Modified Error Bounds for Matrix Completion and Application to RL

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***Abstract*—In matrix completion under noisy measurements, most available results assume that there is an *a priori* bound on the Frobenius norm of the noise, and derive bounds on the Frobenius norm of the residual error. In this letter, we obtain "component-wise" bounds on the residual error, based on a similar bound on the noise. This among the first such results. As in our earlier paper, we choose the locations of the samples to correspond to the edge set of a Ramanujan bigraph. One recent application is to deriving nearly optimal action-value functions in Reinforcement Learning (RL), which is illustrated through examples. The results presented here are only sufficient conditions. Then we illustrate through numerical simulations that there is considerable room for improvement in the sufficient conditions derived here. This is a problem for future research.**

***Index Terms*—Statistical learning, machine learning, iterative learning control.**

I. Introduction

COMPRESSED sensing refers to the recovery of high-dimensional but low-complexity objects from a small number of linear measurements. Recovery of sparse (or nearly sparse) vectors, and recovery of high-dimensional but low-rank matrices are the two most popular applications of compressed sensing. Applications of matrix completion include sensor localization, structure from motion [2], quantum tomography [3]–[7], and most recently, Reinforcement Learning (RL) [8]. Within matrix completion, there are two cases: noise-free measurements, and noisy measurements. With noise-free measurements, one can aspire to recover the unknown matrix exactly. However, with noisy measurements, one can aspire only to recover a good approximation to the unknown matrix. Let X be the unknown low rank matrix, $\mathcal{W}$ be the measurement noise, and $\hat{X}$ be the approximation of X constructed via matrix completion theory. Then current research provides estimates for the Frobenius norm of the recovery error $\|\hat{X}-X\|_F$ in terms of the Frobenius norm $\|\mathcal{W}\|_F$ of the noise. However, in applications such as Reinforcement Learning, one requires bounds on $\|\hat{X}-X\|_{v,\infty}$ in terms of $\|\mathcal{W}\|_{v,\infty}$, where

$$\|M\|_{v,\infty} := \max_{i,j} |M_{ij}|$$

denotes the "vector ℓ_∞-norm," or the maximum modulus of any element of M. Currently available theory does not provide such bounds. One of the main contributions of this letter is to extend *the best available* bounds for matrix completion to derive bounds on $\|\hat{X}-X\|_{v,\infty}$.

II. The Matrix Completion Problem

The matrix completion problem can be stated formally as follows: Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is an unknown matrix whose rank is bounded by a known integer r. We wish to recover X by measuring the components X_{ij} as (i,j) vary over a carefully chosen "sample set" $\Omega = \{(i_1,j_1),\dots,(i_m,j_m)\} \subseteq [n_r]\times[n_c]$. The measurements can either be noise-free or noisy. If they are noise-free, then the measurements can be represented as the Hadamard product $E_\Omega \circ X$,[1] where

$$(E_\Omega)_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \Omega, \\ 0 & \text{if } (i,j) \notin \Omega. \end{cases}$$

In the noisy measurement case, let $\mathcal{W} \in \mathbb{R}^{n_r \times n_c}$ denote the noise matrix. The measurements now consist of $E_\Omega \circ (X + \mathcal{W})$. Note that one can assume, without loss of generality, that $\mathcal{W}_{ij} = 0$ whenever $(i,j) \notin \Omega$.

In order to recover X, in principle we could attempt to find the matrix of minimum rank that is consistent with the measurements. However, this problem is NP-hard [9]. Therefore, a logical approach is to replace the rank function by its convex relaxation, which is the **nuclear norm** $\|\cdot\|_N$ [10]. In the case of noise-free measurements, this leads to formulation

$$\hat{X} := \arg\min_{Z\in\mathbb{R}^{n_r\times n_c}} \|Z\|_N \text{ s.t. } E_\Omega \circ Z = E_\Omega \circ X. \quad (1)$$

In the case of noisy measurements, as suggested in [11], (1) can be modified to

$$\hat{X} := \arg\min_{Z\in\mathbb{R}^{n_r\times n_c}} \|Z\|_N \text{ s.t. } \|E_\Omega \circ Z - E_\Omega \circ X_{\mathcal{W}}\|_F \le \epsilon, \quad (2)$$

where ϵ is a known upper bound on $\|\mathcal{W}\|_F$. For early work on this topic, see [12], which was motivated by the so-called

[1] Recall that the Hadamard product C of two matrices A, B of equal dimensions is defined by $c_{ij} = a_{ij}b_{ij}$ for all i, j.

