Robust Multi-Model Predictive Control via Integral Sliding Modes

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***Abstract*—This letter presents a novel optimal control approach for systems represented by a multi-model, i.e., a finite set of models, each one corresponding to a different operating point. The proposed control scheme is based on the combined use of model predictive control (MPC) and first order integral sliding mode control. The sliding mode control component plays the important role of rejecting matched uncertainty terms possibly affecting the plant, thus making the controlled equivalent system behave as the nominal multi-model. A min-max multi-model MPC problem is solved using the equivalent system without further robustness oriented add-ons. In addition, the MPC design is performed so as to keep the computational complexity limited, thus facilitating the practical applicability of the proposal. Simulation results show the effectiveness of the proposed control approach.**

***Index Terms*—Sliding mode control, model predictive control, multi-model systems, uncertain systems.**

I. Introduction

MANY industrial plants present different behaviours in correspondence of different operating points. When this happens, it is convenient to reformulate the nonlinear model describing the plant as the union of a finite number of linear models. This gives rise to a multi-model representation, also called in the literature "multi-model uncertainty", to underlying the fact that when this modeling approach is adopted, the active system matrices at certain time instant are unknown, but it is known that they belong to finite sets of known matrices. Different approaches have been proposed to design MPCs for systems with multi-model uncertainty, see, e.g., [1]–[3]. The presence of matched disturbances, which are added to the multi-model uncertain term, can make the use of a robust MPC mandatory, which implies a certain level of conservativeness in the control solutions [4], [5].

In the literature, the problem of reducing the conservativeness of MPC schemes has been faced by proposing a combination of MPC with sliding mode control [6], [7]. The underlying idea was to exploit the robustness features of sliding mode control to allow the MPC component of the scheme to be designed relying on a plant with reduced uncertainty. The reduction of the uncertainty effects was performed by the sliding mode control component. Different proposals were studied in recent years. In [8], a nonlinear MPC combined with an integral sliding mode control was proposed for continuous-time sampled data nonlinear systems. This was extended in [9] to the case of networked systems, producing an original asynchronous packetized MPC solution. An MPC and a sliding mode observer are proposed in [10], still for continuous-time systems. Other schemes based on analogous concepts are proposed in [11] and [12] for discrete-time systems. Moreover, such concepts have been also customized for several applications, from robotics to traffic systems (see, e.g., [13], [14] among many others).

In all the previously mentioned schemes, classical single models are considered to represent the plant to control. In this letter we aim at further extending the scope of combined MPC/sliding mode control schemes to make it encompass also the case of plants which need to be described by a multi-model representation affected by matched uncertainty. The proposed solution has the advantage of reducing the conservativeness of the MPC law thanks to the exploitation of the invariance property of sliding mode control. Moreover, by virtue of a suitable transformation, the MPC considered in this letter can be designed relying on a collection of models which results in being of reduced order with respect to the original collection of models describing the plant in its entire multi-model nature.

It should be noticed that the proposed control solution differs from the proposals in [2] and [3] since, in our work, although we consider the additional source of uncertainty due to the matched disturbance terms, we do not need to use any disturbance observer. It also differs from [15, Ch. 6], where

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multi-model systems are considered, but a continuous-time Linear Quadratic (LQ) control approach is adopted in contrast to this letter where we rely on a discrete-time MPC formulation. To the best of the authors' knowledge, this is the first time that a multi-model sliding mode based predictive control is proposed. It fills a scientific gap, since the previously published MPC/sliding mode schemes are not applicable when the plant affected by matched uncertainties has dynamical features only capturable via a multi-model representation.

Notation: The used variables and operators are mostly standard. Some notation for the MPC is instead recalled. Let x be a vector, then x_i refers to its ith entry. Given a signal u, then $u_{t_j|t_k}$ denotes its prediction at t_j, predicted at the time t_k. Moreover, let N_p be the horizon length, $\mathbf{u}_{[t_k,t_{k+N_\mathrm{p}-1}|t_k]}$ indicates the whole input sequence from t_k to $t_{k+N_\mathrm{p}-1}$.

II. Problem Formulation

Consider a system captured by the uncertain continous-time-invariant model

$$\dot{x}(t) = Ax(t) + B(u(t) + \phi_\mathrm{m}(t)), \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the (measurable) state vector, $u \in \mathbb{R}^m$ denotes the control vector, while the term $\phi_\mathrm{m} \in \mathbb{R}^m$ represents parameters uncertainties, external disturbances or unmodelled dynamics of the system. Moreover, the following assumption holds.

$\mathcal{A}_1$: The system is subject to hard constraints on state and control, i.e.,

$$x \in \mathcal{X}, \tag{2a}$$

$$u \in \mathcal{U}, \tag{2b}$$

where $\mathcal{X}$ and $\mathcal{U}$ are compact sets containing the origin as an interior point.

As for the uncertain term ϕ_m, which is unknown and from (1) appears to be matched, it also satisfies the following assumption.

$\mathcal{A}_2$: The matched uncertain term $\phi_\mathrm{m}(t)$ is unknown but bounded as $\|\phi_\mathrm{m}(t)\| \leq \gamma\|x(t)\| + \delta$, with $\gamma \geq 0$ and $\delta > 0$ being known constants.

