Novel Control Approaches Based on Projection Dynamics

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***Abstract*—In this letter, our objective is to explore how two well-known projection dynamics can be used as dynamic controllers for stabilization of nonlinear systems. Combining the properties of projection operators, Lyapunov stability theory and LaSalle's theorem, we confirm that the projection dynamics on the feasible set and tangent cone are Krasovskii passive. To show the effectiveness of the proposed approach, we use the projection dynamics on the tangent cone for stabilizing boost converters in a DC microgrid while satisfying predefined input constraints.**

***Index Terms*—Stability of nonlinear systems, Lyapunov methods, control applications.**

I. Introduction

PASSIVITY theory is a well-known and useful tool to analyze and control complex, nonlinear dynamical systems across multiple domains [1], [2], [3]. Specifically, if a physical plant possesses passivity properties, then one of the most effective control approaches is to design a passive control system and interconnect it with the plant in a power-preserving way. Then, the stability of the closed-loop system can be analyzed relying on Lyapunov theory and invariance principle [4]. Several works on the topic focus on studying the passivity properties of continuous dynamical systems, whereas a limited number investigates the passivity properties of discontinuous dynamics [5]. Among the most typical discontinuous dynamics, projection dynamics are wildly used in mathematical programming, algorithm design [6], and controller design [7], [8]. Due to the deep relation between a projection operator and a variational inequality (VI) [9], projection dynamics are often used to solve mathematical programming problems, such as optimization and game problems [10]. In fact, in [11], the authors show that the equilibrium of the projection dynamics and the solution of a VI coincide. Furthermore, they also establish convergence by employing Lyapunov theory and the properties of the projection operator. With the advent of distributed systems, the focus shifted to combining the projection dynamics with distributed optimization [12] and control [13], [14], [15].

In summary, the research on projection dynamics in the mathematical programming field is exhaustive [16]. However, its potential to control nonlinear systems has not yet been fully explored, e.g., the property of Krasovskii passivity has not been studied in previous works, see [8], [17] for details on the topic. In this letter, we aim at investigating how two types of well-known projection dynamics can be used for controlling nonlinear systems. Particularly, we analyze their passivity properties. The main contributions are as follows.

- Compared to previous results [7], we analyze the interconnection of a nonlinear continuous-time dynamical system with projection dynamics.
- In Section III, under some conditions on the Jacobian of the dynamics, we prove the asymptotic stability of the closed-loop system relying on monotonicity.
- In Section IV, we consider passive nonlinear systems and show that the system dynamics projected on the tangent cone of a polyhedral convex and compact set are Krasovskii passive. Such a property is fundamental to establish convergence of the interconnected closed-loop system via standard passivity arguments.
- We design a new controller for the boost converters of a direct current (DC) microgrid [18], ensuring

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that the control input remains within a predefined set.

II. Preliminaries

To make this letter self-contained, we summarize in this section the main definitions and basic properties of the projection operator and monotone maps that we will use throughout this letter. After that, we state the problems studied in this letter.

A. Projection Operator

Hereafter, define Ω as a nonempty, closed, and convex subset of $\mathbb{R}^n$, Let $s \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$, then the projection of z on Ω is defined as $P_\Omega(z) = \mathrm{argmin}_{s\in\Omega} \|z - s\|$. According to [9, Th. 1.5.5] and [11, Th. 3.2], the following facts hold for any two vectors $z, s \in \mathbb{R}^n$.

Fact 1 (Projection properties):

1) $P_\Omega(z)$ is unique.
2) For all $s \in \Omega$, $< z - P_\Omega(z), s - P_\Omega(z) > \leq 0$.
3) The projection is *non-expansive*, i.e.,
$$\|P_\Omega(z) - P_\Omega(s)\| \leq \|z - s\|, \tag{1}$$
and also *co-coercive*, i.e.,
$$\|P_\Omega(z) - P_\Omega(s)\|^2 \leq \langle z - s, P_\Omega(z) - P_\Omega(s)\rangle. \tag{2}$$
4) The function $D(z) = \frac{1}{2}\|z - P_\Omega(z)\|^2$ is continuously differentiable in z and $\nabla D(z) = z - P_\Omega(z)$.

Let $\mathcal{T}(z, \Omega)$ be the tangent cone of Ω at $z \in \Omega$, as defined in [9, Pg. 15]. Then, $T_\Omega(z, H(z))$ represents the function projecting the vector-valued function $H(z)$ onto $\mathcal{T}(z, \Omega)$. Let $\mathcal{N}_\Omega(z)$ denote a normal cone at $z \in \Omega$ by
$$\mathcal{N}_\Omega(z) \triangleq \{\upsilon \mid \langle \upsilon, s - z\rangle \leq 0, \forall s \in \Omega\}.$$
Then, the inwards normals to Ω at $z \in \Omega$ can be defined as
$$\mathcal{N}^\star_\Omega(z) \triangleq \{\upsilon \in -\mathcal{N}_\Omega(z) \mid \|\upsilon\| = 1\}.$$
The following facts about the tangent cone hold.