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"Netflix" problem, or recommendation engines. An excellent survey of matrix completion can be found in [13].

III. Using Ramanujan Graphs for Sampling

This still leaves the question of how the sampling set Ω is to be chosen. In the earliest work starting with [12], the elements of Ω are chosen at random, either without replacement [12] or with replacement [14]. Practically all of the literature on matrix completion is based on random sampling. In a significant departure, in [1], the present authors suggested choosing Ω to correspond to the edge set of a Ramanujan graph. This provides, for the first time, a *deterministic procedure* for choosing the sample set Ω. Consequently, the results in [1] are applicable to *all* matrices. In contrast, prior results only hold with high probability that can be made close to one, but never equal to one. So we briefly introduce Ramanujan bigraphs. Further details about Ramanujan graphs can be found in [15], [16].

Denote the sampling matrix E_Ω by $B \in \{0,1\}^{n_r \times n_c}$. Then B can be interpreted as the biadjacency matrix of a bipartite graph with n_r vertices on one side and n_c vertices on the other. The matrix B, and the associated bipartite graph, are said to be (d_r, d_c)**-biregular** with row-degree d_r and column-degree d_c, if every row of B has d_r ones and every column of B has d_c ones. Throughout this letter σ_1 and σ_2 will represent the largest and the second largest singular value of the measurement matrix E_Ω. Note that, for *every* (d_r, d_c)-biregular bigraph, we have that $\sigma_1 = \sqrt{d_r d_c}$. However, σ_2 depends on the elements of E_Ω. A (d_r, d_c)-biregular bipartite graph is said to be a **Ramanujan bigraph** if

$$|\sigma_2| \le \sqrt{d_r - 1} + \sqrt{d_c - 1}, \tag{3}$$

where σ_2 represents second largest singular value of the matrix B. It can be shown [17] that the right side of (3) is the *smallest possible* bound for σ_2 over the set of all (d_r, d_c)-biregular graphs. This extends a result for nonbipartite graphs from [18], [19].

In [1], the sample set Ω is chosen as the edge set of a Ramanujan bigraph. The main result on matrix recovery using Ramanujan graphs, to solve the problem (2), is given as [1, Th. 7].

IV. Main Results: Theoretical

In this section we present the main theoretical result of this letter, namely, explicit bounds on the error $\|\hat{X} - X\|_{v,\infty}$ for a modification of (2), namely

$$\hat{X} := \arg\min_{Z \in \mathbb{R}^{n_r \times n_c}} \|Z\|_N \text{ s.t. } \|E_\Omega \circ Z - E_\Omega \circ X_{\mathcal{W}}\|_{v,\infty} \le \epsilon, \tag{4}$$

We remind the reader that, prior to this letter, such bounds were available only for $\|\hat{X} - X\|_F$. For *deterministic* choices of the sample set Ω, the only such result is [1, Th. 7]. Let $X \in \mathbb{R}^{n_r \times n_c}$ be the unknown matrix of rank r or less that we wish to recover. Let $X = U \Sigma V^\top$ be its reduced singular value decomposition (SVD). As is common in matrix completion theory, we make use of the concept of the "coherence" of a matrix, introduced in [12]. Let $P_U = UU^\top \in \mathbb{R}^{n_r \times n_r}$ denote the orthogonal projection of $\mathbb{R}^{n_r}$ onto $U\mathbb{R}^{n_r}$. Finally, let $\mathbf{e}_i \in \mathbb{R}^{n_r}$ denote the i-th canonical basis vector. Then we define

