Hausdorff Dimension Estimates for Interconnected Systems With Variable Metrics

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***Abstract*—In this letter, we develop a framework for estimating the Hausdorff dimension of a compact invariant set for both autonomous and interconnected systems. We first generalize Smith's method for Hausdorff dimension estimates by using variable metrics in linear matrix inequalities. Then, we study open systems with a characterization similar to the differential dissipativity theory. For linear time-invariant systems, we show that our characterization can be considered as a pure input/output property. This fact would be important because it is independent of internal model representations. Finally, we provide an estimate of the attractor dimension for feedback and interconnected systems. Our estimation is scalable in the sense that the components in an interconnected system can be analyzed independently.**

***Index Terms*—Hausdorff dimension, interconnected systems, Lur'e systems, nonlinear systems.**

I. Introduction

IN THE control community, internal and external aspects of dynamical systems have been studied for a long time. However, there is a large theoretical gap between research on *closed* systems (systems without inputs and outputs) and that on *open* systems (systems with inputs and outputs). The purpose of this letter is to generalize a framework for estimating the Hausdorff dimension of a compact invariant set, which represents the degree of freedom of asymptotic behaviors, to open systems. According to [1], dimension-like characteristics such as Hausdorff dimension, fractal dimension, and topological entropy play essential roles in dimension theory. These characteristics are useful to understand dynamical systems from various perspectives [2]. We mention that our approach is inspired by the recent works [3], [4], [5].

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As for closed systems, many researchers have made efforts to understand chaotic phenomena, which are difficult to predict but are important in engineering applications [6]. Although many criteria for global stability are available, such results are not applicable to systems which have more than two equilibria. Also, periodic oscillations are in general difficult to analyze in spite of their importance. Furthermore, complex attractors may not be manifolds, and thus, dimension with non-integer values becomes important in the analysis [7]. In this letter, for these reasons, we are concerned with finding estimates of Hausdorff dimensions of invariant sets. Our study is further motivated by the relation of Hausdorff dimension with entropy [8], which has recently attracted attentions in control theory [9].

In contrast, open systems are studied by input/output properties such as L_2-gain and passivity [10], [11]. A general framework that links internal stability and input/output properties is the celebrated dissipativity theory [12], [13], [14]. This energy-related framework is also useful to analyze stability of large-scale network systems in a scalable way [15]. However, relations between internal *instability* (e.g., chaotic dynamics) and input/output properties are not well understood. Recently, some results in this direction were reported based on the framework of [16], [17]. In [3], the existence and stability of limit cycles for interconnected systems were analyzed by transverse contraction. In [4], classical frequency-domain analysis was extended to systems with multiple equilibria and limit cycles. In [5], the authors utilized the inertia for quadratic storage functions to analyze dominating dynamics. In this letter, we consider another direction to explore interconnected systems with Hausdorff dimension.[1]

Literature Reviews and Contributions: We review some related works on dimension estimates briefly; see also the recent book [19]. In the seminal paper [20], a fundamental method to estimate the Hausdorff dimension of a compact invariant set was developed. The so-called Douady–Oesterlé theorem has been further studied with various tools. In [21], the author proposed a computationally efficient method to estimate Hausdorff dimension. In [22], [23], analogues of Lyapunov's second method were explored to improve the

[1]This letter is based on our conference paper [18] and contains full proofs and additional discussions.

estimate of the Douady–Oesterlé theorem. In [24], [25], compound matrices and logarithmic norms were used to study Hausdorff dimension. These results are closely related to k-contraction [26], [27], which is a natural generalization of standard contraction.

In this letter, we focus on Smith's method in [21] to make the conditions easy to verify. We first generalize the results in [21] by employing time- and state-dependent metrics. Another generalization can be found in [28], where k-contraction with respect to $L_1/L_2/L_\infty$ norms was studied without calculating compound matrices. In contrast to that work, our results allow for variable metrics, which correspond to the Riemannian metric in differential geometry. We then discuss a methodological relation between Smith's method and the differential Lyapunov framework in [17]. This motivates us to extend the framework to open systems. We follow the differential dissipativity theory in [5] to characterize open systems. For linear time-invariant systems, we provide a frequency-domain interpretation of unstable dynamics. Finally, we show that the attractor dimension of an interconnected system can be estimated in a scalable way. We emphasize that our framework is not compatible with that of dominance analysis in [5]. This is because our conditions depend on lower bounds of Jacobian's spectrum while those in [5] specify the separation of stable and unstable spectra.

