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Control Allocation for Windup Mitigation in Weakly Redundant Systems With Input Saturation

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***Abstract*—This letter proposes a control allocation framework to cope with the windup problem that may arise when input limitations are present. Our approach hinges on the design of a dynamic input allocator composed of an optimizer and an annihilator, proposing several policies for designing the dynamics of the former. By doing so, we enable the system to overcome the saturating condition and recover its intended behaviour. The effectiveness of the proposed allocators in mitigating the windup effect has been tested and validated by numerical simulations.**

***Index Terms*—Constrained control, linear systems, optimization algorithms.**

I. Introduction

LINEAR input redundant systems posses a larger number of independent control actions than control variables [1]. These dynamics can be categorized into two types [2], according to the properties of the input matrix B. Considering p as the number of outputs, the term *strongly input redundant* refers to linear systems in which $\text{rank}(B) = p$, whereas the term *weakly input redundant* describes linear systems where $\text{rank}(B) > p$. The control of such dynamical system is commonly referred to as *control allocation problem* [3]. This problem aims at exploiting the intrinsic extra degrees of freedom to achieve secondary objectives and finds applications in various domains such as aerospace [4] or automotive systems [5]. The presence of input constraints introduces additional challenges in designing control algorithm that needs to meet specified behavior in the absence of input saturation [6]. One possible approach is augmenting the nominal controller with a device aimed at helping the system in recovering its nominal behaviour [7]. This approach has been explored both as an $\mathcal{H}_\infty$ optimal control problem [8], analysing its performance as an $\mathcal{L}_2$ gain minimization problem, and as the resolution of a convex optimization problem exploiting linear matrix inequalities [9]. In output regulation problems, the presence of input constraints has been addressed in different ways. In [10] a dynamic allocation scheme is proposed, based on the augmentation of the internal model with an annihilator, while [11] introduces reference governors capable of manipulating the tracking error so to comply with the input constraints.

This letter focuses on the design of output invisible allocators for weakly redundant systems, in the presence of input saturation. Multiple approaches exist for designing an output invisible signal, using either algebraic or geometric methodologies [1], [2], [12]. The main novelty lies in extending the setup of [12] to systems subject to input limitations and showing that the allocator is a viable tool for mitigating the windup issue. In particular, the extra degrees of freedom inherited from input redundancy can be exploited for this purpose. Taking inspiration from the parallel allocator proposed in [12] and from the way of dealing the input saturation block in [13], we have considered the deadzone nonlinearity signal, i.e., the mismatch between the control and its saturated value, as an optimization variable in the optimizer dynamics [12, Sec. IV.A]. Accordingly, we have proposed several optimization objectives to mitigate the windup effect providing a detailed examination of stability and feasibility either in terms of direct Lyapunov analysis [14] or in terms of LMIs conditions [15].

This letter is organized as follows. Section II recalls the theoretical background for the design of an output invisible allocator for weakly redundant linear plants. Section III extends the problem to overactuated systems with input saturation and the proposed solutions are discussed in Sections IV and V. The effectiveness of the different methods is illustrated in Section VI by means of numerical simulations, while the final considerations are collected in Section VII.

Notation: Given a square matrix X, we define $\text{He}(X) := X^T + X$, while given a finite set Ω, the symbol $|\Omega|$ denotes its cardinality.

II. Output Invisible Control Allocation

Consider a weakly input redundant system $\mathcal{P}$, according to the definition provided in [1] and in [2], described by the following state-space representation

$$\mathcal{P} : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

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Algorithm 1 Annihilator Design

- Let $W(s)$ be the transfer matrix of the plant (1).
- Compute a left coprime polynomial factorization $D^{-1}(s)N(s)$ of the transfer matrix $W(s)$.
- Compute the unimodular polynomial matrices $L(s)$ and $R(s)$ such that $L(s)N(s)R(s)$ is in Smith form.
- Define $R_2(s)$ as the last $m-p$ columns of $R_2(s)$.
- Choose $\Psi(s) = \mathrm{diag}(\psi_1(s), \ldots, \psi_{m-p}(s))$ such that each $\psi_h(s)$ has real coefficients, degree not smaller than the highest degree in the hth column of $R_2(s)$ and all roots in a desired set $\mathbb{C}_g \subset \{s : \mathrm{Re}(s) < 0\}$.
- Define the annihilator as any minimal realization of $W_{an}(s) = R_2(s)\Psi^{-1}(s)$.