$$\mu_0(U) := \frac{n_r}{r} \max_{i \in [n_r]} \|P_U \mathbf{e}_i\|_2^2 = \frac{n_r}{r} \max_{i \in [n_r]} \|u^i\|_2^2, \tag{5}$$

where u^i is the i-th row of U. The quantity $\mu_0(V)$ is defined analogously, and

$$\mu_0(X) := \max\{\mu_0(U), \mu_0(V)\}. \tag{6}$$

Let Ω be the biadjacency matrix of a (d_r, d_c)-biregular bipartite graph. We make the following assumptions, which are standard in matrix completion theory; see also [1]:

(A1). There is a known upper bound $\mu_0(X)$ on $\mu_0(U)$ and $\mu_0(V)$ respectively. Hereafter simply write μ_0 for $\mu_0(X)$.

(A2). There is a constant θ such that

$$\left\| \sum_{k \in S} \frac{n_r}{d_c} (U^{k\top} U^k) - I_r \right\|_S \le \theta, \ \forall S \subseteq [n_r], |S| = d_c, \tag{7}$$

$$\left\| \sum_{k \in S} \frac{n_c}{d_r} (V^{k\top} V^k) - I_r \right\|_S \le \theta, \ \forall S \subseteq [n_c], |S| = d_r, \tag{8}$$

where $U^{k\top}$ is shorthand for $(U^k)^\top$, $V^{k\top}$ is shorthand for $(V^k)^\top$, d_r, d_c are the degrees of the Ramanujan bigraph Ω, and $\|\cdot\|_S$ denotes the spectral norm, i.e., the largest singular value of a matrix.

The bounds below make use of the constants μ and θ, which depend on the unknown matrix X. The interpretation is that nuclear norm minimization does not recover *all* matrices of rank r or less, but only some subset thereof. This aspect is common to *all* papers in matrix completion.

Next, define

$$\alpha = \frac{d_c}{n_r} = \frac{d_r}{n_c} = \sqrt{\frac{d_r d_c}{n_r n_c}}, \tag{9}$$

and note that α is the fraction of elements of X that are being sampled. Finally, define

$$\phi := \frac{\sigma_2}{\sigma_1} \mu_0 r. \tag{10}$$

Observe that ϕ depends on the ratio σ_2/σ_1 which depends on the sampling matrix E_Ω, and the quantity μ_0 which depends on the unknown matrix X.

Theorem 1: With the above notation, suppose without loss of generality that $n_r \le n_c$. Suppose that $\theta + \phi < 1$, and that

$$\alpha > \frac{r(\theta^2 + \phi^2)}{(1 - (\theta + \phi))(1 - \phi)^2}. \tag{11}$$

Define constants c, γ as

$$c := (1 - \phi) - [r\alpha(1 - (\theta + \phi))(\theta^2 + \phi^2)]^{1/2}, \tag{12}$$

$$\gamma := 2\frac{n_r \sqrt{d_r}}{c} \left(1 + \frac{1}{\alpha(1 - k_2)}\right)^{1/2}, \tag{13}$$

and note that $c > 0$ as a consequence of (11). Then every solution $\hat{X}$ of (2) satisfies, for every $\delta > 0$, the bound

$$\|\hat{X} - X\|_{v,\infty} \le (\gamma + \delta)\epsilon. \tag{14}$$

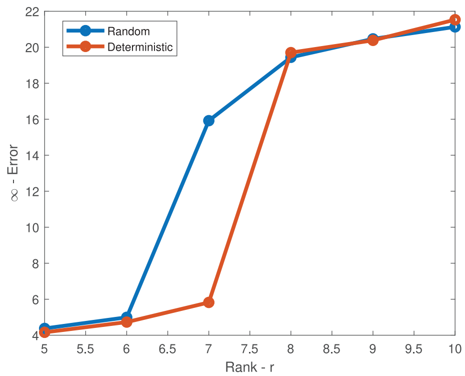

Fig. 1. Numerical computation results using guaranteed Ramanujan graphs with $d_r = 53$, $d_c = 20$ [20].