Fact 2 (Tangent cone properties):

1) From [19, Lemma 2.1], it follows that
$$T_\Omega(z, H(z)) = \lim_{\alpha\to 0} \frac{P_\Omega(z + \alpha H(z)) - z}{\alpha}. \tag{3}$$
2) From [20, Lemma 2.1], one attains
$$T_\Omega(z, H(z)) = \begin{cases} H(z) & \text{if } z \in \mathrm{int}(\Omega) \\ H(z) + \beta(z)\alpha(z) & \text{if } z \in \mathrm{bnd}(\Omega) \end{cases}$$
where
$$\alpha(z) = \arg\max_{\omega\in\mathcal{N}^\star_\Omega(z)} \langle H(z), -\omega\rangle, \tag{4a}$$
$$\beta(z) = \max\{0, \langle H(z), -\alpha(z)\rangle\}. \tag{4b}$$

B. Monotonicity and Variational Inequality

We next recall some definitions on monotone operators [9, Definition 2.3.1]. For all $z, s \in \Omega$, a map $H : \mathbb{R}^n \to \mathbb{R}^n$ is

- *monotone*, if $< H(z) - H(s), z - s > \geq 0$,
- *strictly monotone*, if $< H(z) - H(y), z - s >> 0$, $z \neq s$,
- *ξ-monotone* with $\xi > 1$, if there exists $c \in \mathbb{R}_+$ such that
$$\langle H(z) - H(s), z - s\rangle \geq c\|z - s\|^\xi, \ \forall z, s \in \Omega.$$

If $\xi = 2$, we say that it is *strongly monotone*.

Now, we recall the definition of the *variational inequality* [9, Definition 1.1.1]. Given a feasible set $\Omega \subset \mathbb{R}^n$ and a map $H : \Omega \to \mathbb{R}^n$, VI$(H, \Omega)$ amounts to the problem of finding all $z \in \Omega$ satisfying $< s >\geq 0$, $\forall s \in \Omega$. The solution set of VI(H, Ω) is denoted by Sol(H, Ω).

Monotonicity is often used to establish the existence and uniqueness of the solution set of the VI. We next introduce some classic results from [9, Th. 2.3.3 and 2.3.5].

Fact 3 (Variational inequality properties):

1) If H is monotone on Ω, then Sol(H, Ω) is convex.
2) if H is ξ-monotone on Ω, then Sol(H, Ω) is a singleton.

C. Problem Formulation

Consider the nonlinear plant dynamics
$$\dot{x} = f(x, u), \quad y = h(x) \tag{5}$$
where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ denote the state, control input, and output, respectively. The maps $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable. Now, we formulate two control problems, which are solved in Sections III and IV, respectively.

Pr. 1: Given the dynamical system (5), consider a projection-based controller associated with a natural map C [9, Sec. 1.5]
$$\dot{u} = C(y, u) = P_\Omega(u - F(y, u)) - u, \tag{6}$$
where $F : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ is a vector-valued map. Derive a condition such that a desired equilibrium (x^*, u^*) of the closed-loop system is asymptotically stable.

Pr. 2: Given the dynamical system (5) with $m = p$, establish Krasovskii passivity properties for the projection-based dynamic controller [19, eq. (2.2)], [5, eq. (5)]
$$\dot{u} = G(y, u) = T_\Omega(u, -F(y, u)), \tag{7}$$
where $G : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$.

In Problems 1 and 2, we investigate different projection-based controllers, and their relations are explained below. In Problem 1, we provide a control design method that is applicable to a general class of plants (5). On the other hand, sometimes control design is simplified by utilizing the fundamental properties of plants. A representative method is passivity-based control, which illustrates that passive plants can be stabilized by passive feedback controllers. Motivated by this, we mention in Problem 2 that the controller (7) has kinds of passivity properties, which can be beneficial for the stabilization of a plant possessing the corresponding passivity property. The following remark introduces the relationship between the above two projection-based controllers.

Remark 1: According to the definition of the convex set, we can deduce that $C(y, u) \in \mathcal{T}(u, \Omega)$ holds for all $u \in \Omega$. If Ω is a polyhedron, then the feasible cone of Ω at arbitrary $u \in \Omega$ coincides with $\mathcal{T}(u, \Omega)$ (refer to [9, Lemma 3.3.6]). Hence, if $u - F(y, u) \notin \mathcal{T}(u, \Omega)$, then $P_\Omega(u - F(y, u))$ must be on the boundary of $\mathcal{T}(u, \Omega)$, which further implies that
$$\frac{G(y, u)}{\|G(y, u)\|} = \frac{C(y, u)}{\|C(y, u)\|}. \tag{8}$$

Moreover, if $u - F(y, u) \in \Omega$, then the following equality

$$C(y, u) = G(y, u) = -F(y, u) \tag{9}$$

holds. The above equalities (8) and (9) show that the two controllers (6) and (7) are similar in some specific cases.

III. Projection Dynamics on the Feasible Set

In this section, we solve Problem 1. To this end, we rely on the following blanket assumptions, where $A \succ 0$ ($A \succcurlyeq 0$) implies that the symmetric part of the matrix A is positive definite (semi definite).