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ with $m > p$. A general controller $\mathcal{C}$ for the system (1) is given by

$$\mathcal{C} : \begin{cases} \dot{x}_c = A_c x_c + B_c u_c \\ y_c = C_c x_c \end{cases} \tag{2}$$

The interconnection between (1) and (2) is expressed by

$$u_c = \omega - y \quad u = y_c + \bar{u} \tag{3}$$

where ω is a suitable measurable exogenous input, e.g., a reference signal, and $\bar{u}$ is the output of a device able to exploit the weak input redundancy of (1).

Assumption 1: The system $\{A, B, C\}$ is minimum phase and the closed-loop system (1)-(3) is asymptotically stable with $\bar{u} = 0$ and $\omega = 0$.

Regarding the output invisible allocator we refer to the one proposed in [12], which is designed as a cascade of an *optimizer* and an *annihilator*. The former is described by the dynamics

$$Opt : \begin{cases} \dot{x}_{opt} = -\alpha \nabla_{x_{opt}} J(x_{opt}, u_{opt}) \\ v \quad = C_{opt} x_{opt} \end{cases} \tag{4}$$

where the function $J(x_{opt}, u_{opt})$ encodes secondary optimization objectives, u_{opt} is the optimizer input that depends on the quantity being minimized in the cost function and $\alpha \in \mathbb{R}_{>0}$ is the rate of convergence towards the minimum of the function. The annihilator is designed to render the overall action of the allocator completely invisible for the plant output. We report one possible synthesis algorithm based on factorization of the plant transfer function.

The resulting annihilator can be described by the following state-space representation

$$An : \begin{cases} \dot{x}_{an} = A_{an} x_{an} + B_{an} u_{an} \\ y_{an} = C_{an} x_{an} + D_{an} u_{an} \end{cases} \tag{5}$$

where $\{A_{an}, B_{an}, C_{an}, D_{an}\}$ is any realization of the annihilator transfer function $W_{an}(s)$, defined in the bottom item of Algorithm 1 and u_{an}, according to the cascade architecture of the allocator, coincides with the output of the optimizer, i.e., $u_{an} = v$. Moreover, if we choose a quadratic function $J(x_{opt}, u_{opt})$ in (4), resulting in a linear gradient, then the optimizer can also be expressed in a linear state-space representation of the form

$$Opt : \begin{cases} \dot{x}_{opt} = -\alpha\left(A_{opt} x_{opt} + B_{opt} u_{opt}\right) \\ y_{opt} = C_{opt} x_{opt} \end{cases} \tag{6}$$

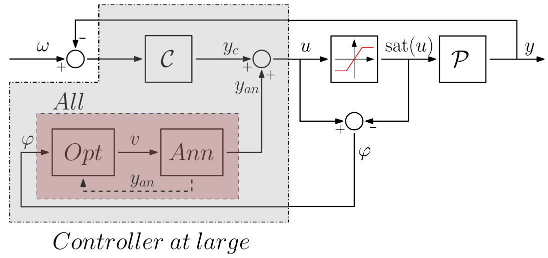


Fig. 1. Control allocation scheme. The dashed line in the allocator sub-system refers to the possibility of feeding the control y_{an} into the optimization function that governs the optimizer dynamics.

III. Input Constrained Control Allocation

In this section we consider the output invisible control allocation problem for weakly redundant systems subject to input limitations. The plant taken into account is

$$\mathcal{P} : \begin{cases} \dot{x} = Ax + B\mathrm{sat}(u) \\ y = Cx \end{cases} \tag{7}$$

where $\mathrm{sat}(\cdot)$ is a decentralized and symmetric saturation function, ensuring that $u \in \mathcal{U}$ with

$$\mathcal{U} = \{u \in \mathbb{R}^m : -\rho_i \le u_i \le \rho_i, i = 1, \ldots, m\}.$$

The formal definition of the problem is introduced in the following statement

Problem 1: Design an output invisible allocator for the weakly redundant systems (7) able to counter-balance the windup effect due to the saturation.

The proposed solution involves designing an optimizer (4) that effectively manages the input constraints characterizing (7). This is achieved by exploiting the extra control components, by means of the additional control action y_{an}, to help driving the system away from the saturation condition, as shown in Figure 1.

Dealing with saturating inputs is a multi-faced job and, for this reason, we have considered different optimization objectives aimed at mitigating the windup effect as solutions to Problem 1:

- *Steady-state optimization*: taking inspiration from [12], we want to minimize the norm of φ at steady-state, to avoid the possible damaging coming from the use of actuators at their limit in long-term applications.
- *Transient optimization*: we want to minimize the norm of φ not only at steady-state but also along the transient.
- *Mixed optimization*: we finally consider a situation in between the previous solution and the optimization goal proposed in [12], minimizing a combination of the norm of φ during transient and the norm of the total control at steady-state.