Now, assuming that model (1) presents multiple operating points corresponding to N linear models, then, depending on the current point, system (1) can be represented as follows

$$\dot{x}^\alpha(t) = A^\alpha x^\alpha(t) + B(u(t) + \phi_\mathrm{m}(t)), \quad x^\alpha(0) = x_0, \tag{3}$$

where the state $x^\alpha(t)$ is measurable, $A^\alpha \in \mathbb{R}^{n\times n}$ is the corresponding dynamics matrix, $B \in \mathbb{R}^{n\times m}$ is fixed and common to all subsystems. The following assumption holds.

$\mathcal{A}_3$: The active system matrix A^α at certain time instant is unknown, but it is known that it belongs to a finite set of N known matrices, that is $A^\alpha \in \{A^1, A^2, \dots, A^N\}$, $\alpha \in \mathcal{A}$, with $\mathcal{A} := \{1, 2, \dots, N\}$. Moreover, the control matrix $B \in \mathbb{R}^{n\times m}$ is a bounded known full rank matrix, i.e., $\mathrm{rank}(B) = m$.

We are now in a position to state the control problem. We want to design a control law $u(t)$ capable of guaranteeing the asymptotic stability of the multi-model system (3), without the knowledge of the active subsystem at generic time instant t, while fulfilling state and input constraints (2) despite the presence of matched uncertain terms affecting the plant.

III. The Proposed Control Approach

In this letter, to solve the problem formulated in Section II, we propose a min-max MPC combined with an integral sliding mode control capable of rejecting the matched uncertain terms affecting the system.

Making reference to [16], [17], assume a control signal split into two components, that is

$$u(t) = u_1(t) + u_0(t), \tag{4}$$

such that the element u_1 is devoted to eliminate the matched perturbations, and u_0 is a stabilizing controller based on an MPC law. In the following sections, the two components will be discussed in detail.

A. Integral Sliding Mode Control Component

Let us design the sliding mode control law $u_1(t)$. Consider system (3) and design the so-called sliding variable as

$$s(t) = B^+\big(x^\alpha(t) - x_0\big) - \int_0^t u_0(\tau)\mathrm{d}\tau, \quad s(0) = 0, \tag{5}$$

where B^+ is the left inverse of B, which guarantees that possible unmatched uncertainty terms are minimal and not amplified [18]. The sliding mode control component is then selected as

$$u_1(t) = -\beta(x^\alpha(t))\frac{s(t)}{\|s(t)\|}, \tag{6}$$

where the control amplitude $\beta(x^\alpha(t))$ is a positive scalar function such that there exists $\bar{\beta} \geq \beta$, and it is designed to dominate the matched uncertain term affecting the system. Specifically, the following result proves the invariance property provided by the sliding mode component in front of the matched uncertain term ϕ_m.

Proposition 1: Given the multi-model system (3) with u_1 as in (6), and the sliding variable s in (5), if $\mathcal{A}_2$ holds and a sliding mode $s = 0$ is enforced, with $\dot{s} = 0$ in Filippov's sense, then the equivalent control $\tilde{u}_1$ is given by

$$\tilde{u}_1 = -B^+A^\alpha x^\alpha(t) - \phi_\mathrm{m}(t), \tag{7}$$

with equivalent dynamics on the sliding mode

$$\dot{x}^\alpha(t) = \bar{A}^\alpha x^\alpha(t) + Bu_0, \tag{8}$$

where $\bar{A}^\alpha := (I - BB^+)A^\alpha$.

Proof: Compute the time-derivative of (5) so that

$$\begin{aligned}\dot{s}(t) &= B^+\dot{x}^\alpha(t) - u_0(t)\\ &= B^+A^\alpha x^\alpha(t) + u_1(t) + \phi_\mathrm{m}(t).\end{aligned} \tag{9}$$

Since in sliding mode one has $s = 0$ with $\dot{s} = 0$ in Filippov's sense, relying on the so-called *equivalent control* concept (see, e.g., [7, Ch. 1]), by posing (9) equal to zero, (7) is obtained. Applying the latter to the multi-model system (3), its equivalent dynamics constrained to the sliding mode is given by

$$\begin{aligned}\dot{x}^\alpha(t) &= (I - BB^+)A^\alpha x^\alpha(t) + Bu_0\\ &= \bar{A}^\alpha x^\alpha(t) + Bu_0,\end{aligned} \tag{10}$$

with $\bar{A}^\alpha = (I - BB^+)A^\alpha$, which concludes the proof. ■

Instead, the following proposition provides the needed condition to guarantee finite-time convergence of the sliding variable and its derivative to zero. Let $a = \max_{\alpha\in\mathcal{A}}\{\|A^\alpha\|\}$.

Proposition 2: Given the sliding variable (5) and its dynamics (9), with u_1 as in (6), if $\mathcal{A}_2$ holds and $\beta(x^\alpha(t)) > (\|B^+\|a+\gamma)\|x^\alpha(t)\| + \delta$, then a sliding mode $s(t)=0$ is enforced for any $t \geq 0$, with $\dot{s}(t) = 0$ in Filippov's sense.