Assumption 1 (Initial Value and Continuity): The initial value satisfies $u(t_0) \in \Omega$, and the vector-valued function $\mathrm{col}(f(x, u), F(h(x), u))$ is locally Lipschitz continuous on $K_\Omega \triangleq \mathbb{R}^n \times \Omega$ and differentiable for all $(x, u) \in K_\Omega$.

Assumption 2 (Positive Semi Definiteness of the Jacobian): The Jacobian matrix of $\mathrm{col}(f(x, u), F(y, u))$, denoted by

$$J(x, u) \triangleq \begin{bmatrix} -\frac{\partial f(x,u)}{\partial x} & -\frac{\partial f(x,u)}{\partial u} \\ \left.\frac{\partial F(y,u)}{\partial x}\right|_{y=h(x)} \frac{\partial h(x)}{\partial x} & \left.\frac{\partial F(y,u)}{\partial u}\right|_{y=h(x)} \end{bmatrix}$$

satisfies $J(x, u) \succcurlyeq 0$ for all $(x, u) \in K_\Omega$.

Assumption 3 (Positive Definiteness of the Jacobian): The Jacobian matrix $J(x, u) \succ 0$ holds for all $(x, u) \in K_\Omega$.

Before proceeding with the closed-loop analysis, we explain that the set of equilibrium points (x^*, u^*) of the closed-loop system (5)-(6) is non-empty, and it coincides with the following set

$$\Phi = \left\{ (x^*, u^*) \in K_\Omega \,\middle|\, \begin{array}{c} f(x^*, u^*) = \mathbf{0}, \\ \langle F(x^*, u^*), u - u^* \rangle \geq 0, \forall u \in \Omega \end{array} \right\}.$$

According to Fact 3.1 and [9, Proposition 2.3.2], Assumption 2 ensures that Φ is non-empty, bounded, and convex. Combining the fact that Ω is convex with [9, Proposition 1.5.8], we conclude that Φ coincides with the set of equilibria of the closed-loop system. Next, we introduce the convergence results of the closed-loop system (5)-(6).

Theorem 1 (Convergence): If Assumptions 1-2 hold, then the closed-loop system (5)-(6) is positively invariant in K_Ω and its trajectory $(x(t), u(t))$ approaches to Φ as $t \to \infty$.

Proof: We first show that K_Ω is a positively invariant set for the system (5)-(6). We consider the distance of $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ to $K_\Omega = \mathbb{R}^n \times \Omega$. This distance is equivalent to a distance of $u \in \mathbb{R}^m$ to Ω, which is nothing but $D(u)$ introduced in Fact 1.4. By virtue of Fact 1.4 and taking the time derivative of $D(u)$ along (6), it follows that

$$\begin{aligned} \dot{D}(u) &= \langle u - P_\Omega(u), P_\Omega(u - F(y, u)) - u \rangle \\ &\leq \langle u - P_\Omega(u), P_\Omega(u - F(y, u)) - P_\Omega(u) \rangle \leq 0, \end{aligned} \tag{10}$$

where the last inequality holds by virtue of Fact 1.2 and the fact that $P_\Omega(u - F(y, u)) \in \Omega$. Inequality (10) implies that $D(u(t))$ is non-increasing for all $t \geq t_0$. Combining this with Assumption 1, we can conclude that $D(u(t_0)) = 0$ and $u(t) = P_\Omega(u(t))$ holds for all $t \geq t_0$, which implies that K_Ω is a positively invariant set for the closed-loop system (5)-(6). Next, construct the following storage function

$$\begin{aligned} S_I(x, u) = &-\frac{1}{2}\|C(y, u) + F(y, u)\|^2 + \frac{1}{2}\|f(x, u)\|^2 \\ &+ \frac{1}{2}\|F(y, u)\|^2 + \frac{1}{2}\|u - u^*\|^2 + \frac{1}{2}\|x - x^*\|^2, \end{aligned} \tag{11}$$

where (x^*, u^*) is an arbitrary point in Φ. The above analysis shows that if $u(t_0) \in \Omega$, then $u(t) \in \Omega$ holds for all $t \geq t_0$. Based on such a fact, we can deduce that

$$\begin{aligned} &-\langle C(y, u), F(y, u) \rangle - \frac{1}{2}\|C(y, u)\|^2 \\ &\quad = -\langle C(y, u), F(y, u) + C(y, u) \rangle + \frac{1}{2}\|C(y, u)\|^2 \geq 0 \end{aligned} \tag{12}$$

holds for all $t \geq t_0$, where the last inequality is based on Fact 1.2. By substituting (12) into (11), it follows that $S_I(x, u) \geq 0$ holds for $(x, u) \in K_\Omega$. Also, if $(x, u) \in K_\Omega$ and $\|x\| \to +\infty$, we can deduce that $S_I(x, u) \to +\infty$. Combining this result with the fact that $S_I(x, u)$ is continuous, we can further deduce that, for arbitrary bounded initial value $(x(t_0), u(t_0)) \in K_\Omega$, there always exists an $r > 0$ such that $S_I(x, u) > S_I(x(t_0), u(t_0))$ whenever $\|\mathrm{col}(x, u)\| > r$. As a consequence, we can deduce that