The dynamics of the total system $\mathcal{H} = \{\mathcal{P}, \mathcal{C}, Opt, An\}$ is described by the state $x_{cl} := [x_p, x_c, x_{opt}, x_{an}]^T$ with inputs the exogenous signal ω and the deadzone nonlinearity φ. Along with the interconnections between the plant and the controller in (3), we consider the following relationship within the allocator subsystems, setting $\bar{u} = y_{an}$

$$\begin{aligned} u_{an} &= v, & u_{opt} &= \varphi, \\ u &= y_c + y_{an}, & \varphi &= u - \mathrm{sat}(u) \end{aligned} \tag{8}$$

Let us first define the state-space representation of the allocator considering its state as $x_{all} = [x_{opt}, x_{an}]^T$

$$All : \begin{cases} \dot{x}_{all} = A_{all}x_{all} + B_{all}\varphi \\ y_{all} = C_{all}x_{all} \end{cases} \tag{9}$$

where

$$A_{all} = \begin{bmatrix} -\alpha A_{opt} & 0_{n_{opt}\times n_{an}} \\ B_{an} & A_{an} \end{bmatrix} \quad B_{all} = \begin{bmatrix} -\alpha B_{opt} \\ 0_{n_{an}\times m} \end{bmatrix}$$
$$C_{all} = \begin{bmatrix} D_{an} & C_{an} \end{bmatrix}$$

Based on the allocator structure, the overall system dynamics reads as

$$\mathcal{H} : \begin{cases} \dot{x}_{cl} = A_{cl}x_{cl} + B_{cl,\omega}\omega + B_{cl,\varphi}\varphi \\ y \;\; = C_{cl}x_{cl} \\ u \;\; = C_{cl,u}x_{cl} + D_{cl,u\omega}\omega + D_{cl,u\varphi}\varphi \\ \varphi \;\; = u - \mathrm{sat}(u) \end{cases} \tag{10}$$

where

$$A_{cl} = \begin{bmatrix} A & BC_c & BC_{all} \\ -B_cC & A_c & 0_{n_c\times n_{all}} \\ 0_{n_{all}\times n_p} & 0_{n_{all}\times n_c} & A_{all} \end{bmatrix}$$
$$B_{cl,w} = \begin{bmatrix} 0_{n_p\times p} \\ B_c \\ 0_{n_{all}\times p} \end{bmatrix} \; B_{cl,\varphi} = \begin{bmatrix} -B \\ 0_{n_c\times m} \\ B_{all} \end{bmatrix}$$
$$C_{cl} = \begin{bmatrix} C & 0_{p\times n_c} & 0_{p\times n_{all}} \end{bmatrix} \; C_{cl,u} = \begin{bmatrix} 0_{m\times n_p} & C_c & C_{all} \end{bmatrix}$$

As anticipated, there is still freedom in the design of the optimizer dynamics, and this will be addressed in the following sections.

IV. Optimizer Design: Steady-State Optimization

In this section we present the first solution to Problem 1, designing an optimizer whose ultimate goal is the minimization of the norm of the deadzone at steady-state.[1] To this end, with the aim of enabling for a cascade interconnection of annihilator and optimizer, let us introduce the quantity

$$\xi = y_c + W_{an}(0)x_{opt} \tag{11}$$

where the matrix $W_{an}(0) = \begin{bmatrix} W_1^T \cdots W_{n_y}^T \end{bmatrix}^T \in \mathbb{R}^{n_y\times n_x}$ is the static gain of (5), which is always well defined since Algorithm 1 provides an asymptotically stable annihilator. The signal ξ represents the steady-state of the input provided by the *controller at large*, as long as both y_c and x_{opt} are at steady-state. Accordingly, we define the auxiliary variable $\varphi_{ss} = \xi - \mathrm{sat}(\xi)$, which can be interpreted as a *pre-steady-state* of the deadzone φ and can be expanded as follows

$$\varphi_{ss} = \begin{bmatrix} \sum_{i=1}^{n_x} W_{1,i}x_{opt,i} + y_1 - \mathrm{sat}(\xi_1) \\ \vdots \\ \sum_{i=1}^{n_x} W_{n_y,i}x_{opt,i} + y_{n_y} - \mathrm{sat}(\xi_{n_y}) \end{bmatrix} \tag{12}$$

In this case the optimizer dynamics is formally governed by the gradient of the following quadratic cost function

$$J(\varphi_{ss}) = \frac{1}{2}\varphi_{ss}^T\varphi_{ss} \rightarrow \nabla J(\varphi_{ss}) = \varphi_{ss}^T\frac{\partial\varphi_{ss}}{\partial x_{opt}} \tag{13}$$

[1]Let us point out that, whenever the global minimum $\varphi = 0$ is reached, we are in the desirable situation of non-saturating steady-state inputs.

