with an enlarged view of the output x_4. The results on the left have been obtained by using the controller (10), while those on the right by using the controller (15). **(b)** Comparison between the control inputs generated by (10) and (15).

the controller (10). Figure 2(b) shows the comparison between the control inputs generated by (10) and (15).

VI. Conclusion

In this letter, we have studied stabilization of positive bilinear systems based on Lyapunov theory. By virtue of positivity, a max-separable function is employed with respect to the state, enabling to estimate a region of attraction as a box. With respect to the input, we have used two different functions: a max-separable function and a quadratic one. Each function leads to a different dynamic stabilizing controller, and the latter achieves additionally output regulation. In theory of positive linear systems, max-separable Lyapunov functions enable sclable analysis and control design for large-scale networks. Future work includes investigating scalable control design for a network of heat exchangers. On the other hand, max-separable Lyapunov functions lead to controllers having a similar structure as sliding mode controllers. Another future work is to investigate these connections.

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