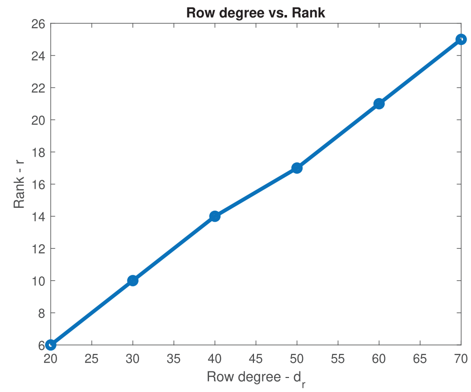

Fig. 2. Numerical computation results using random graphs that are verified to nearly satisfy the Ramanujan property.

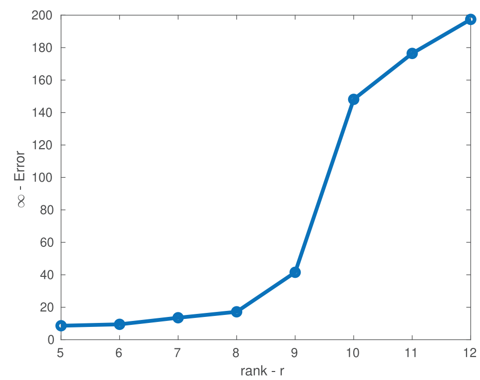

Fig. 3. Numerical computation results using guaranteed Ramanujan graphs with $d_r = 385$, $d_c = 13$ [20].

Note that in the bound (14), the quantity ϵ is a bound on $\|\mathcal{W}\|_{v,\infty}$, which can be substantially smaller than $\|\mathcal{W}\|_F$. To illustrate, suppose every element of $\mathcal{W}$ is a random variable uniformly distributed over $[-\epsilon, \epsilon]$, and every component is independent of every other component. Then $\|\mathcal{W}\|_{v,\infty} \leq \epsilon$. However, the expected value of $\|\mathcal{W}\|_F$ is $(\sqrt{(n_r n_c)/6})\epsilon$, which can be orders of magnitude larger than ϵ, depending on the size of the problem at hand. Thus the bound (14) is quite useful.

The proof of Theorem 1 is given as an Appendix.

V. Main Results: Computational

The sufficient condition in Theorem 1 is very restrictive. If one were to work through all the details, it works out to $\min\{d_c, d_r\} = O(r^3)$. However, the simulations below show that $\min\{d_r, d_c\} \geq 3r$ appears to be sufficient. This extends the observations of [1, Sec. VI].

We generated several random matrices of dimensions $2,500 \times 1,000$ of low rank r for various values of r, These dimensions were chosen to match those in [8]. Then we perturbed each matrix by adding random noise $\mathcal{W}$ where $\|\mathcal{W}\|_{v,\infty}$ is bounded *a priori*, using randomly generated numbers in $[-1, 1]$ and then scaling by some multiplicative constant. Then we attempted to recover the underlying matrix using nuclear norm minimization.

First, using the construction in [20], we constructed a Ramanujan bigraph of size $2,809 \times 1,060$, with degrees 20 and 53 respectively. These dimensions are the nearest for which it is possible to construct a graph that is *guaranteed* to be Ramanujan graph. As can be seen, the recovery is very good until $r = 7 \approx (1/3)\min\{d_r, d_c\}$.

Next, we generated random (d_r, d_c)-biregular graphs of dimension $2,500 \times 1,000$ for various values of d_r, d_c, and random matrices of rank r, and chose one that was a Ramanujan graph. It follows from [21] that such graphs have the Ramanujan property with probability close to one. Then we used this graph together with nuclear norm minimization to recover the unknown matrix. The results are shown in Figure 2. Again, as in [1, Sec. VI], the maximum rank is roughly given by $r \approx (1/3)\min\{d_r, d_c\}$.