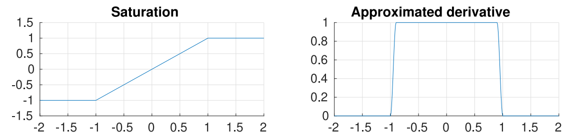


Fig. 2. Saturation function and approximated derivative.

The choice of the cost function (13) introduces in the optimizer dynamics (4) the derivative of the saturation function, which is not differentiable. Consequently, we adopt a smooth approximation, denoted by $\frac{\tilde{\partial}\mathrm{sat}(\zeta_i)}{\partial\zeta_i}$, having the following basic properties:

- $\frac{\tilde{\partial}\mathrm{sat}(\zeta_i)}{\partial\zeta_i} \in [0, 1]$ and is everywhere continuous;
- $\frac{\tilde{\partial}\mathrm{sat}(\zeta_i)}{\partial\zeta_i} = 0$ in $(-\infty, -\rho_i] \cup [\rho_i, \infty)$;
- there exists $\epsilon > 0$ with $\frac{\tilde{\partial}\mathrm{sat}(\zeta_i)}{\partial\zeta_i} = 1$ in $(-\rho_i + \epsilon, \rho_i - \epsilon)$

An example is shown in Figure 2, obtained by third-order polynomial approximation. Accordingly, by replacing the deadzone derivative $\frac{\partial\varphi_{ss}}{\partial x_{opt}}$ with the smooth approximation

$$\frac{\tilde{\partial}\varphi_{ss}}{\partial x_{opt}} := \left[I - \frac{\tilde{\partial}\mathrm{sat}(\xi)}{\partial\xi}\right]W_{An}(0),$$

we define the optimizer dynamics

$$Opt_{ss} : \begin{cases} \dot{x}_{opt} = -\alpha\varphi_{ss}^T\frac{\tilde{\partial}\varphi_{ss}}{\partial x_{opt}} \\ v \;\;\; = C_{opt}x_{opt} \end{cases} \tag{14}$$

It is worth noticing that the approximated derivative of the deadzone enjoys the useful property:

$$\frac{\tilde{\partial}\varphi_{ss}}{\partial x_{opt}} = 0 \implies \frac{\partial\varphi_{ss}}{\partial x_{opt}} = 0. \tag{15}$$

A. Input-to-State Boundedness (BIBS)

The optimizer dynamics (14), along with (12) and (13), can be rewritten as

$$\dot{x}_{opt} = -\alpha\sum_{i=1}^{n_y} W_i^T\big[W_ix_{opt} + (y_i \pm \rho_i)\big]\chi(\varphi_{ss,i}) \tag{16}$$

where ρ_i is the saturation limit of the ith component of the control and

$$\chi(s) = \begin{cases} 0 & \text{if } s = 0 \\ 1 & \text{if } s \neq 0 \end{cases} \tag{17}$$

is an *activation function* selecting the saturating components of the control. In fact, the dynamics of the optimizer (16) is only driven by the saturating components whereas the contributions from the others disappear.

Theorem 1: Consider a control allocation architecture (9) with an optimizer defined by (14). Under Assumption 1, for any bounded nominal control input y_c, the optimizer state x_{opt}, driven by the dynamics (16), remains bounded.

Proof: Consider the Lyapunov function candidate

$$V = \frac{1}{2}x_{opt}^Tx_{opt} \tag{18}$$

and let us evaluate its time derivative along the optimizer dynamics (16)

$$\begin{aligned}\dot V &= -\alpha x_{opt}^T \sum_{\ell=1}^{n_s} W_{i_\ell}^T\big[W_{i_\ell} x_{opt} + (y_{i_\ell} \pm \rho_{i_\ell})\big] \\ &= -\alpha x_{opt}^T \sum_{\ell=1}^{n_s} (W_{i_\ell}^T W_{i_\ell}) x_{opt} - \alpha x_{opt}^T \sum_{\ell=1}^{n_s} W_{i_\ell}^T (y_{i_\ell} \pm \rho_{i_\ell}) \quad (19)\end{aligned}$$