Proof: Consider the candidate Lyapunov function $V(x) = \frac{1}{2}s^\top s$. Then, its derivative has the form

$$\begin{aligned}\dot{V}(s) &= s^\top\big(B^+A^\alpha x^\alpha(t) + u_1(t) + \phi_{\mathrm{m}}(t)\big)\\ &\leq -\|s\|\big(\beta(x^\alpha(t)) - \big(\|B^+\|a+\gamma\big)\|x^\alpha(t)\| - \delta\big).\end{aligned}$$

Hence, if $\beta(x^\alpha(t)) > (\|B^+\|a+\gamma)\|x^\alpha(t)\|+\delta$, the *reachability condition* (see, e.g., [7, Ch. 1] for a definition) holds, that is, s tends to zero in finite time, with $\dot{s}=0$ in Filippov's sense. Since $s(0)=0$, then $s(t)=\dot{s}(t)=0$ for any $t\geq 0$. ■

B. Min-Max Optimal Control Problem

The matched perturbations are rejected under the sliding mode control action, thus generating an equivalent multi-model system without uncertainties. This considerably simplifies the design of the min-max MPC component. Since the whole control law is given by (4), in order to fulfil the input constraint (2b), that is to guarantee the feasibility of the finite horizon optimal control problem (FHOCP), the MPC input constraint must be redefined considering the amplitude of the components u_1. Then, a new set can be accordingly found as

$$\mathcal{U}_0 := \{u_0 \mid u_0 \in \mathcal{U}_0 \Rightarrow u \in \mathcal{U}\}. \tag{11}$$

Defining u_1 as in (6), a quantity equal to $\bar{\beta}$ must be subtracted to each component of the control bounds to determine the bounds of the components of the MPC. We assume that $\mathcal{U}_0$ is a non-empty set, that is even for the worst realization of the uncertainty terms, a residual amplitude for the MPC is guaranteed.

Then, relying on (8), the min-max optimal control problem consists in finding $u_0(t)$, in the interval $0\leq t\leq \bar{t}$, such that

$$\begin{aligned}\min_{u_0\in\mathcal{U}_0}\max_{\alpha\in\mathcal{A}}\ & J(x^\alpha, u_0)\\ \text{s.t.}\ & \dot{x}^\alpha(t) = \bar{A}^\alpha x^\alpha(t) + Bu_0(t),\\ & x^\alpha(t)\in\mathcal{X},\ x^\alpha(0) = x(0),\end{aligned} \tag{12}$$

with the cost given by

$$\begin{aligned}J^\alpha(x^\alpha,u_0) &= \tfrac{1}{2}x^{\alpha\top}(\bar{t})P^\alpha x^\alpha(\bar{t}) + \tfrac{1}{2}\int_0^{\bar{t}}\Big[x^{\alpha\top}(t)Q^\alpha x^\alpha(t)\\ &\quad + \big(u_0(t) - B^+\bar{A}^\alpha x^\alpha(t)\big)^\top R^\alpha\big(u_0(t) - B^+\bar{A}^\alpha x^\alpha(t)\big)\Big]\mathrm{d}t\\ &= \tfrac{1}{2}x^{\alpha\top}(\bar{t})P^\alpha x^\alpha(\bar{t})\\ &\quad + \tfrac{1}{2}\int_0^{\bar{t}}\Bigg[x^{\alpha\top}(t)\underbrace{\Big(Q^\alpha + \bar{A}^{\alpha\top}B^{+\top}R^\alpha B^+\bar{A}^\alpha\Big)}_{\bar{Q}^\alpha}x^\alpha(t)\\ &\quad - 2x^{\alpha\top}(t)\underbrace{\bar{A}^{\alpha\top}B^{+\top}R^\alpha}_{S^\alpha}u_0(t) + u_0^\top(t)R^\alpha u_0(t)\Bigg]\mathrm{d}t,\end{aligned} \tag{13}$$

where Q^α and R^α are symmetric semi-positive definite and positive definite matrices, respectively, and P^α is the positive definite solution of the corresponding algebraic Riccati equation for the α-subsystem.

At this point, in order to solve the min-max optimal control problem, the robust maximum principle [19] is used. More precisely, the controller is defined by identifying the plants for which the maximum cost is reached during the optimization procedure. Therefore, consider the following definition.

Definition 1 (Simplex [19]): The simplex of dimension $N-1$ is the set

$$\mathcal{S}^N := \left\{\mu\in\mathbb{R}^N \mid \sum_{\alpha\in\mathcal{A}}\mu_\alpha = 1,\ \mu_\alpha\geq 0\right\}.$$

We need now to consider an extended system to take into account all the families of the model constituting the plant. Hence, let $\mathbf{x}(t) := [x^{1\top}(t)\,x^{2\top}(t)\,\ldots,\,x^{N\top}(t)]^\top$ be the extended state vector, such that the extended system takes the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_0(t), \tag{14}$$

with $\mathbf{x}(0) = [x^{1\top}(0)\,x^{2\top}(0)\,\ldots,\,x^{N\top}(0)]^\top$, while matrices $\mathbf{A} := \mathrm{diag}(\bar{A}^1,\bar{A}^2,\ldots,\bar{A}^N)\in\mathbb{R}^{nN\times nN}$, and $\mathbf{B} := \mathrm{col}(B,B,\ldots,B)\in\mathbb{R}^{nN\times m}$.