$$\Omega_c = \{(x, u) \in K_\Omega \mid S_I(x, u) \leq S_I(x(t_0), u(t_0))\}$$

is a compact set for arbitrary bounded $(x(t_0), u(t_0)) \in K_\Omega$. Next, we show that $\dot{S}_I(x, u) \leq 0$ holds for all $t \geq t_0$. By employing Fact 1.4 and the chain rule, it follows that

$$\begin{aligned} \dot{S}_I(x, u) = &-\langle \mathrm{col}(\dot{x}, \dot{u}), J(x, u)\mathrm{col}(\dot{x}, \dot{u}) \rangle \\ &+ \langle C(y, u), F(y, u) \rangle + \big\langle C(y, u), u - u^* \big\rangle \\ &+ \|C(y, u)\|^2 + \big\langle f(x, u), x - x^* \big\rangle. \end{aligned} \tag{13}$$

Note that, by combining the fact that $u(t) \in \Omega$ for all $t \geq t_0$, the inequality (2), and Assumption 1, it follows that the sum of the second, third, and fourth elements in (13) satisfies

$$\big\langle C(y, u) + F(y, u), C(y, u) + u - u^* \big\rangle \leq 0. \tag{14}$$

Moreover, Assumption 2 guarantees that the first item on the right-hand side of (13) is negative semi-definite. Combining this fact with (13)-(14), we can conclude that

$$\dot{S}_I(x, u) \leq -\big\langle F(y, u), u - u^* \big\rangle + \big\langle f(x, u), x - x^* \big\rangle. \tag{15}$$

Recall that Ω is nonempty, closed, and convex, and Assumption 2 guarantees that the inequality $<F(y^*, u^*), u - u^*> \geq 0$ holds for all $u \in \Omega$ and $(x^*, u^*) \in \Phi$, where $y^* = h(x^*)$. Therefore, we have

$$\begin{aligned} \dot{S}_I(x, u) \leq &-\big\langle F(y, u) - F\big(y^*, u^*\big), u - u^* \big\rangle \\ &+ \big\langle f(x, u) - f\big(x^*, u^*\big), x - x^* \big\rangle. \end{aligned} \tag{16}$$

By virtue of Assumption 2 and using [9, Proposition 2.3.2], we deduce $\dot{S}_I(x, u) \leq 0$ for all $t \geq t_0$. Finally, we show the convergence. Combing (14)-(16) with Assumption 2, we can deduce that the first item and the sum of the later four items in (13) are non-positive, which further implies that $\dot{S}_I(x, u) = 0$ if and only if $\dot{x} = \mathbf{0}$ and $\dot{u} = \mathbf{0}$. Thus, for a given bounded $(x(t_0), u(t_0)) \in K_\Omega$, we can always find a bounded Ω_c such that the trajectory starting at it converges to $\Phi \in \Omega_c$. Then, we complete the proof by combining the LaSalle's theorem introduced in [4, Th. 4.4]. ■

Now, by virtue of Assumption 3, the next theorem shows that the equilibrium of the system (5)-(6) is unique and asymptotically stable.

Theorem 2 (Asymptotic Stability): If Assumptions 1 and 3 hold, the equilibrium of the closed-loop system (5)-(6) is unique and asymptotically stable.

Proof: Combining (11)-(12) and the analysis in the proof of Theorem 1, it follows that $S_I(x, u) > 0$ holds for all $(x, u) \in K_\Omega \backslash \Phi$ and $S_I(x, u) = 0$ for $(x, u) \in \Phi$. Since $J(x, u) \succ 0$ holds for all $(x, u) \in K_\Omega$, then it follows that Φ is a singleton by using Fact 3.2 and [9, Proposition 2.3.2]. Following the analysis in (13)-(16) and using [9, Proposition 2.3.2] again, we deduce that $\dot{S}_I(x, u) < 0$ holds for all $(x, u) \in K_\Omega \backslash \Phi$. Employing Lyapunov's stability theorem [4, Th. 4.1], we complete the proof. ■

IV. Projection Dynamics on the Tangent Cone

In this section, we solve Problem 2. Namely, we mention that the projection-based dynamic controller (7) on the tangent cone of Ω can be either shifted or Krasovskii passive under suitable assumptions. Consider the projection-based dynamic controller (7) on the tangent cone of Ω with,

$$F(y, u) = g(u) - y,$$

where $g : \mathbb{R}^m \to \mathbb{R}^m$ is differentiable. We assume that the following two assumptions hold for the controller dynamics.

Assumption 4 (Non-Empty Equilibrium Set): The equilibrium set $\bar{\Phi}$ of (7), i.e., the set of $(y^*, u^*) \in \mathbb{R}^m \times \Omega$ satisfying $T_\Omega(u^*, -F(y^*, u^*)) = 0$ is not empty.

In general, the equilibrium sets $\bar{\Phi}$ for this setup is different from Φ defined in Section III.

Assumption 5 (Monotonicity): The function $g(u)$ is monotone for all $u \in \Omega$.

Remark 2: Assumptions 4 and 5 are weaker than Assumption 2, since they only guarantees that $\bar{\Phi}$ is non-empty and $\mathrm{Sol}(F(y^*, \cdot), \Omega)$ is convex. According to in [9, Proposition 2.3.2], Assumption 5 holds if and only if $\partial g(u)/\partial u \succcurlyeq 0$ holds for all $u \in \Omega$.