Notations: Throughout this letter, we employ the following notations. Consider a matrix $A \in \mathbb{R}^{n\times n}$. Let $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ be the eigenvalues of $(AA^\mathsf{T})^{1/2}$ arranged in decreasing order. Also, let $\eta_1(A) \geq \cdots \geq \eta_n(A)$ be the eigenvalues of $(A + A^\mathsf{T})/2$ arranged in decreasing order. For a real number $d \in [0, n]$, we define the *truncated determinant of order d* (*d-determinant*) of A by

$$\det_d A := \begin{cases} 1 & \text{if } d = 0, \\ \sigma_1(A)\cdots\sigma_{d_0}(A)\sigma_{d_0+1}(A)^s & \text{otherwise,} \end{cases}$$

and the *truncated trace of order d* (*d-trace*) of A by

$$\mathrm{tr}_d A := \begin{cases} 0 & \text{if } d = 0, \\ \eta_1(A) + \cdots + \eta_{d_0}(A) + s\eta_{d_0+1}(A) & \text{otherwise,} \end{cases}$$

where $d_0 \in \{0, 1, \ldots, n-1\}$ and $s \in (0, 1]$ are chosen such that $d = d_0 + s$.

We denote by $\mathsf{D}\varphi$ the derivative of a smooth map φ. Given N column vectors $v_1, \ldots, v_N$, we write $\mathrm{col}(v_1, \ldots, v_N)$ to represent the vector stacking v_1 through v_N in one column. Given N matrices $M_1, \ldots, M_N$, we define $M_1 \oplus \cdots \oplus M_N$ as the block-diagonal matrix composed of M_1 through M_N. Also, $\mathbb{R}_+$ denotes the set of nonnegative real numbers, and $\mathcal{S}_+^n$ denotes the set of positive-definite symmetric matrices of size n.

Let $K \subset \mathbb{R}^n$ be a compact set. Given $d \in [0, n]$ and $\varepsilon > 0$, we define

$$\mu(K, d, \varepsilon) := \inf\left\{\sum_i (r_i)^d : K \subset \bigcup_i \mathcal{B}_i,\ r_i \leq \varepsilon\right\},$$

where the infimum is taken over all finite coverings of K by balls $\mathcal{B}_i$ with radii r_i. The *d-dimensional Hausdorff measure* of K is defined by

$$\mu_H(K, d) := \lim_{\varepsilon\to 0^+} \mu(K, d, \varepsilon) = \sup_{\varepsilon>0} \mu(K, d, \varepsilon).$$

Then, the *Hausdorff dimension* of K is defined by

$$\dim_H(K) := \inf\{d \in [0, n] : \mu_H(K, d) = 0\}.$$

We note that the Hausdorff dimension can be well defined and is less than or equal to n for every compact set in $\mathbb{R}^n$.

II. Dimension Estimates via Smith's Method

In this section, we extend Smith's method on Hausdorff dimension estimates in [21] and discuss its connection with the differential Lyapunov framework in [17].

A. Generalized Liouville's Formula

Consider a linear system

$$\dot{x} = A(t)x, \tag{1}$$

where $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a continuous function. Recall that a fundamental matrix is an $n \times n$ matrix whose columns are linearly independent solutions of (1). A generalization of the well-known Liouville's formula was first obtained in [21] and is stated below.

Lemma 1: Let $X : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ be a fundamental matrix of (1). Then, for each $k \in \{1, \ldots, n\}$, we have

$$\det_k X(t) \leq [\det_k X(0)] \exp \int_0^t \mathrm{tr}_k A(\tau)\, \mathrm{d}\tau, \quad t \geq 0.$$

The k-determinant of $X(t)$ can be interpreted as the k-dimensional volume of the parallelepiped spanned by certain k solutions [27]. This result plays a fundamental role in dimension theory [1]. However, in general, calculating the k-trace of $A(t)$ is computationally hard. The following proposition generalizes [21, Corollary 1.2], which avoids the computation of eigenvalues.

Proposition 1: Let $X : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ be a fundamental matrix of (1) normalized as $X(0) = I$. Suppose that there exist a continuous function $\lambda : \mathbb{R}_+ \to \mathbb{R}$ and a continuously differentiable function $P : \mathbb{R}_+ \to \mathcal{S}_+^n$ such that for all $t \geq 0$,

$$\dot{P}(t) + A(t)^\mathsf{T} P(t) + P(t)A(t) + 2\lambda(t)P(t) \succeq 0. \tag{2}$$

Then, for each $k \in \{1, \ldots, n\}$, we have

$$\det_k X(t) \leq \beta_k(t) \exp \int_0^t [\mathrm{tr}\, A(\tau) + (n-k)\lambda(\tau)]\, \mathrm{d}\tau \tag{3}$$

for all $t \geq 0$ with $\beta_k(t) := [\det_{n-k} P(0)^{-1} \cdot \det_{n-k} P(t)]^{1/2}$.