By introducing the parameter $\mu\in\mathcal{S}^N$, the min-max problem (12) is recast into a μ-dependent optimal control problem of the following form

$$\begin{aligned}\min_{u_0\in\mathcal{U}_0}\max_{\mu\in\mathcal{S}^N}\ & J(\mathbf{x},u_0,\mu)\\ \text{s.t.}\ & \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_0(t),\ \mathbf{x}(t)\in\mathcal{X}^N,\\ & \mathbf{x}(0) = \Big[x^{1\top}(0)\,x^{2\top}(0)\ \ldots,\ x^{N\top}(0)\Big]^\top,\end{aligned} \tag{15}$$

with the cost in (13) given by

$$\begin{aligned}J(\mathbf{x},u_0,\mu) &= \tfrac{1}{2}\mathbf{x}^\top(\bar{t})\mathbf{P}(\mu)\mathbf{x}(\bar{t}) + \tfrac{1}{2}\int_0^{\bar{t}}\Big[\mathbf{x}^\top(t)\mathbf{Q}(\mu)\mathbf{x}(t)\\ &\quad - 2\mathbf{x}^\top(t)\mathbf{S}(\mu)u_0(t) + u_0^\top(t)\mathbf{R}(\mu)u_0(t)\Big]\mathrm{d}t,\end{aligned} \tag{16}$$

where

$$\begin{aligned}\mathbf{P}(\mu) &:= \mathrm{diag}(P^1\mu_1, P^2\mu_2,\ldots,P^N\mu_N)\in\mathbb{R}^{nN\times nN},\\ \mathbf{Q}(\mu) &:= \mathrm{diag}(\bar{Q}^1\mu_1,\bar{Q}^2\mu_2,\ldots,\bar{Q}^N\mu_N)\in\mathbb{R}^{nN\times nN},\\ \mathbf{S}(\mu) &:= \Big[S^{1\top}\mu_1\,S^{2\top}\mu_2\ \ldots\ S^{N\top}\mu_N\Big]^\top\in\mathbb{R}^{nN\times m},\\ \mathbf{R}(\mu) &:= \sum_{\alpha=1}^N\mu_\alpha R^\alpha\in\mathbb{R}^{m\times m}.\end{aligned}$$

C. MPC Design

We are now in a position to design the MPC component. To this end, it is worth highlighting that, by virtue of the integral sliding mode control action, since the initial time instant, the equivalent multi-model (8) does not contain any residual uncertainty term (note that this beneficial effect would not be possible in case of use of discrete-time sliding mode control, see, e.g., [20]).

It is also worth noticing that the previous min-max optimal control problem has dimension nN, and this aspect could significantly affect the computational complexity of the MPC. This letter uses a procedure to reduce the dimension of the extended system, by exploiting a state transformation method [15, Ch. 6]. Specifically, consider the following transformation matrix

$$T := \begin{bmatrix} B^{\perp} \\ B^{+} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{17}$$

where $B^{\perp}$ is the orthogonal complement matrix with $\mathrm{rank}(B^{\perp}) = n - m$, and let

$$z^{\alpha} = Tx^{\alpha} = \begin{bmatrix} z_1^{\alpha} \\ z_2^{\alpha} \end{bmatrix} = \begin{bmatrix} T_{11}x_1^{\alpha} + T_{12}x_2^{\alpha} \\ T_{21}x_1^{\alpha} + T_{22}x_2^{\alpha} \end{bmatrix}, \tag{18}$$

with $x_1^{\alpha}, z_1^{\alpha} \in \mathbb{R}^{n-m}$ and $x_2^{\alpha}, z_2^{\alpha} \in \mathbb{R}^{m}$. Then, the multi-model system can be written as

$$\dot{z}^{\alpha}(t) = \tilde{A}^{\alpha} z^{\alpha}(t) + \tilde{B}u_0(t), \quad z^{\alpha}(0) = z_0^{\alpha} = Tx_0^{\alpha}, \tag{19}$$

with $\tilde{A}^{\alpha} = T\bar{A}^{\alpha}T^{-1}$ and $\tilde{B} = TB$. More precisely, the proposed transformation is such that

$$T(I - BB^{+}) = \begin{bmatrix} B^{\perp} \\ 0 \end{bmatrix},$$

thus implying that $\dot{z}_2^{\alpha}(t) = u_0(t)$, and the new system matrices become

$$\tilde{A}^{\alpha} = \begin{bmatrix} \tilde{A}_{11}^{\alpha} & \tilde{A}_{12}^{\alpha} \\ 0 & 0 \end{bmatrix}, \text{ and } \tilde{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

Hence, the extended system dimension can be reduced by defining $\mathbf{z}(t) := [z_1^1(t)\, z_1^2(t)\, \dots\, z_1^N\, z_2(t)]$, with $z_2 = z_2^{\alpha}$, so that the reduced version of the system is

$$\dot{\mathbf{z}}(t) = \tilde{\mathbf{A}}\mathbf{z}(t) + \tilde{\mathbf{B}}u_0(t), \tag{20}$$

with matrices $\tilde{\mathbf{A}} \in \mathbb{R}^{((n-m)N+m)\times((n-m)N+m)}$ and $\tilde{\mathbf{B}} \in \mathbb{R}^{((n-m)N+m)\times m}$ as

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{A}_{11}^1 & 0 & \dots & \tilde{A}_{12}^1 \\ 0 & \tilde{A}_{11}^2 & \dots & \tilde{A}_{12}^2 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I \end{bmatrix}.$$