Finally, we addressed the matrix completion problem when the number of rows n_r is much smaller than n_c. This situation arises in Reinforcement Learning (discussed in the next section), wherein it is common for the state space to have much larger cardinality than the action space. Using the construction of [20], we constructed a graph of size 169×5005, with row degree 385 and column degree 13. The results are shown in Figure 3. Once again, good recovery occurs for $r \leq (1/3)\min\{d_r, d_c\}$.

VI. *Q*-Learning as a Matrix Completion Problem

A recent paper [8] recasts Q-learning, which is one of the hottest areas in Machine learning, as a matrix completion problem with noisy measurements. However, the solution method proposed there is somewhat ad hoc. In this section, we show that the problem formulated in [8] can be solved using the present results, thus adding to the list of applications of matrix completion theory.

Q-learning is a standard topic and can be found in many texts, for example [22], [23]. Consider Markov Decision Processes (MDPs) defined on a finite state space $\mathcal{X} = \{x_1, \dots, x_n\}$ and a finite action space $\mathcal{U} = \{u_1, \dots, u_m\}$. There is a family of state transition matrices $A^{(u_k)} \in [0, 1]^{n \times n}$ on $\mathcal{X}$ indexed by $u_k \in \mathcal{U}$, where

$$a_{ij}^{(u_k)} = \Pr\{X_{t+1} = x_j | X_t = x_i \& U_t = u_k\}. \tag{15}$$

In addition, there is a deterministic reward function $R : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, as well as a discount factor $\gamma < 1$. The central concept in Q-learning is an "optimal action-value" function $Q^* : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, which satisfies the following implicit relationship:

$$Q^*(x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} \max_{u_k \in \mathcal{U}} Q^*(x_i, u_k). \quad (16)$$

Define a operator G that takes a map $Q : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and returns another map $GQ : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ defined via

$$[GQ](x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} \max_{w_l \in \mathcal{U}} Q(x_j, w_l).$$

It can be shown that G is a contraction with respect to the norm $\| \cdot \|_{v,\infty}$. So it is possible to find the unique solution of (16) by starting with an arbitrary function $Q_0 : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and just iterating.

In a recent paper [8], a novel "sample-efficient" approach to Q-learning is advanced, building in prior work in [25]. These authors study the case where the state space $\mathcal{X}$ and action space $\mathcal{U}$ are compact sets in some finite-dimensional space, for example $\mathcal{X} = [-1, 1]^{d_1}$ and $\mathcal{U} = [-1, 1]^{d_2}$ for appropriate integers d_1, d_2. For such a situation, mild assumptions ensure that the optimal action-state function Q^* is Lipschitz-continuous, and thus has an expansion of the form

$$Q^*(x, u) = \sum_{k=1}^{\infty} \sigma_k f_k(x) g_k(u). \quad (17)$$

In [8], this is truncated to a "low rank" approximation of the form

$$Q_r^*(x, u) = \sum_{k=1}^{r} \sigma_k f_k(x) g_k(u). \quad (18)$$

The heart of the approach in [8] is to construct *approximations* of $Q_r^*(x, u)$ at some "anchor points" $(x_i, u_j) \in \mathcal{X} \times \mathcal{U}$, interpreted as noisy measurements $Q_r^*(x_i, u_k) + \mathcal{W}_{ij}$. Then, using these values, the authors try to reconstruct an approximation $\bar{Q}$ for the entire low rank matrix Q_r^*. In [8], it observed that there is an *element-wise* upper bound on the measurement noise $\mathcal{W}$. Thus one can assume that $\|\mathcal{W}\|_{v,\infty} \leq \epsilon$ for some known constant ϵ. In turn, one must find an upper bound for $\|\bar{Q} - Q_r^*\|_{v,\infty}$, which is *uniform over all state-action pairs*.

In [8], the authors *do not use* nuclear norm minimization to achieve matrix completion. Instead, they use some *ad hoc* methods to solve the problem, and to find some error bounds on $\|\bar{Q} - Q_r^*\|_{v,\infty}$; see [8, eq. (6)]. However, the results presented here lead to rigorous bounds for the quantity $\|\bar{Q} - Q_r^*\|_{v,\infty}$. Therefore the present results extend the scope of the matrix completion problem to encompass Q-learning.