Proof: Let $Q(t) := P(t)^{1/2}$. By the coordinate transformation $y = Q(t)x$, we transform the system (1) into

$$\dot{y} = \tilde{A}(t)y, \tag{4}$$

where $\tilde{A}(t) := [Q(t)A(t) + \dot{Q}(t)]Q(t)^{-1}$. Then, the matrix inequality (2) can be rewritten as

$$\tilde{A}(t)^\mathsf{T} + \tilde{A}(t) + 2\lambda(t)I \succeq 0,$$

which implies that $\eta_i(\tilde{A}(t)) + \lambda(t) \geq 0$ for all $i \in \{1, \ldots, n\}$. Thus, we have

$$\mathrm{tr}_k\, \tilde{A}(t) = \mathrm{tr}\, \tilde{A}(t) - \sum_{i=k+1}^n \eta_i(\tilde{A}(t)) \leq \mathrm{tr}\, \tilde{A}(t) + (n-k)\lambda(t). \tag{5}$$

Since $Y(t) := Q(t)X(t)$ is a fundamental matrix of (4) such that $Y(0) = Q(0)$, Lemma 1 and (5) imply that

$$\det{}_k Y(t) \leq [\det{}_k Q(0)] \exp \int_0^t [\mathrm{tr}\, \tilde{A}(\tau) + (n-k)\lambda(\tau)]\, \mathrm{d}\tau.$$

Notice that $\mathrm{tr}\, \tilde{A}(t) = \mathrm{tr}\, A(t) + \mathrm{tr}\, \dot{Q}(t)Q(t)^{-1}$. From Jacobi's formula, we have

$$\mathrm{tr}\, \dot{Q}(t)Q(t)^{-1} = \frac{(\mathrm{d}/\mathrm{d}t)\det Q(t)}{\det\, Q(t)}.$$

We integrate both sides over time to obtain

$$\int_0^t \mathrm{tr}\, \dot{Q}(\tau)Q(\tau)^{-1}\, \mathrm{d}\tau = \log\det Q(t) - \log\det Q(0).$$

Therefore,

$$\det{}_k Y(t) \leq \det{}_k Q(0) \cdot \det Q(0)^{-1} \cdot \det Q(t) \cdot \exp \int_0^t [\mathrm{tr}\, A(\tau) + (n-k)\lambda(\tau)]\, \mathrm{d}\tau.$$

Finally, it follows from Horn's inequality that

$$\det{}_k X(t) \leq \det{}_k Q(t)^{-1} \cdot \det{}_k Y(t),$$

and hence,

$$\det{}_k X(t) \leq \det{}_{n-k} Q(0)^{-1} \cdot \det{}_{n-k} Q(t) \cdot \exp \int_0^t [\mathrm{tr}\, A(\tau) + (n-k)\lambda(\tau)]\, \mathrm{d}\tau.$$

Since $Q(t) = P(t)^{1/2}$, we complete the proof. ■

Remark 1: By limiting the metric $P(t)$ to be a constant matrix, Theorem 1 reduces to [21, Corollary 2.1]. Here, $P(t)$ specifies the transformation of (1) into (4). However, the obtained estimate of the exponential growth rate of the k-dimensional volume is associated with the coefficient matrix in (1) and not in (4). This estimate can be replaced with

$$\int_0^t [\mathrm{tr}\, A(\tau) + \mathrm{tr}\, \dot{Q}(\tau)Q(\tau)^{-1} + (n-k)\lambda(\tau)]\, \mathrm{d}\tau,$$

which may or may not be less conservative than the original one in (3). It is advantageous to use the estimate in (3) since we can avoid matrix calculations. This point is different from the results in [23], and our bound of the exponential growth rate is easy to calculate. The tightness of the obtained estimate depends on the location of the $n-k$ smallest eigenvalues of $\tilde{A}(t)$, as can be seen from (5). Thus, the result in Proposition 1 can be conservative especially when k is small.

B. Smith's Method With Variable Metrics

Consider a nonlinear autonomous system

$$\dot{x} = f(x), \tag{6}$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. Let $J(x) := \mathsf{D}f(x)$ denote the Jacobian matrix of $f(x)$. A fundamental result for dimension estimates is the so-called Douady–Oesterlé theorem [20]. This theorem can be stated as follows: For a given smooth mapping φ, if there exists a real number d such that $\det_d \mathsf{D}\varphi(x) < 1$ holds on a compact invariant set, then its Hausdorff dimension is smaller than d. Note that this estimate can also be used as an upper bound of the fractal dimension [29].

Here, we follow the approach in [21] and provide an extension with variable metrics.[2] The obtained estimate of the Hausdorff dimension can be conservative if d is small for the reason explained in Remark 1. In the theorem below, we use the following notation:

$$\dot{P}(x) := \begin{bmatrix} \mathcal{L}_f P_{11}(x) & \cdots & \mathcal{L}_f P_{1n}(x) \\ \vdots & \ddots & \vdots \\ \mathcal{L}_f P_{n1}(x) & \cdots & \mathcal{L}_f P_{nn}(x) \end{bmatrix},$$

where $\mathcal{L}_f$ is the Lie derivative of a real-valued function with respect to the vector field f. If $P(x)$ is constant, then we recover [21, Corollary 2.2].