A. Shifted Passivity

To make this letter self-consistent, we present the following proposition, proven in [5, Sec. "Projected-gradient play and passivity"], that ensures that the projection-based dynamic controller (7) has the following shifted passivity property [1, Sec. 4.7].

Proposition 1 (Shifted Passivity): If Assumptions 1, 4, and 5 hold for the controller dynamics (7), then $S_s(u) = \frac{1}{2}\|u - u^*\|^2$ satisfies

$$\dot{S}_s(u) \le -\langle y - y^*, u - u^* \rangle \tag{17}$$

for any $(y, u) \in \mathbb{R}^m \times \Omega$ and any $(y^*, u^*) \in \bar{\Phi}$.

By Proposition 1, the controller dynamics (7) is shifted passive with respect to the storage function $S_s(z)$, input $-y$, and output u. Now, we suppose that the system (5) is shifted passive with respect to the input u and output y, i.e., satisfies $\dot{V}_s(x) \le \langle y - y^*, u - u^* \rangle$ for some storage function $V_s(x)$. Then, the closed-loop system satisfies $\dot{V}_s(x) + \dot{S}_s(u) \le 0$. Based on this inequality, one can proceed with the closed-loop stability analysis by invoking the results of the passive interconnection analysis, e.g., Lyapunov and LaSalle's theorems.

B. Krasovskii Passivity

When Ω is a polyhedral set, we can also show that the controller (7) is Krasovskii passive [8, Definition 2.8].

Theorem 3 (Krasovskii Passivity): If Ω is a polyhedron, and Assumptions 1, 4, and 5 hold for the controller dynamics (7), then the following scalar-valued function

$$S_k(y, u) = \frac{1}{2}\|T_\Omega(u, -F(y, u))\|^2 \tag{18}$$

satisfies

$$\dot{S}_k(y, u) \le -\langle \dot{y}, \dot{u} \rangle \tag{19}$$

in the sense of Carathéodory for almost all $t \ge t_0$, all $u(t_0) \in \Omega$, and all continuously differentiable $y : \mathbb{R} \to \mathbb{R}^m$.

Proof: According to the analysis in [19, Pg. 27], if $u(t_0) \in \Omega$ holds, then we have $u(t) \in \Omega$ holds for all $t \ge t_0$ and for all continuously differentiable $y(t) \in \mathbb{R}^m$, $t \ge t_0$. Next, we analyze all the possible scenarios for the trajectory of (7). First, if $u(t) \in \mathrm{int}(\Omega)$, then it follows that

$$\dot{S}_k(y, u) = -\left\langle \frac{\partial g(u)}{\partial u}\dot{u} + \dot{y}, \dot{u} \right\rangle. \tag{20}$$

Combining Assumption 5 with (20), we obtain (19). Second, we consider the case that $u(t)$ located on $\mathrm{bnd}(\Omega)$ is moving to $\mathrm{int}(\Omega)$. In such a case, the right-hand side of (7) is continuous, and thus (19) follows from the analysis in (20). Third, if $u(t)$ is switching from $\mathrm{int}(\Omega)$ to $\mathrm{bnd}(\Omega)$, the storage function $S_k(y, u)$ is not differentiable. However, following the analysis in [21, Lemma 4.2] and combining (1), we can deduce that $S_k(y, u)$ is non-increasing during the switching. Since Ω is polyhedral, and $F(y, u)$ is continuous, the number of switching from $\mathrm{int}(\Omega)$ to $\mathrm{bnd}(\Omega)$ is finite. Finally, according to Fact. 2.2, when $u(t)$ is moving on $\mathrm{bnd}(\Omega)$, the dynamic (7) reduces to

$$\dot{u} = -A_p(u, \Omega)F(y, u), \tag{21}$$

where $A_p(u, \Omega)$ represents the projection matrix projecting $-F(y, u)$ on $\mathrm{bnd}(\Omega)$. Using this representation (21), we have

$$\dot{S}_k(y, u) \le -\left\langle A_p(u, \Omega)\left(\frac{\partial G(u)}{\partial u}\dot{u} + \dot{y}\right), \dot{u} \right\rangle. \tag{22}$$

Combining the fact that $A_p(u, \Omega)$ is idempotent with (22) and using Assumption 5, we complete the proof. ■

Theorem 3 implies that the controller dynamics (7) is Krasovskii passive with respect to the storage function $S_k(y, u)$, input $-\dot{y}$, and output $\dot{u}$. A similar discussion as shifted passivity holds for the closed-loop stability analysis. That is, suppose the system (5) is Krasovskii passive with respect to the input $\dot{u}$ and output $\dot{y}$, i.e., satisfies $\dot{V}_k(x, u) \le <\dot{y}, \dot{u}>$ for some storage function $V_k(x, u)$. Then, the closed-loop system satisfies $\dot{V}_k(x, u) + \dot{S}_k(y, u) \le 0$, and we can proceed with the stability analysis based on this inequality and invoke Lyapunov