Consider now the new weight matrices

$$\tilde{P}^{\alpha} = T^{-1\top}PT^{-1} = \begin{bmatrix} \tilde{P}_{11}^{\alpha} & \tilde{P}_{12}^{\alpha} \\ \tilde{P}_{21}^{\alpha} & \tilde{P}_{22}^{\alpha} \end{bmatrix},$$
$$\tilde{Q}^{\alpha} = T^{-1\top}Q^{\alpha}T^{-1} = \begin{bmatrix} \tilde{Q}_{11}^{\alpha} & \tilde{Q}_{12}^{\alpha} \\ \tilde{Q}_{21}^{\alpha} & \tilde{Q}_{22}^{\alpha} \end{bmatrix},$$
$$\tilde{S}^{\alpha} = T^{-1\top}S^{\alpha} = \begin{bmatrix} \tilde{S}_1^{\alpha} \\ \tilde{S}_2^{\alpha} \end{bmatrix},$$

then the reduced min-max problem equivalent to (16) is

$$\begin{aligned} \min_{u_0 \in \mathcal{U}_0} \max_{\mu \in \mathcal{S}^N} \quad & J(\mathbf{z}, u_0, \mu) \\ \text{s.t.} \quad & \dot{\mathbf{z}}(t) = \tilde{\mathbf{A}}\mathbf{z}(t) + \tilde{\mathbf{B}}u_0(t),\ T^{-1}\begin{bmatrix} z_1^{\alpha}(t) \\ z_2(t) \end{bmatrix} \in \mathcal{X}, \\ & \mathbf{z}(0) = \left[z_1^{1\top}(0)\, z_1^{2\top}(0)\, \dots\, z_2^{\top}(0)\right]^{\top}, \end{aligned} \tag{21}$$

with the cost redefined as

$$\begin{aligned} J(\mathbf{z}, u_0, \mu) = {} & \tfrac{1}{2}\mathbf{z}^{\top}(\bar{t})\tilde{\mathbf{P}}(\mu)\mathbf{z}(\bar{t}) + \tfrac{1}{2}\int_0^{\bar{t}} \Big[\mathbf{z}^{\top}(t)\tilde{\mathbf{Q}}(\mu)\mathbf{z}(t) \\ & - 2\mathbf{z}^{\top}(t)\tilde{\mathbf{S}}(\mu)u_0(t) + u_0^{\top}(t)\mathbf{R}(\mu)u_0(t)\Big]\mathrm{d}t. \end{aligned} \tag{22}$$

In (22), the matrices $\tilde{\mathbf{P}} \in \mathbb{R}^{((n-m)N+m)\times((n-m)N+m)}$, $\tilde{\mathbf{Q}} \in \mathbb{R}^{((n-m)N+m)\times((n-m)N+m)}$, $\tilde{\mathbf{S}} \in \mathbb{R}^{((n-m)N+m)\times m}$ are

$$\tilde{\mathbf{P}}(\mu) = \begin{bmatrix} \tilde{P}_{11}^1\mu_1 & 0 & \dots & \tilde{P}_{12}^1\mu_1 \\ 0 & \tilde{P}_{11}^2\mu_2 & \dots & \tilde{P}_{12}^2\mu_2 \\ \dots & \dots & \dots & \dots \\ \tilde{P}_{21}^1\mu_1 & \tilde{P}_{21}^2\mu_2 & \dots & \sum_{\alpha=1}^N \tilde{P}_{22}^{\alpha}\mu_{\alpha} \end{bmatrix},$$
$$\tilde{\mathbf{Q}}(\mu) = \begin{bmatrix} \tilde{Q}_{11}^1\mu_1 & 0 & \dots & \tilde{Q}_{12}^1\mu_1 \\ 0 & \tilde{Q}_{11}^2\mu_2 & \dots & \tilde{Q}_{12}^2\mu_2 \\ \dots & \dots & \dots & \dots \\ \tilde{Q}_{21}^1\mu_1 & \tilde{Q}_{21}^2\mu_2 & \dots & \sum_{\alpha=1}^N \tilde{Q}_{22}^{\alpha}\mu_{\alpha} \end{bmatrix},$$
$$\tilde{\mathbf{S}}(\mu) = \begin{bmatrix} \tilde{S}_1^1\mu_1 \\ \tilde{S}_1^2\mu_2 \\ \dots \\ \sum_{\alpha=1}^N \tilde{S}_2^{\alpha}\mu_{\alpha} \end{bmatrix}.$$

Now, since we want to design an MPC, a discretization of the reduced system (20) is needed. Hence, relying on (8), let T_{s} be the sampling time, and t_k, with $k \in \mathbb{N}_{\geq 0}$, be the discrete-time instant, such that

$$\mathbf{z}_{\mathrm{d}}(t_{k+1}) = \tilde{\mathbf{A}}_{\mathrm{d}}\mathbf{z}_{\mathrm{d}}(t_k) + \tilde{\mathbf{B}}_{\mathrm{d}}u_0(t_k), \tag{23}$$