where n_s represents the number of components that are saturated and $\mathcal{I}_{n_s} := (i_1, \ldots, i_{n_s})$ is the multi-index encoding the labels of the saturating components. The natural idea to prove boundedness would be invoking an ISS theorem [14]. However, this is not possible to be applied such a procedure because the matrix $\sum_{\ell=1}^{n_s}(W_{i_\ell}^T W_{i_\ell})$ in (19) is typically not full-rank, unless $n_s = n_y$. To overcome this issue, we will then resort to a weaker version of the ISS condition. To this end, let us begin by analysing the evolution restricted onto $\ker(W_{\mathcal{I}_{n_s}})$, where $W_{\mathcal{I}_{n_s}} := \left[W_{i_1}^T, \ldots W_{i_{n_s}}^T\right]^T$. The projection on $\ker(W_{\mathcal{I}_{n_s}})$ of a generic x_{opt} is defined as

$$\tilde x = \Pi_{W_{\mathcal{I}_{n_s}}^\perp} x_{opt} \quad (20)$$

where $\Pi_{W_{\mathcal{I}_{n_s}}^\perp}$ is the linear projection operator

$$\Pi_{W_{\mathcal{I}_{n_s}}^\perp} = (I - W_{\mathcal{I}_{n_s}}^T (W_{\mathcal{I}_{n_s}} W_{\mathcal{I}_{n_s}}^T)^{-1} W_{\mathcal{I}_{n_s}}) \quad (21)$$

By a simple calculation, the dynamics along the projection of (20) reads as

$$\begin{aligned}\dot{\tilde x} &= \Pi_{W_{\mathcal{I}_{n_s}}^\perp} \dot x_{opt} \\ &= \Pi_{W_{\mathcal{I}_{n_s}}^\perp} W_{\mathcal{I}_{n_s}}^T \big[W_{\mathcal{I}_{n_s}} x_{opt} + (y_{\mathcal{I}_{n_s}} \pm \rho_{\mathcal{I}_{n_s}})\big] = 0 \quad (22)\end{aligned}$$

This proves that x_{opt} never evolves along directions belonging to the null-space of $W_{\mathcal{I}_{n_s}}$, which are *invisible* for $\dot V$, and therefore any increase of $|x_{opt}|$ will always contribute with a negative term in the derivative of the Lyapunov function. Bearing this in mind, we can manipulate $\dot V$ as follows

$$\begin{aligned}\dot V \le &-\alpha\, \overbrace{x_{opt}^T \sum_{\ell=1}^{n_s} (W_{i_\ell}^T W_{i_\ell}) x_{opt}}^{=:\Psi(x_{opt})} \\ &+ \alpha \underbrace{\left(\max_{\ell=1,\ldots,n_s} \left|x_{opt}^T W_{i_\ell}^T\right|\right)}_{=:\Phi(x_{opt})} \underbrace{\left(\max_{\ell=1,\ldots,n_s} (|y_{i_\ell}| - \rho_{i_\ell})\right)}_{=:\Gamma(y)} \quad (23)\end{aligned}$$

We see that, whenever x_{opt} is such that

$$\left(\frac{\alpha - \varepsilon}{\alpha}\right) \Phi^{-1}(x_{opt}) \Psi(x_{opt}) > \Gamma(y) \quad (24)$$

with $\varepsilon \in \mathbb{R}$ and $\varepsilon < \alpha$, then

$$\dot V \le -\varepsilon x_{opt}^T \sum_{\ell=1}^{n_s} (W_{i_\ell}^T W_{i_\ell}) x_{opt} \le 0 \quad (25)$$

thus proving the claimed boundedness property. ∎

Remark 1: The boundedness of y_c is always ensured for stable plants. Conversely, for open-loop exponentially unstable plants, boundedness can be expected only for operating conditions within a limited region, so that the allocator needs to be fast enough to anticipate the windup of the system.

B. Convergence Analysis

Let us know establish the convergence properties of the allocator (5)-(14).

Theorem 2: Consider the input-constrained weakly redundant plant (1), a control allocation architecture (9) and an optimizer defined by (14). Under Assumption 1, for any value of the parameter $\alpha \in \mathbb{R}_{>0}$, the allocator guarantees that φ_{ss} asymptotically approaches the set $\{\varphi_{ss}^T \frac{\partial \varphi_{ss}}{\partial x_{opt}} = 0\}$.