Theorem 1: Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact invariant set of (6). Suppose that there exist a continuous function $\lambda: \mathbb{R}^n \to \mathbb{R}$ and a continuously differentiable function $P: \mathbb{R}^n \to \mathcal{S}_+^n$ such that for all $x \in \mathcal{A}$,

$$\dot{P}(x) + J(x)^\top P(x) + P(x)J(x) + 2\lambda(x)P(x) \succeq 0.$$

If there exists $d \in (0, n]$ such that

$$\mathrm{tr}\, J(x) + (n-d)\lambda(x) < 0, \quad x \in \mathcal{A},$$

then $\dim_H(\mathcal{A}) < d$.

Proof: The proof is similar to that in [21]. Consider a fundamental matrix X of the variational equation $\delta\dot{x} = J(x(t))\delta x$ associated with a solution x of (6) starting in $\mathcal{A}$. We apply Proposition 1 to obtain for each $k \in \{1, \ldots, n\}$,

$$\det{}_k X(t) \leq \beta_k(t) \exp \textstyle\int_0^t [\mathrm{tr}\, J(x(\tau)) + (n-k)\lambda(x(\tau))]\, \mathrm{d}\tau,$$

where β_k is as in Proposition 1. Note that β_k is a bounded function due to the compactness of $\mathcal{A}$. If $d = d_0 + s$ with $d_0 \in \{0, \ldots, n-1\}$ and $s \in (0, 1]$, then

$$\det{}_d X(t) = [\det{}_{d_0} X(t)]^{1-s} [\det{}_{d_0+1} X(t)]^s.$$

By setting $\beta_d(t) := \beta_{d_0}(t)^{1-s} \beta_{d_0+1}(t)^s$, we obtain

$$\det{}_d X(t) \leq \beta_d(t) \exp \int_0^t [\mathrm{tr}\, J(x(\tau)) + (n-d)\lambda(x(\tau))]\, \mathrm{d}\tau.$$

Thus, the hypothesis in the theorem implies that the exponential growth rate of $\det_d X(t)$ is negative, and thus, there is a finite time t for which $\det_d X(t) < 1$. Since every linearized flow map on $\mathcal{A}$ satisfies the condition of the Douady–Oesterlé theorem, we conclude the proof. ■

Remark 2: It must be noted that Theorem 1 does not guarantee convergence of the system's trajectories to the invariant set. However, many systems in nature are dissipative in the sense of Levinson [30], and hence, every trajectory of such a system converges to a compact set. Thus, the Hausdorff dimension of the largest invariant set can be used as the maximum degree of freedom of asymptotic behaviors. Several consequences of Theorem 1 are as follows:

1) When $d = 1$, the system cannot possess any compact invariant set except for a single equilibrium point.
2) When $d = 2$, the system cannot possess any limit cycle. This is because if there is a closed orbit, then there must be a surface which is invariant under the flow.
3) When $d > 2$, chaotic attractors may appear like the famous Lorenz system.

[2]Time-varying periodic systems were also considered in [21], and it is not difficult to extend our result to such periodic systems.

C. Methodological Remarks

Our results in this letter are closely related to the differential framework in [17] and its generalization in [5]. In particular, under the hypothesis in Theorem 1, the function $V(x, \delta x) = \delta x^{\mathsf{T}} P(x)\delta x$ satisfies the inequality

$$\dot{V}(x, \delta x) + 2\lambda(x)V(x, \delta x) \geq 0 \tag{7}$$

along the solutions to the system

$$\begin{cases} \dot{x} = f(x), \\ \delta\dot{x} = J(x)\delta x. \end{cases}$$

From this fact, it is not difficult to generalize the framework on Hausdorff dimension estimates to open systems with the help of the differential dissipativity theory in [5].

Our characterization is different from such an approach based on the differential Lyapunov/dissipativity framework for several reasons. First, our condition in (7) seems to be related to the reversed Lyapunov inequality. However, the term $\lambda(x)$ does not serve as the rate of contraction/expansion because $\lambda(x)$ measures the degree of stability of the unstable system. The same argument holds true for the dissipativity-like condition to be explored in (10). By using the reversed inequality, we can estimate the Hausdorff dimension based on the trace of the Jacobian, which can be computed efficiently. On the other hand, it would be meaningful to refer to the method in [22], where the authors studied Hausdorff dimension estimates with an analogue of a Lyapunov function. However, it is not easy to generalize their framework to open systems because it depends on the spectrum of the Jacobian, which may change under interconnections.

III. Dimension Estimates for Feedback and Interconnected Systems

In this section, we consider a generalization of the results obtained so far to open systems and their interconnections.

A. Prolonged Systems With Inputs and Outputs

Consider an open nonlinear system

$$\begin{cases} \dot{x} = f(x) + Bu, \\ y = Cx, \end{cases} \tag{8}$$

where $f:\mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function and $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are constant matrices. The associated variational system is described by

$$\begin{cases} \delta\dot{x} = J(x)\delta x + B\delta u, \\ \delta y = C\delta x. \end{cases} \tag{9}$$

Recall that $J(x)$ is the Jacobian matrix of $f(x)$. The system (8) together with (9) is called the *prolonged system* [31], [32].