with discrete-time matrices defined as $\tilde{\mathbf{A}}_{\mathrm{d}} = e^{\tilde{\mathbf{A}}T_{\mathrm{s}}}$, $\tilde{\mathbf{B}}_{\mathrm{d}} = \int_0^{T_{\mathrm{s}}} e^{\tilde{\mathbf{A}}t}\tilde{\mathbf{B}}\,\mathrm{d}t$. The MPC relies on the solution of a FHOCP, which consists in minimizing, at any sampling time t_k, a cost function with respect to the control sequence $\mathbf{u}_{0,[t_k,t_{k+N_{\mathrm{p}}-1}|t_k]} := [u_{0,0}(t_k)\, u_{0,1}(t_k), \dots, u_{0,N_{\mathrm{p}}-1}(t_k)]$, with $N_{\mathrm{p}} \geq 1$ being the so-called prediction horizon. In our case, the cost function used to compute the MPC law u_0 is

$$\begin{aligned} & J(\mathbf{z}_{\mathrm{d}}(t_k), \mathbf{u}_{0,[t_k,t_{k+N_{\mathrm{p}}-1}|t_k]}, N_{\mathrm{p}}, \mu^{\mathrm{o}}) \\ & = \tfrac{T_{\mathrm{s}}}{2}\sum_{j=0}^{N_{\mathrm{p}}-1}\chi^{\top}(t_{k+j})\boldsymbol{\Gamma}(\mu^{\mathrm{o}})\chi(t_{k+j}) + \tfrac{1}{2}\mathbf{z}_{\mathrm{d}}^{\top}(t_{k+N_{\mathrm{p}}})\tilde{\mathbf{P}}(\mu^{\mathrm{o}})\mathbf{z}_{\mathrm{d}}(t_{k+N_{\mathrm{p}}}), \end{aligned} \tag{24}$$

subject to the hard constraints represented by the reduced discrete-time dynamics (23), and constraints on states and input given by

$$T^{-1}\begin{bmatrix} z_{\mathrm{d}_1}^{\alpha}(t_{k+j}) \\ z_{\mathrm{d}_2}(t_{k+j}) \end{bmatrix} \in \mathcal{X}, \tag{25a}$$
$$u_0(t_{k+j}) \in \mathcal{U}_0. \tag{25b}$$

Moreover, in (24), one has $\chi = [\mathbf{z}_d^{\top}\, u_0]^{\top}$, while

$$\boldsymbol{\Gamma}(\mu) = \begin{bmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} \\ \boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{bmatrix},$$

with entries

$$\boldsymbol{\Gamma}_{11} = \tilde{\mathbf{A}}_{\mathrm{d}}^{\top}(\mu)\tilde{\mathbf{Q}}(\mu)\tilde{\mathbf{A}}_{\mathrm{d}}(\mu),$$
$$\boldsymbol{\Gamma}_{12} = \tilde{\mathbf{A}}_{\mathrm{d}}^{\top}(\mu)(\tilde{\mathbf{Q}}(\mu)\tilde{\mathbf{B}}_{\mathrm{d}} - \tilde{\mathbf{S}}(\mu)),$$
$$\boldsymbol{\Gamma}_{21} = (\tilde{\mathbf{B}}_{\mathrm{d}}^{\top}\tilde{\mathbf{Q}}(\mu) - \tilde{\mathbf{S}}^{\top}(\mu))\tilde{\mathbf{A}}_{\mathrm{d}}(\mu),$$

$$\mathbf{\Gamma}_{22} = \tilde{\mathbf{B}}_{\mathrm{d}}^{\top}\tilde{\mathbf{Q}}(\mu)\tilde{\mathbf{B}}_{\mathrm{d}} - \tilde{\mathbf{B}}_{\mathrm{d}}^{\top}\tilde{\mathbf{S}}(\mu) - \tilde{\mathbf{S}}^{\top}(\mu)\tilde{\mathbf{B}}_{\mathrm{d}} + \mathbf{R}(\mu).$$

The optimal value of μ is in turn given as

$$\mu^{\mathrm{o}} = \arg\max_{\mu\in\mathcal{S}^N} J(\mathbf{z}_{\mathrm{d}}(t_k), \mathbf{u}_{0,[t_k,t_{k+N_{\mathrm{p}}-1}|t_{k-1}]}, N_{\mathrm{p}}, \mu), \quad (26)$$

subject to the dynamics (23). More precisely, following [19] the optimal parameter μ can be obtained by solving the finite horizon optimal problem

$$\mu^{\mathrm{o}} = \arg\max_{\mu\in\mathcal{S}^N} \frac{1}{2}\mathbf{z}(t_k)^{\top}\mathbf{\Pi}(0,\mu)\mathbf{z}(t_k), \quad (27)$$

with $\mathbf{\Pi}(0,\mu)$ being the backward solution of the discrete-time Riccati equation

$$\begin{aligned}\mathbf{\Pi}(k,\mu) = {} & \tilde{\mathbf{Q}}(\mu) + \tilde{\mathbf{A}}_{\mathrm{d}}^{\top}\mathbf{\Pi}(k+1,\mu)\tilde{\mathbf{A}}_{\mathrm{d}} \\ & - \left(\tilde{\mathbf{S}}(\mu) + \tilde{\mathbf{B}}_{\mathrm{d}}^{\top}\mathbf{\Pi}(k+1,\mu)\tilde{\mathbf{A}}_{\mathrm{d}}\right)^{\top} \\ & \times \left(\mathbf{R}(\mu) + \tilde{\mathbf{A}}_{\mathrm{d}}^{\top}\mathbf{\Pi}(k+1,\mu)\tilde{\mathbf{B}}_{\mathrm{d}}\right)^{-1} \\ & \times \left(\tilde{\mathbf{S}}(\mu) + \tilde{\mathbf{B}}_{\mathrm{d}}^{\top}\mathbf{\Pi}(k+1,\mu)\tilde{\mathbf{A}}_{\mathrm{d}}\right), \end{aligned} \quad (28)$$

with boundary condition $\mathbf{\Pi}(N_{\mathrm{p}},\mu) = \tilde{\mathbf{P}}(\mu)$.