Proof: We choose as candidate Lyapunov function the cost function J defined in (13) itself, with $\dot J = \varphi_{ss}^T \dot\varphi_{ss}$. Now, using the notation (11), the expressions of φ_{ss} and of the derivative $\dot\varphi_{ss}$ read as

$$\varphi_{ss} = \xi - \mathrm{sat}(\xi) \quad \dot\varphi_{ss} \in \left[I - \frac{\partial \mathrm{sat}(\xi)}{\partial \xi}\right] W_{An}(0) \dot x_{opt} \quad (26)$$

with the differential inclusion reducing to a simple equality everywhere except at the saturation limits, i.e., $|\xi_i| = \rho_i$, where a Filippov-like condition can be invoked [16, Sec. 2.7, Corollary 2, Th. 1], by means of the generalized differential

$$\frac{\partial \mathrm{sat}(\xi_i)}{\partial \xi_i} \in \mathrm{co}\{0, 1\} \quad (27)$$

with co standing for the convex hull. Substituting (14) in (26), and replacing (26) in $\dot J = \varphi_{ss}^T \dot\varphi_{ss}$, we get

$$\begin{aligned}\dot J &\in -\alpha \varphi_{ss}^T \underbrace{\left[I - \frac{\partial \mathrm{sat}(\xi)}{\partial \xi}\right] W_{An}(0)}_{= \frac{\partial \varphi_{ss}}{\partial x_{opt}}} \varphi_{ss}^T \frac{\tilde\partial \varphi_{ss}}{\partial x_{opt}} \\ &\le -\alpha \left[\varphi_{ss}^T \frac{\partial \varphi_{ss}}{\partial x_{opt}}\right]^2 \le 0 \quad (28)\end{aligned}$$

where condition (15), combined with (27), has been used to infer the majorization. In particular, the latter condition implies that the set $\{\varphi_{ss}^T \frac{\partial \varphi_{ss}}{\partial x_{opt}} = 0\}$ is attractive and forward invariant. This concludes the proof. ∎

Remark 2: The selection of the cost function (13) aims at minimizing the norm of the deadzone nonlinearity at steady-state, through the minimization of $|\varphi_{ss}|$, to the greatest extent possible, considering the actuators' capabilities. In the most favourable situations, the effect of the optimizer (14) would then result in $\lim_{t\to\infty} |\varphi_{ss}| = 0$.

V. Optimizer Design: Transient Optimization

In this section we propose the second solution to Problem 1, choosing as cost function the norm of the deadzone nonlinearity φ along the whole transient

$$J(\varphi) = \frac{1}{2}\varphi^T \varphi \to \nabla J(\varphi) = \varphi^T \dot\varphi \quad (29)$$

where $u = y_c + y_{an}$ and $\varphi = u - \mathrm{sat}(u)$ and its time derivative w.r.t. x_{opt} is

$$\dot\varphi = \underbrace{\left[I - \left.\frac{\partial \mathrm{sat}(\zeta)}{\partial \zeta}\right|_{\zeta = u}\right]}_{=:\mathcal{M}} D_{an} \quad (30)$$

where $\mathcal{M} \in \Theta = \{\mathrm{diag}(v) : v \in \{0, 1\}^m\}$, i.e., $\mathcal{M}_{ii} = 1$ if the ith component of ζ is saturated and $\mathcal{M}_{ii} = 0$ otherwise. The adoption of (29) as cost function aims at driving the plant (1) out of the saturating condition not only at steady-state, as for (14), but also in the transient phase. It is important to stress that the choice of (29) introduces a feedback loop within the allocator structure, see Figure 1, violating the stability condition in [12, Sec. III], which is based on a cascade argument. Bearing this in mind, we can know provide the tools for the stability analysis of the overall system $\mathcal{H}$.

A. Stability Analysis: LMI Formulation

In this section we are going to state the sufficient conditions for stability and performance of the closed-loop system $\mathcal{H}$ with cost function (29) in terms of LMI conditions. First, let us define $n_\Theta = |\Theta|$ and $\hat{B}_{cl}$ the set of all possible matrices $B_{cl,\varphi}$ resulting from all the possible saturating conditions described by the elements of Θ.