Following [5], we consider the quadratic storage function $V:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of the form[3]

$$V(x, \delta x) = \delta x^{\mathsf{T}} P(x)\delta x,$$

where $P:\mathbb{R}^n \to \mathcal{S}^n_+$ is a continuously differentiable function. Also, we consider the quadratic supply rate $w:\mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ of the form

$$w(\delta u, \delta y) = \begin{bmatrix} \delta y \\ \delta u \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} \delta y \\ \delta u \end{bmatrix},$$

where $Q \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$, and $R \in \mathbb{R}^{m\times m}$ are constant matrices with Q and R symmetric. The energy-like property in (7) can now be generalized to the following inequality:

$$\dot{V}(x, \delta x) + 2\lambda(x)V(x, \delta x) \geq w(\delta u, \delta y), \tag{10}$$

where $\lambda:\mathbb{R}^n \to \mathbb{R}$ is a continuous function.

In what follows, we show that the result in Theorem 1 can easily be extended to interconnected systems. This is based on the fact that our condition depends on the diagonal part of the Jacobian. Recall that the trace of the sum of matrices is equal to the sum of their traces. Hence, the trace of the Jacobian of an interconnected system can be replaced by the sum of the traces of the Jacobians of its components.

B. Dimension Estimates for Lur'e Systems

Here, we concentrate on the linear time-invariant system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{11}$$

where A, B, and C are constant matrices of appropriate sizes. It is clear that the inequality (10) can be converted to

$$\begin{bmatrix} \dot{x} \\ x \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & P \\ P & 2\lambda P \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \geq \begin{bmatrix} y \\ u \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \tag{12}$$

where λ and P do not depend on x. We notice that (12) can be rewritten as

$$\begin{bmatrix} A^{\mathsf{T}}P + PA + 2\lambda P & PB \\ B^{\mathsf{T}}P & 0 \end{bmatrix} - \begin{bmatrix} C^{\mathsf{T}}QC & C^{\mathsf{T}}S \\ S^{\mathsf{T}}C & R \end{bmatrix} \succeq 0.$$

Remark 3: From the KYP lemma [33], the above LMI is equivalent to the following frequency-domain inequality:

$$\begin{bmatrix} G(\mathrm{j}\omega - \lambda) \\ I \end{bmatrix}^{\mathsf{H}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} G(\mathrm{j}\omega - \lambda) \\ I \end{bmatrix} \preceq 0, \ \ \omega \in \mathbb{R} \cup \{\infty\},$$

where $G(s) := C(sI - A)^{-1}B$ for $s \in \mathbb{C}$. We notice that the trace of A is equal to the sum of all poles of $G(s)$ as long as there is no pole/zero cancellation. Thus, the inequality (12) can be regarded as a pure input/output property. We point out that input/output characterization of unstable dynamics has received little attention, and it can be of interest to utilize it in the control-theoretic methods such as data-driven verification.

Now, we consider the memoryless feedback of the form

$$u = \phi(y), \tag{13}$$

where $\phi:\mathbb{R}^p \to \mathbb{R}^m$ is a continuously differentiable function. The composite system is called the Lur'e system, which is an important class of nonlinear systems. The following result is motivated from [34] and addresses a more general setting.

Theorem 2: Consider the linear time-invariant system (11) such that (12) holds. Suppose that the memoryless

[3]In this letter, we do not allow V to take a negative value; it may be interesting to investigate more general signed storages as in [5].

feedback (13) satisfies

$$\begin{bmatrix} I \\ \mathsf{D}\phi(y) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} I \\ \mathsf{D}\phi(y) \end{bmatrix} \succeq 0, \quad y \in C\mathcal{A}, \tag{14}$$

where $\mathcal{A}$ is an invariant set of the closed-loop system. If there exists $d \in (0, n]$ such that for all $x \in \mathcal{A}$,

$$\text{tr } A + \text{tr } B\mathsf{D}\phi(Cx)C + (n-d)\lambda < 0,$$

then $\dim_H(\mathcal{A}) < d$.

Proof: The variational equation of the closed-loop system is described by $\delta\dot{x} = [A + B\mathsf{D}\phi(Cx(t))C]\,\delta x$. The inequality (12) together with (14) reads as

$$\begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & P \\ P & 2\lambda P \end{bmatrix} \begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix} \succeq 0.$$

Thus, the closed-loop system satisfies the condition in Theorem 1. The proof is complete. ■

The following example illustrates the usefulness of the trace in the dimension estimate instead of the truncated trace.

Example 1: Consider a 2-dimensional system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

with a memoryless feedback $u = \phi(y)$ of (13) such that

$$\begin{bmatrix} I \\ \mathsf{D}\phi(y) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} I \\ \mathsf{D}\phi(y) \end{bmatrix} \succeq 0, \quad y \in \mathbb{R}.$$

Because of the system structure, we have tr $B\mathsf{D}\phi(y)C \equiv 0$, and thus, the closed-loop attractor dimension can be estimated independent of ϕ. This example clarifies robustness of our attractor dimension estimate with respect to uncertainty in ϕ.