Finally, according to the receding horizon strategy, the applied piecewise-constant control law is the following

$$u_0(t) = u_{0,0}^{\mathrm{o}}(t_k), \quad t \in [t_k, t_{k+1}), \quad (29)$$

that is the first element of the optimal control sequence $\mathbf{u}^{\mathrm{o}}_{0,[t_k,t_{k+N_{\mathrm{p}}-1}|t_k]}$ is selected and applied to the plant.

To conclude this section, notice that the MPC that has been designed, by relying on the nominal equivalent dynamics (14), is a standard MPC, so that the proof of feasibility and stability of the overall control scheme follows from classical arguments. Moreover, also consider that, even if continuous-time state constraints are considered and the extended system (14) is continuous-time as well, this is only conceptual, because in practice a numerical implementation would need a time discretization of that system, and the constraints satisfaction could be checked only at the integration time instants. Yet, as clarified in [8], this is not a limitation. According to [21, Th. 3], by choosing an integration step τ_{i} small enough (i.e., $\tau_{\mathrm{i}} \ll T_{\mathrm{s}}$), to emulate the continuous-time extended system, one has that the convergence properties of the control system are preserved, without increasing the conservativeness due to the numerical approximation.

IV. Illustrative Example

In this section, the proposal is assessed in simulation on a four-tanks system, which is a multivariable laboratory plant of interconnected tanks with nonlinear dynamics subject to state and input constraints (see, [2]).

The linearized model corresponding to (1) is given by:

$$\dot{x}(t) = \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{1}{\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{1}{\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x(t) + \begin{bmatrix} \frac{\gamma_a}{S} & 0 \\ 0 & \frac{\gamma_b}{S} \\ 0 & \frac{1-\gamma_b}{S} \\ \frac{1-\gamma_a}{S} & 0 \end{bmatrix} (u(t) + \phi_{\mathrm{m}}(t)),$$

where $x_i = h_i - h_i^{\mathrm{o}}$, $i \in \{1,2,3,4\}$ is the ith state deviation variable, with h_i being the water level of tank i and h_i^{o} its linearization level, and $u_j = q_j - q_j^{\mathrm{o}}$, $j \in \{a,b\}$ is the input deviation variable, with q_j being the inlet flows of the valves, and q_j^{o} the linearization flows. Moreover, $S = 0.06\,\mathrm{m}^2$ is the cross section of tanks, $\tau_i = S/a_i\sqrt{2h_i^{\mathrm{o}}/g} \geq 0$ is the time constant of tank i, with $a_1 = 1.31\times10^{-4}\,\mathrm{m}^2$, $a_2 = 1.51\times10^{-4}\,\mathrm{m}^2$, $a_3 = 9.27\times10^{-5}\,\mathrm{m}^2$, $a_4 = 8.82\times10^{-5}\,\mathrm{m}^2$ being the discharge constants, $g = 9.81\,\mathrm{m\,s}^{-2}$, while $\gamma_a = 0.3$ and $\gamma_b = 0.4$ are the ratios of the three-way valves of pumps a and b. The matched disturbance is instead $\phi_{\mathrm{m}}(t) = \left[\begin{smallmatrix} 0.2\sin(\pi/8\cos(3\pi t)+\pi/7\sin(5\pi t)+0.04 \\ -0.2\cos(\pi/8\cos(4\pi t)+\pi/9t\cos(3\pi t)-0.03 \end{smallmatrix}\right]$ (see Fig. 1, bottom). As in [2] the multi-model system has been obtained by linearizing the plant in four (i.e., $\alpha \in \{1,2,3,4\}$) different operation points (see Tab. I), while the initial conditions are $x_0 = [0.032\ \ 0.033\ \ 0.032\ \ 0.033]^{\top}$.

TABLE I

Linearization and Steady State Points

α	h_1^{o}	h_2^{o}	q_a^{o}	q_b^{o}	h_{s_1}	h_{s_2}	h_{s_3}	h_{s_4}	q_{s_a}	q_{s_b}
1	0.42	0.47	1.48	1.52	0.42	0.47	0.42	0.47	1.48	1.52
2	0.29	0.33	1.24	1.28	0.29	0.33	0.29	0.33	1.24	1.28
3	0.86	0.57	1.04	2.69	0.86	0.57	0.85	0.57	1.04	2.698
4	0.65	0.66	1.63	2	0.65	0.66	0.65	0.66	1.62	2

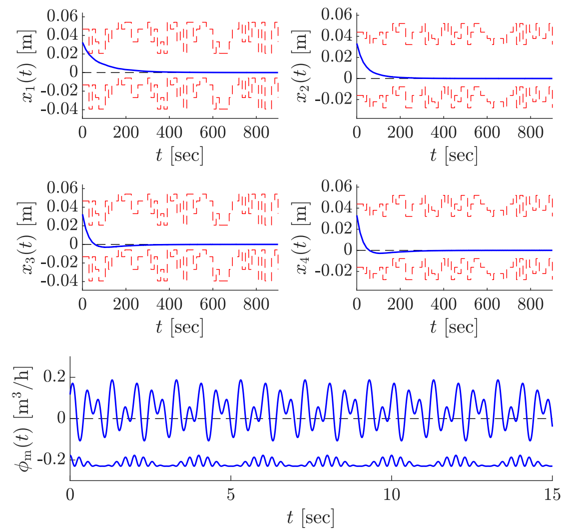

Fig. 1. Time evolution of the state $x(t)$ (top) with constraints (dashed red lines), and of the matched uncertainty $\phi_{\mathrm{m}}(t)$ (bottom).