TABLE I
USED VARIABLES AND THEIR PHYSICAL DESCRIPTION

I	Output current vector	u	Control input vector
V	Load voltage vector	R_l	Line resistance matrix
R	Filter resistance matrix	I_l	Line current vector
L	Filter inductance matrix	V_o	Voltage source vector
C	Shunt capacitor matrix	L_l	Line inductance matrix
I_L	Load current vector	Z_L	Load impedance matrix

and LaSalle's theorems. Note that Krasovskii passivity is a property of $\dot{u}$ and $\dot{y}$. Thus, instead of the original output $h(x)$ of the system (5), we can feed its shifted signal $h(x) + \bar{y}$ by arbitrary constant $\bar{y}$ into the input y of the controller dynamics (7); the closed-loop system can still be interpreted as the feedback interconnection of a Krasovskii passive plant and controller. A similar discussion holds for u.

V. SIMULATION

In this section, we use the projection dynamics (7) and Krasovskii passivity to design a controller for a DC microgrid with 4 nodes in a ring topology, where each node includes a boost converter supplying a constant impedance and a constant current load. Let $z \in \mathbb{R}^n$, then we define $[z] \triangleq \mathrm{diag}(z_1, \ldots, z_n)$. The dynamics of the considered microgrid can be expressed as in [18], i.e.,

$$L\dot{I} = -(\boldsymbol{I} - [u])\, V - RI + V_o, \tag{23a}$$
$$C\dot{V} = (\boldsymbol{I} - [u])\, I - I_L + BI_l - Z_L^{-1} V, \tag{23b}$$
$$L_l \dot{I}_l = -R_l I_l - B^\top V, \tag{23c}$$

where the used symbols are explained in Table I. The network is described by an undirected ring graph with incidence matrix denoted by $B \in \mathbb{R}^{4\times4}$. For convenience, let $x \triangleq \mathrm{col}(I, V, I_l)$ and rewrite (23) as the following compact form

$$Q\dot{x} = \underbrace{\begin{bmatrix} \mathbf{0} & [u] & \mathbf{0} \\ -[u] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\triangleq Y(u)} x - \underbrace{\begin{bmatrix} R & \boldsymbol{I} & \mathbf{0} \\ -\boldsymbol{I} & Z_L^{-1} & -B \\ \mathbf{0} & B^\top & R_l \end{bmatrix}}_{\triangleq A} x + \underbrace{\begin{bmatrix} V_o \\ -I_L \\ \mathbf{0} \end{bmatrix}}_{\triangleq d}, \tag{24}$$

where $Q \succ 0$ is a diagonal matrix that can be easily attained by inspection of (23). Next, we analyze the passivity property of (24). Consider the storage function $H_k(x) = \frac{1}{2}\|\dot{x}\|_Q^2$, which satisfies

$$\dot{H}_k(x) = \langle \dot{x}, (Y(u) - A)\dot{x} \rangle + \left\langle \dot{x}, \frac{\partial (Y(u)x)}{\partial u} \dot{u} \right\rangle.$$

Our goal is to design a controller such that (24) converges to a desired equilibrium and the control input solves the following optimization problem

$$\min_{u \in \Omega} \quad \tfrac{1}{2}\|u - u_r\|_E^2 + \varepsilon [I][V]u \quad \text{s.t.} \quad u = u_r, \tag{25}$$

where $u_r \in \Omega$ denotes the reference control input, $E \succ 0$, and $\varepsilon > 0$. Note that (25) has a unique solution $u = u_r$.

Now, we show the closed-loop system consisting of (24) and a controller based on (7), i.e.,

$$\dot{x} = Q^{-1}(Ax + Y(u)x + d), \tag{26a}$$
$$\dot{u} = T_\Omega(u, -E(u - u_r) - \varepsilon[I]V - \lambda), \tag{26b}$$
$$\dot{\lambda} = u - u_r, \tag{26c}$$

where $\lambda \in \mathbb{R}^n$ is the Lagrange multiplier, and $\dot{u}$ and $\dot{\lambda}$ are interconnected in a passive way. Next, we construct the following storage function

$$V_k(x, u, \lambda) = \varepsilon H_k(x) + \frac{1}{2}\|\dot{u}\|^2.$$

Invoking Theorem 3, $V_k(x, u, \lambda)$ satisfies

$$\begin{aligned} \dot{V}_k(x, u, \lambda) \le & -\|\dot{u}\|_E^2 - \langle \dot{u}, \dot{\lambda} \rangle + \langle \dot{\lambda}, \dot{u} \rangle - \varepsilon \|\dot{V}\|_{Z_l^{-1}}^2 \\ & + \varepsilon \langle \dot{I}, [V]\dot{u} \rangle - \varepsilon \langle \dot{V}, [I]\dot{u} \rangle \\ & - \varepsilon \langle \dot{u}, [V]\dot{I} \rangle - \varepsilon \langle \dot{u}, [I]\dot{V} \rangle, \end{aligned} \tag{27}$$

where the inequality follows from $A - Y(u) \succ 0$. Let $\sigma_{\min}(\cdot)$ and $\sigma_{\max}(\cdot)$ denote the smallest and largest eigenvalue of a matrix, respectively. If