C. Dimension Estimates for Interconnected Systems

Finally, we investigate an interconnected system composed of the following N open systems:

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i) + B_i u_i, \\ y_i = C_i x_i, \end{cases} \tag{15}$$

where $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, and $C_i \in \mathbb{R}^{p_i \times n_i}$ are as before for each $i \in \{1, \ldots, N\}$. Assume that each subsystem Σ_i satisfies the inequality (10) with the shift parameter $\lambda_i(x_i)$, the storage function $V_i(x_i, \delta x_i) = \delta x_i^{\mathsf{T}} P_i(x_i) \delta x_i$, and the supply rate

$$w_i(\delta u_i, \delta y_i) = \begin{bmatrix} \delta y_i \\ \delta u_i \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q_i & S_i \\ S_i^{\mathsf{T}} & R_i \end{bmatrix} \begin{bmatrix} \delta y_i \\ \delta u_i \end{bmatrix}.$$

We define $x := \text{col}(x_1, \ldots, x_N)$, $u := \text{col}(u_1, \ldots, u_N)$, and $y := \text{col}(y_1, \ldots, y_N)$. The interconnection that we consider is of the form

$$u = My, \tag{16}$$

where $M \in \mathbb{R}^{(m_1 + \cdots + m_N) \times (p_1 + \cdots + p_N)}$ is a matrix such that the block-diagonal parts associated with the subsystems are zeros. This assumption does not lose any generality since the block-diagonal parts can be considered as self-feedbacks.

Let $f(x) := \text{col}(f_1(x_1), \ldots, f_N(x_N))$, $B := B_1 \oplus \cdots \oplus B_N$, and $C := C_1 \oplus \cdots \oplus C_N$. Then, the n-dimensional closed-loop system is described by

$$\dot{x} = f(x) + BMCx. \tag{17}$$

Notice that tr $BMC = 0$ holds because of the structure of M, and hence, the trace of the Jacobian is invariant under the interconnection. This is a key point since the truncated trace of the Jacobian in general varies after interconnections.

In the following theorem, we write $\Lambda(x) := \lambda_1(x_1) I_{n_1} \oplus \cdots \oplus \lambda_N(x_N) I_{n_N}$, $P(x) := P_1(x_1) \oplus \cdots \oplus P(x_N)$, and the same notations are used for Q, S, and R.

Theorem 3: Consider the N subsystems (15) such that the hypotheses mentioned above hold. Suppose that the interconnection in (16) satisfies $Q + SM + M^{\mathsf{T}} S^{\mathsf{T}} + M^{\mathsf{T}} RM \succeq 0$. For each $i \in \{1, \ldots, N\}$, let $\hat{d}_i \in (0, n_i]$ be such that

$$\text{tr } J_i(x) + (n_i - \hat{d}_i)\lambda_i(x) < 0, \quad x \in \mathbb{R}^n.$$

Then, for any compact invariant set $\mathcal{A} \subset \mathbb{R}^n$ of (17), it holds true that $\dim_H(\mathcal{A}) < d$, where $d \in (0, n]$ is a solution of

$$\text{tr}_{n-d}\Lambda(x) \le \sum_{i=1}^{N} (n_i - \hat{d}_i)\lambda_i(x_i), \quad x \in \mathcal{A}.$$

Proof: We sum up all storage functions to obtain

$$\sum_{i=1}^{N} [\dot{V}_i(x_i, \delta x_i) + \lambda_i(x_i) V_i(x_i, \delta x_i)]$$
$$\ge \delta y^{\mathsf{T}} (Q + SM + M^{\mathsf{T}} S^{\mathsf{T}} + M^{\mathsf{T}} RM) \delta y.$$

It follows from the assumption in the theorem that

$$\begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & P(x) \\ P(x) & \Lambda(x)P(x) \end{bmatrix} \begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix} \succeq 0.$$

Using the commutativity between $P(x)$ and $\Lambda(x)$, we conclude that the dimension estimate d is such that tr $J(x) + \text{tr}_{n-d}\Lambda(x) < 0$, where $J(x) := \mathsf{D}f(x)$. From the definition of $\hat{d}_i$, we can observe that tr $J(x) + \sum_{i=1}^{N} (n_i - \hat{d}_i)\lambda_i(x_i) < 0$, and the proof is complete. ■

Remark 4: The above result is scalable in the sense that the attractor dimension is estimated from $\hat{d}_i$, which can be computed individually. In addition, $\text{tr}_{n-d}\Lambda(x)$ can be easily computed since $\Lambda(x)$ is a diagonal matrix. Yet, we must be careful with the dependence on x_i of $J_i(x_i)$ and $\lambda_i(x_i)$. Thus, we need to solve an infinite system of LMIs. If the system under consideration has a special structure, then the problem can be relaxed (see [35]). In that case, we only need to solve a finite set of LMIs.