As for the sliding mode component of the control law (4), its amplitude is chosen as $\beta(x^{\alpha}) = 3.51\|x^{\alpha}\| + 0.24$, such that $\bar{\beta} = 0.74$. In order to design the min-max MPC, the weight matrices are instead chosen as $Q^{\alpha} = I_{4\times4}$ and $R^{\alpha} = 0.01I_{2\times2}$. The FHOCP is then solved with prediction horizon $N_{\mathrm{p}} = 6$, sampling time $T_{\mathrm{s}} = 15\,\mathrm{s}$, subject to state and input constraints defined as $(0.012 - 0.06h_{\mathrm{s}_i}) \leq x_i \leq (0.072 - 0.06h_{\mathrm{s}_i})$, $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \leq u \leq \left[\begin{smallmatrix}3.26\\4\end{smallmatrix}\right]$, and $-\frac{1}{2}\left[\begin{smallmatrix}q_{\mathrm{s}a}\\q_{\mathrm{s}b}\end{smallmatrix}\right] \leq u_0 \leq \frac{1}{2}\left[\begin{smallmatrix}3.26\\4\end{smallmatrix}\right] - \frac{1}{2}\left[\begin{smallmatrix}q_{\mathrm{s}a}\\q_{\mathrm{s}b}\end{smallmatrix}\right]$, respectively, where h_{s_i}, $i \in \{1,2,3,4\}$ and q_{s_j}, $j \in \{a,b\}$ are the steady state level and flows of the plant (see Tab. I). In Fig. 1, the time evolution of the state $x(t)$ is illustrated, which is regulated to zero despite the presence of the matched disturbance $\phi_{\mathrm{m}}(t)$ and the variation of the active subsystem. Fig. 2 illustrates the MPC control signal $u_0(t)$, which is always inside the constraints, and the sliding variable $s(t)$ zeroed from the

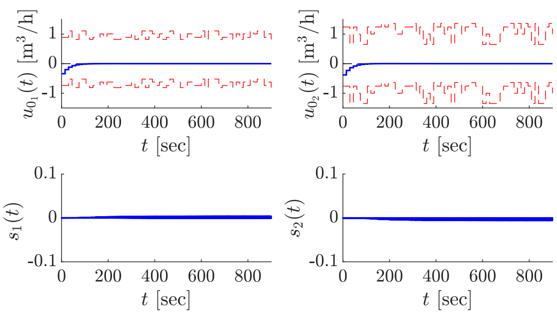

Fig. 2. Time evolution of the MPC input signals $u_0(t)$ (top) with constraints (dashed red lines), and of the sliding variable $s(t)$ (bottom).

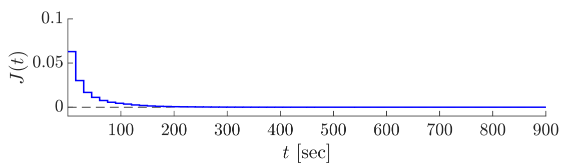

Fig. 3. Time evolution of the cost $J(t)$.

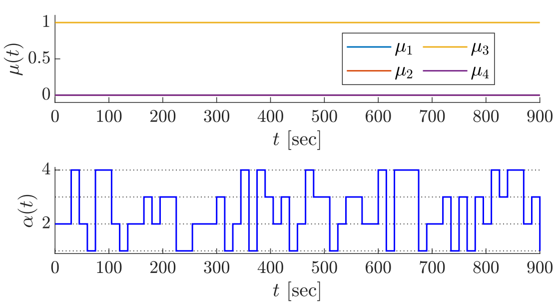

Fig. 4. Time evolution of the optimal parameter $\mu(t)$ (top), under the time varying active subsystem $\alpha(t)$ (bottom).

beginning. Fig. 3 reports the time evolution of the cost $J(t)$, which is minimized according to the proposed min-max MPC law, while the active model, indicated by $\alpha(t)$, is reported in Fig. 4, together with the optimal parameter $\mu(t)$. Overall, as expected, by virtue of the proposed sliding mode control action, the system controlled by the MPC is not affected by the presence of the matched perturbation and behaves as the nominal one.

V. Conclusion

A novel multi-model predictive control combined with an integral sliding-mode control is presented in this letter. The proposal exploits the advantages provided by the sliding mode controller of rejecting the matched uncertainty which affects the plant. As a consequence the MPC component is designed relying on the solution to a min-max optimization problem, and considering the nominal multi-model system.

Future works will be devoted to the extension of the proposal to more complex situations, in particular those characterized by unmatched uncertainty terms, and multi-model uncertainty also on the control effectiveness matrix.

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