$$0 < \varepsilon \le \frac{\sigma_{\min}(Z_L^{-1})\, \sigma_{\min}(E)}{\sigma_{\max}([I]^2)}, \tag{28}$$

for all $t \ge t_0$, then $\dot{V}_s(x, u, \lambda) \le 0$ holds for almost all $t \ge t_0$, since $E \succ 0$ and $Z_L^{-1} \succ 0$. Note that, since (26b) has a unique equilibrium $u^* = u_r$, then x^* and $\lambda^* = -\varepsilon[I^*]V^*$ are also unique. By combining the above analysis and the result in [4, Lemma 4.1], we then prove that (26) converge to the unique equilibrium in the sense of Carathéodory if ε satisfies (28) for all $t \ge t_0$. Next, we introduce the detailed parameter settings. The parameters of all the distributed generator units (DGUs) are reported in Table II. The resistance and inductance of the transmission lines are selected as $R_l = \mathrm{col}(70, 50, 80, 60)(\mathrm{m}\Omega)$ and $L_l = \mathrm{col}(2.1, 2.0, 3.0, 2.2)(\mu\mathrm{H})$, respectively. Moreover, we set the desired voltage and the voltage source as $V^* = \mathrm{col}(381.5, 382, 382.5, 383)(\mathrm{V})$ and $V_o = \mathrm{col}(270, 270, 270, 270)(\mathrm{V})$, respectively, and we select $\varepsilon = 10^{-5}$. Considering that R is negligible in practice, we set $u_r = \mathbf{1} - [V^*]^{-1} V_0$ as an approximated value of u^*, which is obtained by solving the equality $-(\boldsymbol{I} - [u_r])\, V^* + V_o = \mathbf{0}$. Although such a selection may cause a deviation from the desired value, it avoids to require information on I^*. Then, we obtain $u_r = \mathrm{col}(0.2923, 2932, 0.2941, 0.2950)$. Moreover, we set $\Omega = [u_r - 10^{-3}\mathbf{1}, u_r + 10^{-3}\mathbf{1}]$. From Figure 1(a), we observe that the control input u is always within the feasible set Ω. Figure 1(b) shows that λ converges to 0 after a short transient, which also confirms that the deviation between the control input and the reference value converges to 0. Figure 2 shows that the currents and voltages of all the DGUs converge towards the desired values within a short time.

TABLE II
THE PARAMETERS OF THE NODE IN THE 4 BUS CASE

DGU	L_i (mH)	C_i (mF)	R_i (mΩ)	$Z_{L,i}$ (Ω)	$I_{L,i}$ (A)
1	1.8	2.2	20	16	30
2	2.0	1.9	18	50	15
3	3.0	2.5	16	16	30
4	2.2	1.7	15	20	26

Figure 3(a) shows the transmission line currents. Since we neglect R in the computation of u^r, we observe from

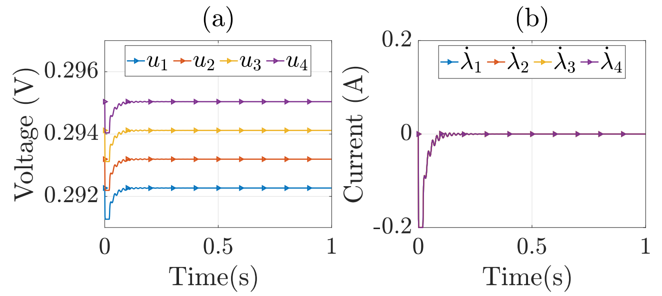

Fig. 1. (a) Control input. (b) The deviation with respect to the reference control input.

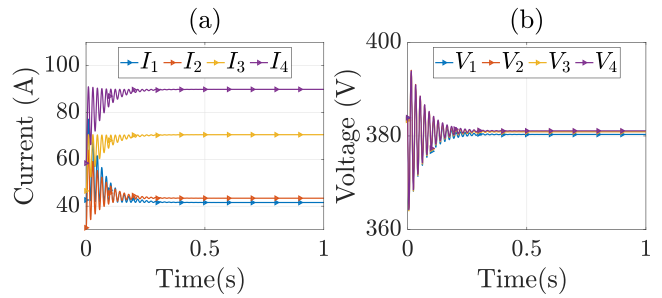

Fig. 2. (a) Microgrid currents. (b) Microgrid voltages.

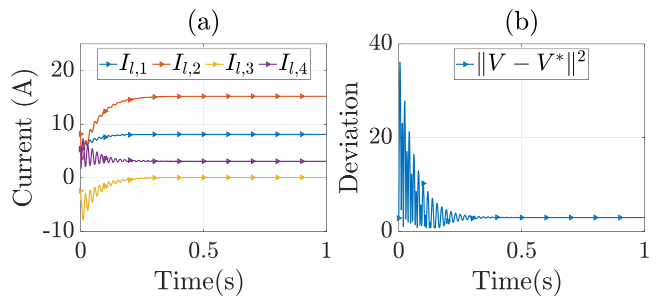

Fig. 3. (a) Transmission line currents. (b) Deviation between the microgrid state and the reference state.

Figure 3(b) a steady-state bias between the microgrid voltages and the desired value V^*. Remarkably, such a deviation is very small compared to V^*.