We here provide an example to demonstrate applicability of the above result. It is known that coupled chaotic systems exhibit various phenomena depending on the coupling parameters [36]. Moreover, even if there is no coupling between two chaotic systems, the Hausdorff dimension of the Cartesian product of the two attractors can in general be greater than their sum (see, e.g., [37]). Thus, our analysis of Hausdorff dimension for interconnected systems is meaningful.

Example 2: Consider the following coupled Lorenz system:

$$\begin{cases} \dot{x}_1 = \sigma(y_1 - x_1) - x_2, \\ \dot{y}_1 = \rho x_1 + x_1 z_1 - y_1, \\ \dot{z}_1 = x_1 y_1 - \beta z_1, \end{cases} \quad \begin{cases} \dot{x}_2 = \sigma(y_2 - x_2) + x_1, \\ \dot{y}_2 = \rho x_2 + x_2 z_2 - y_2, \\ \dot{z}_2 = x_2 y_2 - \beta z_2, \end{cases}$$

where $(\sigma, \rho, \beta) = (10, 28, 8/3)$. Since it is difficult to satisfy the inequality (10) on the whole state space, we verify the conditions on the following ranges: $x_i \in [-20, 30]$, $y_i \in [-30, 30]$, $z_i \in [-10, 50]$. Note that the closed-loop attractor can be over-approximated by the above ranges. We consider the storage functions with $P_i = I$ and the supply rates with $(Q_i, S_i, R_i) = (0, 1, 0)$ for both subsystems. Then, we can choose $\lambda_i = 35$ to satisfy the inequality (10). Also, $\hat{d}_i$ in Theorem 3 are given by $d_1 = d_2 = 2.61$ on the regions mentioned above. This value is conservative since the actual Hausdorff dimension of the Lorenz attractor is about 2.06 [38], but it can be made less conservative by using more appropriate storage functions and supply rates. Because the interconnection matrix is skew symmetric, the condition in Theorem 3 immediately holds. Therefore, the attractor dimension of the closed-loop system can be estimated as $d_1 + d_2 = 5.22$. To the best of our knowledge, there is no other result to estimate the attractor dimension of coupled chaotic systems based on a bottom-up approach.

IV. Conclusion

We have studied Hausdorff dimension involving asymptotic behaviors of dynamical systems by employing the differential Lyapunov and dissipativity frameworks. The developed framework is useful in estimation of the attractor dimension for autonomous and interconnected systems. Our method is also useful to study entropy as similar approaches are employed in [39], [40]. There are several directions to extend this research. First, the use of Hausdorff dimension in control problems has not been well studied. Second, relations with various entropy concepts proposed in the context of networked control systems have not been understood.

References

[1] V. A. Boichenko, G. A. Leonov, and V. Reitmann, *Dimension Theory for Ordinary Differential Equations*. Wiesbaden, Germany: Vieweg+Teubner Verlag Wiesbaden, 2005.

[2] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed. New York, NY, USA: Springer, 1997.

[3] I. R. Manchester and J.-J. E. Slotine, "Transverse contraction criteria for existence, stability, and robustness of a limit cycle," *Syst. Control Lett.*, vol. 63, pp. 2–38, Jan. 2014.

[4] F. A. Miranda-Villatoro, F. Forni, and R. Sepulchre, "Analysis of Lur'e dominant systems in the frequency domain," *Automatica*, vol. 98, pp. 76–85, Dec. 2018.

[5] F. Forni and R. Sepulchre, "Differential dissipativity theory for dominance analysis," *IEEE Trans. Autom. Control*, vol. 64, no. 6, pp. 2340–2351, Jun. 2019.

[6] A. N. Pisarchik and U. Feudel, "Control of multistability," *Phys. Rep.*, vol. 540, no. 4, pp. 167–218, 2014.

[7] S. Zelik, "Attractors then and now," 2022, *arXiv:2208.12101*.

[8] P. Thieullen, "Entropy and the Hausdorff dimension for infinite-dimensional dynamical systems," *J. Dyn. Diff. Equat.*, vol. 4, no. 1, pp. 127–159, 1992.

[9] C. Kawan, *Invariance Entropy for Deterministic Control Systems: An Introduction*. Cham, Switzerland: Springer, 2013.

[10] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. Cambridge, MA, USA: Academic Press, 1975.

[11] A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 3rd ed. Cham, Switzerland: Springer, 2017.

[12] J. C. Willems, "Dissipative dynamical systems—Part I: General theory," *Arch. Ration. Mech. Anal.*, vol. 45, no. 5, pp. 321–351, 1972.

[13] J. C. Willems, "Dissipative dynamical systems—Part II: Linear systems with quadratic supply rates," *Arch. Ration. Mech. Anal.*, vol. 45, no. 5, pp. 352–393, 1972.