Theorem 3: Fix $\alpha > 0$ and assume the existence of a unique symmetric positive definite matrix $P \in \mathbb{R}^{n_{cl}\times n_{cl}}$, non-negative diagonal matrices $U_j \in \mathbb{R}^{m\times m}$ and a positive scalar γ such that the set of LMIs

$$\mathrm{He}\left(\begin{bmatrix} PA_{cl} & P\hat{B}_{cl,j} + C_{cl,u}^T U_j & PB_{cl,w} & C_{cl}^T \\ 0 & U_j D_{cl,u\varphi} - U_j & U_j D_{cl,u\omega} & 0 \\ 0 & 0 & -\frac{\gamma}{2} I_{n_\omega} & 0 \\ 0 & 0 & 0 & -\frac{\gamma}{2} I_{n_y} \end{bmatrix}\right) < 0$$

for $j = 1, \dots, n_\Theta$ are satisfied. Then the system $\mathcal{H}$ (10) is asymptotically stable with $\mathcal{L}_2$ gain from the exogenous input ω to the output y less than γ for all $j = 1, \dots, n_\Theta$.

Proof: Without loss of generality, let us consider the jth saturating condition among the n_Θ possible scenarios. First of all we inscribe the decentralized deadzone nonlinearity into a conic region [14], yielding in the condition

$$\varphi^T U_j (u - \varphi) \geq 0 \tag{31}$$

where $U_j \in \mathbb{R}^{m\times m}$ is an arbitrary positive diagonal matrix. Considering the candidate Lyapunov function $V = x_{cl}^T P x_{cl}$, with $P = P^T > 0$, external stability is guaranteed if the following inequality holds

$$\dot{V} \leq -\gamma \left[\frac{1}{\gamma^2} y^T y - \omega^T \omega\right] \tag{32}$$

Then, along with (31) the condition on the time derivative translates to the condition

$$\varphi^T U_j (u - \varphi) \geq 0 \Rightarrow \dot{V} \leq -\gamma \left[\frac{1}{\gamma^2} y^T y - \omega^T \omega\right] \tag{33}$$

which can be rewritten, as implication to the S-procedure, as

$$\dot{V} + \frac{1}{\gamma} y^T y - \gamma \omega^T \omega + 2\varphi^T U_j (u - \varphi) < 0 \tag{34}$$

Then, using the definition of y in (10), (34) is equivalent to

$$\mathrm{He}\left(\begin{bmatrix} PA_{cl} & P\hat{B}_{cl,j} + C_{cl,u}^T U_j & PB_{cl,w} \\ 0 & U_j D_{cl,u\varphi} - U_j & U_j D_{cl,u\omega} \\ 0 & 0 & -\frac{\gamma}{2} I_{n_\omega} \end{bmatrix}\right) + \frac{1}{\gamma}\begin{bmatrix} C_{cl}^T \\ 0 \\ 0 \end{bmatrix}\left[C_{cl}\ 0\ 0\right] < 0 \tag{35}$$

Applying the Schur complement we can rewrite (35) as the following LMIs

$$\mathrm{He}(\Psi_j) < 0 \quad U_j = U_j^T > 0 \quad P = P^T > 0 \tag{36}$$

with

$$\Psi_j = \begin{bmatrix} PA_{cl} & P\hat{B}_{cl,j} + C_{cl,u}^T U_j & PB_{cl,w} & C_{cl}^T \\ 0 & U_j D_{cl,u\varphi} - U_j & U_j D_{cl,u\omega} & 0 \\ 0 & 0 & -\frac{\gamma}{2} I_{n_\omega} & 0 \\ 0 & 0 & 0 & -\frac{\gamma}{2} I_{n_y} \end{bmatrix}$$

In conclusion, if the set of LMIs (36) admit a unique solution P for all $j = 1, \dots, n_\Theta$, then system $\mathcal{H}$ (10) is asymptotically stable for the given choice of $\alpha > 0$ in the optimizer dynamics (6), with $\mathcal{L}_2$ gain from the exogenous input ω to the output y less than γ. ■

B. Optimizer Design: Mixed Optimization

It is worth reporting also a variation of the previous solution, consisting of a cost function $J(\varphi)$ given by the sum of the norm of φ in the transient and the norm of the control at steady-state, that is

$$J(\varphi) = \frac{1}{2}\left(\beta ||u_{ss}||^2 + ||\varphi||^2\right) \tag{37}$$

In this case the objective of the minimization corresponds to a trade-off between minimizing overall control effort and windup mitigation. The design parameter $\beta \in \mathbb{R}_{>0}$ has to be chosen to adjust the priority in the optimization goal. The resulting closed-loop system $\mathcal{H}$ in the form (10) can be proved to be asymptotically stable for a given $\alpha > 0$ and $\beta > 0$ with a similar reasoning to the one used for proving Theorem 3, i.e., by suitably modifying the LMI conditions.