It is worth noting that shifted passivity can also be used for control design. However, such a controller requires information on the equilibrium x^* including the load parameters I_L and Z_L that are usually unknown. In contrast, the proposed Krasovskii passivity-based controller can be implemented only by knowing the desired voltage. Compared with the controller in [18], the proposed controller does not require information on $\dot{x}$, which makes it easier to be implemented and less sensitive to measurement noises.

VI. Conclusion

Projection dynamics can be connected to a differentiable dynamical system, ensuring asymptotic stability of the closed-loop system under certain assumptions on the associated Jacobian matrix. The projection dynamics on the tangent cone of a convex set are passive and, in the case of a polyhedral set, Krasovskii passive. These findings are valuable for designing controllers for nonlinear systems, such as power converters, to meet operational constraints. However, further investigation is needed to understand the impact of relaxing these assumptions on the convergence of the closed-loop system

References

[1] A. J. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*. Cham, Switzerland: Springer, 2000.
[2] C. Wu, A. J. van der Schaft, and J. Chen, "Stabilization of port-Hamiltonian systems based on shifted passivity via feedback," *IEEE Trans. Autom. Control*, vol. 66, no. 5, pp. 2219–2226, May 2021.
[3] P. Borja, R. Ortega, and J. M. A. Scherpen, "New results on stabilization of port-Hamiltonian systems via PID passivity-based control," *IEEE Trans. Autom. Control*, vol. 66, no. 2, pp. 625–636, Feb. 2020.
[4] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.
[5] L. Pavel, "Dissipativity theory in game theory: On the role of dissipativity and passivity in Nash equilibrium seeking," *IEEE Control Syst. Mag.*, vol. 42, no. 3, pp. 150–164, Jun. 2022.
[6] Y. Xia and J. Wang, "A general projection neural network for solving monotone variational inequalities and related optimization problems," *IEEE Trans. Neural Netw.*, vol. 15, no. 2, pp. 318–328, Mar. 2004.
[7] G. Bianchin, J. Cortés, J. I. Poveda, and E. Dall'Anese, "Time-varying optimization of LTI systems via projected primal-dual gradient flows," *IEEE Trans. Control Netw. Syst.*, vol. 9, no. 1, pp. 474–486, Mar. 2022.
[8] Y. Kawano, K. C. Kosaraju, and J. M. A. Scherpen, "Krasovskii and shifted passivity-based control," *IEEE Trans. Autom. Control*, vol. 66, no. 10, pp. 4926–4932, Oct. 2021.
[9] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*. New York, NY, USA: Springer, 2003.
[10] C. Cenedese, G. Belgioioso, S. Grammatico, and M. Cao, "An asynchronous distributed and scalable generalized Nash equilibrium seeking algorithm for strongly monotone games," *Eur. J. Control*, vol. 58, pp. 143–151, Mar. 2021.
[11] M. Fukushima, "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems," *Math. Program.*, vol. 53, pp. 99–110, Jan. 1992.
[12] P. Yi and L. Pavel, "An operator splitting approach for distributed generalized Nash equilibria computation," *Automatica*, vol. 102, pp. 111–121, Apr. 2019.
[13] Z. Fu, M. Cucuzzella, C. Cenedese, W. Yu, and J. M. Scherpen, "A distributed control framework for the optimal operation of DC microgrids," in *Proc. IEEE 61st Conf. Decis. Control (CDC)*, 2022, pp. 4585–4590.
[14] G. Belgioioso et al., "Online feedback equilibrium seeking," 2022, *arXiv:2210.12088*.
[15] A. Hauswirth, S. Bolognani, G. Hug, and F. Dörfler, "Timescale separation in autonomous optimization," *IEEE Trans. Autom. Control*, vol. 66, no. 2, pp. 611–624, Feb. 2020.
[16] T. Yang et al., "A survey of distributed optimization," *Annu. Rev. Control*, vol. 47, pp. 278–305, 2019.
[17] K. C. Kosaraju, M. Cucuzzella, J. M. A. Scherpen, and R. Pasumarthy, "Differentiation and passivity for control of Brayton–Moser systems," *IEEE Trans. Autom. Control*, vol. 66, no. 3, pp. 1087–1101, Mar. 2020.
[18] M. Cucuzzella, R. Lazzari, Y. Kawano, K. C. Kosaraju, and J. M. A. Scherpen, "Robust passivity-based control of boost converters in DC microgrids," in *Proc. IEEE 58th Conf. Decis. Control (CDC)*, 2019, pp. 8435–8440.
[19] A. Nagurney and D. Zhang, *Projected Dynamical Systems and Variational Inequalities With Applications*, vol. 2. New York, NY, USA: Springer, 1995.
[20] K. Nabetani, P. Tseng, and M. Fukushima, "Parametrized variational inequality approaches to generalized Nash equilibrium problems with shared constraints," *Comput. Optim. Appl.*, vol. 48, no. 3, pp. 423–452, 2011.
[21] P. Yi, Y. Hong, and F. Liu, "Distributed gradient algorithm for constrained optimization with application to load sharing in power systems," *Syst. Control Lett.*, vol. 83, pp. 45–52, Sep. 2015.

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