[14] D. J. Hill and P. J. Moylan, "Dissipative dynamical systems: Basic input–output and state properties," *J. Frankl. Inst.*, vol. 309, no. 5, pp. 327–357, 1980.

[15] M. Arcak, C. Meissen, and A. Packard, *Networks of Dissipative Systems*. Cham, Switzerland: Springer, 2016.

[16] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.

[17] F. Forni and R. Sepulchre, "A differential Lyapunov framework for contraction analysis," *IEEE Trans. Autom. Control*, vol. 59, no. 3, pp. 614–628, Mar. 2014.

[18] R. Kato and H. Ishii, "Dimension analysis via differential Lyapunov and dissipation inequalities," in *Proc. IFAC World Congr.*, 2023.

[19] N. Kuznetsov and V. Reitmann, *Attractor Dimension Estimates for Dynamical Systems: Theory and Computation*. Cham, Switzerland: Springer, 2021.

[20] A. Douady and J. Oesterlé, "Dimension de Hausdorff des attracteurs," *C. R. Acad. Sci. Paris S'er. A*, vol. 290, no. 24, pp. 1135–1138, 1980.

[21] R. A. Smith, "Some applications of Hausdorff dimension inequalities for ordinary differential equations," *Proc. R. Soc. Edinburgh Sect. A: Math.*, vol. 104, nos. 3–4, pp. 235–259, 1986.

[22] G. A. Leonov and V. A. Boichenko, "Lyapunov's direct method in the estimation of the Hausdorff dimension of attractors," *Acta. Appl. Math.*, vol. 26, no. 1, pp. 1–60, 1992.

[23] A. Y. Pogromsky and H. Nijmeijer, "On estimates of the Hausdorff dimension of invariant compact sets," *Nonlinearity*, vol. 13, no. 3, pp. 927–945, 2000.

[24] V. A. Boichenko and G. A. Leonov, "Lyapunov functions, Lozinskii norms, and the Hausdorff measure in the qualitative theory of differential equations," in *Proc. St. Petersburg Math. Soc. Vol. V*, 1999, pp. 1–26.

[25] C. Wu, R. Pines, M. Margaliot, and J.-J. E. Slotine, "Generalization of the multiplicative and additive compounds of square matrices and contraction theory in the Hausdorff dimension," *IEEE Trans. Autom. Control*, vol. 67, no. 9, pp. 4629–4644, Sep. 2022.

[26] Y. S. Il'yashenko, "Weakly contracting systems and attractors of Galerkin approximations of Navier–Stokes equations on the two-dimensional torus," *Sel. Math. Sov.*, vol. 11, no. 3, pp. 203–239, 1992.

[27] C. Wu, I. Kanevskiy, and M. Margaliot, "*k*-contraction: Theory and applications," *Automatica*, vol. 136, Feb. 2022, Art. no. 110048.

[28] O. Dalin, R. Ofir, E. Bar-Shalom, A. Ovseevich, R. Bullo, and M. Margaliot, "Verifying *k*-contraction without computing *k*-compounds," 2022, *arXiv:2209.01046*.

[29] B. R. Hunt, "Maximum local Lyapunov dimension bounds the box dimension of chaotic attractors," *Nonlinearity*, vol. 9, no. 4, pp. 845–852, 1996.

[30] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*. Providence, RI, USA: Amer. Math. Soc., 1988.

[31] F. Forni and R. Sepulchre, "On differentially dissipative dynamical systems," in *Proc. 9th IFAC Symp. Nonlinear Control Syst.*, 2013, pp. 15–20.

[32] A. J. van der Schaft, "On differential passivity," in *Proc. 9th IFAC Symp. Nonlinear Control Syst.*, 2013, pp. 21–25.

[33] A. Rantzer, "On the Kalman–Yakubovich–Popov lemma," *Syst. Control Lett.*, vol. 28, no. 1, pp. 7–10, 1996.

[34] G. A. Leonov, D. V. Ponomarenko, and V. B. Smirnova, *Frequency-Domain Methods for Nonlinear Analysis: Theory and Applications*. Singapore: World Sci., 1996.

[35] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: SIAM, 1994.

[36] H.-C. Ma, C.-C. Chen, and B.-W. Chen, "Dynamics and transitions of the coupled Lorenz system," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 56, pp. 1550–1555, Aug. 1997.

[37] K. Falconar, *Fractal Geometry*, 3rd ed. Hoboken, NJ, USA: Wiley, 2014.

[38] P. Grassberger and I. Procaccia, "Measuring the strangeness of strange attractors," *Physica D*, vol. 9, nos. 1–2, pp. 189–208, 1983.

[39] A. Y. Pogromsky and A. S. Matveev, "Estimation of topological entropy via the direct Lyapunov method," *Nonlinearity*, vol. 24, pp. 1937–1959, May 2011.

[40] M. S. Tomar and M. Zamani, "Modular computation of restoration entropy for networks of systems: A dissipativity approach," *IEEE Control Syst. Lett.*, vol. 6, pp. 3289–3294, 2022.