VI. Illustrative Example

In this section we illustrate the performance of the proposed allocation-based windup mitigation scheme, comparing the different options for the optimizer design with the framework proposed in [1, Sec. 4], adapted to the weakly redundant case. We consider a rather simple academic example, yet capable of catching several interesting features of the proposed allocator in a fairly challenging saturating scenario.

Let us consider then a weakly redundant plant described by

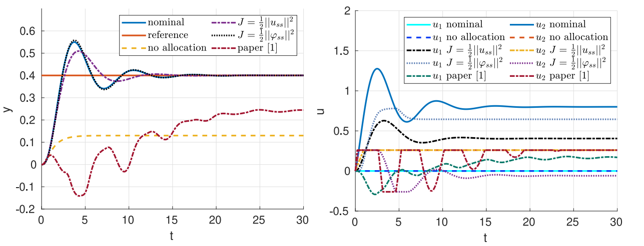
$$A = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -3 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and a controller described by

$$A_c = \begin{bmatrix} 0 & 2.22 \\ 0 & -3 \end{bmatrix} B_c = \begin{bmatrix} 0 \\ 2 \end{bmatrix} C_c = \begin{bmatrix} 0 & 0 \\ 1.468 & 0.66 \end{bmatrix}$$

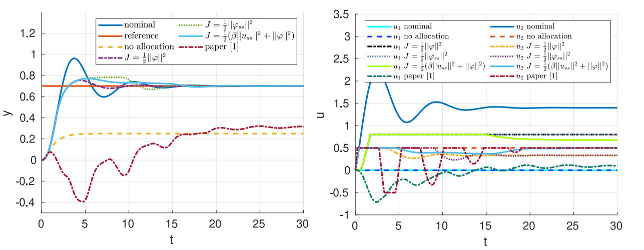
In nominal conditions, i.e., without allocation, this controller uses just the second component of the control of the plant (1) for a regulation task, leaving the other component available for achieving secondary objectives.[2] Moreover, the allocator is composed by the cascade of the annihilator, designed following Algorithm 1, and of the optimizer, which

[2]This choice is somewhat naive but is useful for better illustrating the exploitation of the input redundancy for anti-windup purposes.



(a) Output (b) Control

Fig. 3. Comparison between the minimization of the control at steady-state [12] and the minimization of φ at steady-state.



(a) Output (b) Control

Fig. 4. Comparison between the all the three types of optimizer proposed.

dynamics depends on the type of optimization we want to apply (13), (29) or (37). In all the simulations we are going to present, we have compared the proposed method with the nominal condition, i.e., without input limitations and with the case where no allocation is applied.

A. Steady-State Optimization

Here we present the results of our simulations using the optimizer (14), comparing its performances with the one proposed in [12], which aims to minimize the norm of the total control at steady-state. In Figure 3(b), we can observe the improvement achieved by the proposed optimizer in addressing the windup issues caused by saturation. Specifically, the second component of the control signal is no longer saturated, in contrast to the optimizer used in [12], where the second actuator remains saturated. Similarly, the action of the anti-windup compensator in [1] cannot prevent the saturation of the second component. The improved mitigation effect is also reflected in the system's output, as shown in Figure 3(a), yielding in an almost exact reproduction of the nominal behaviour.

B. Transient and Mixed Optimization

Here we show the outcomes of our simulations where we utilized the cost functions (29) and (37) as optimization objective, comparing their performances with the previous optimizer. In the case of (37) we have chosen $\beta < 1$ to prioritize minimizing $|\varphi|^2$ over $|u_{ss}|^2$. In Figure 4(b), we can see the further improvement achieved by using (29) compared to (13), as the second component of the control exits the saturating condition earlier than with the previous method. Conversely, the mixed optimization approach (37) reflects a situation that lies between steady-state optimization and transient optimization.

VII. Conclusion

In this letter we have addressed the problem of designing an output invisible allocator for weakly redundant plants subject to input limitations. The proposed solution focuses on selecting a suitable cost function for the optimizer dynamics to mitigate the windup effect caused by saturation. We have proposed several solutions to the problem, providing stability conditions for the designed optimizer. These conditions are expressed either in terms of Lyapunov analysis, referring to the steady-state optimization, or in terms of LMIs, which relate to the transient optimization. In all the simulations presented, the proposed allocator outperforms both the frameworks proposed in [12] and in [1] in terms of windup mitigation. Future works will be devoted to analyse the co-design [17] of anti-windup schemes and control allocation in weakly input redundant plants and to the application of the results in multi-agent scenarios.

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