VII. Conclusion and Future Work

In this letter we have derived new component-wise error bounds for matrix completion when nuclear norm minimization is used. Previously such bounds were available only for the Frobenius norm of the error matrix. This letter improves upon the approach suggested in [1], [8], and extends the applicability of matrix completion to Q-learning, a part of Reinforcement Learning. We then established the utility of our approach by numerical simulations, and showed that the practical limits on the applicability of the approach are far higher than the sufficient conditions proven here. For future research, it would be highly desirable to establish rigorous proofs for these observations arrived at via simulation. Also, as of now, for "highly skewed" situations wherein $n_c < \sqrt{n_r}$, there are no techniques for constructing bigraphs that are *guaranteed* to have the Ramanujan property. This situation is common in RL, where the number of actions is much smaller than the number of states. For example, it is shown in [23] that for the blackjack problem, $m = 2$ whereas $n \sim 2^{100}$. This problem is also worthy of further study.

Appendix
Proof of Theorem 1

Define $\mathcal{T} \subseteq \mathbb{R}^{n_r \times n_c}$ to be the subspace spanned by all matrices of the form $UB^\top$ and $CV^\top$. It is easy to show that the projection operator $\mathcal{P}_\mathcal{T}$ equals

$$\begin{aligned}\mathcal{P}_\mathcal{T} Z &= UU^\top Z + ZVV^\top - UU^\top ZVV^\top \\ &= UU^\top Z + U_\perp U_\perp^\top ZVV^\top \\ &= UU^\top ZV_\perp V_\perp^\top + ZVV^\top,\end{aligned}$$

where $U_\perp, V_\perp$ are chosen such that such that $[U \ \ U_\perp]$ and $[V \ \ V_\perp]$ are square and orthogonal. This ensures that $U_\perp U_\perp^\top = I_{n_r} - UU^\top$ and $V_\perp V_\perp^\top = I_{n_c} - VV^\top$.

Now we recall three lemmas from [1].

Lemma 1 (See [1, Lemma 6]): Suppose $E_\Omega \{0, 1\}^{n_r \times n_c}$ is a (d_r, d_c)-biregular sampling matrix, and that $Z \in \mathcal{T}$. Define

$$B^\top = U^\top Z, C = U_\perp U_\perp^\top ZV, \quad (19)$$

so that $Z = UB^\top + CV^\top$. Next, define

$$\begin{aligned}\bar{Z} &:= (1/\alpha) P_\mathcal{T} E_\Omega \circ Z - Z, & (20)\\ \bar{B}^\top &= U^\top \bar{Z}, \bar{C} = U_\perp U_\perp^\top \bar{Z}. & (21)\end{aligned}$$

Then

$$\begin{aligned}\|\bar{B}\|_F &\leq \theta \|B\|_F + \phi \|C\|_F, & (22)\\ \|\bar{C}\|_F &\leq \phi \|B\|_F + \theta \|C\|_F, & (23)\end{aligned}$$

where θ, ϕ are defined in (7) and (10) respectively.

Lemma 2 (See [1, Lemma 3]): Suppose $E_\Omega \in \{0, 1\}^{n_r \times n_c}$ is a (d_r, d_c)-biregular sampling matrix, that $Z \in \mathcal{T}$, and define $\bar{Z}$ as per (20). Then

$$\|\bar{Z}\|_F \leq (\theta + \phi) \|Z\|_F. \quad (24)$$

Lemma 3 (See [1, Lemma 7]): Suppose $M \in \mathbb{R}^{n_r \times n_c}$, $A \in \mathbb{R}^{n_r \times r}$, and $B \in \mathbb{R}^{n_c \times r}$. Suppose further that

$$\|A^i\|_2^2 \leq a^2, \|B^i\|_2^2 \leq b^2. \quad (25)$$

Then

$$\|M \circ (AB^\top)\|_S \leq ab \|M\|_S, \quad (26)$$

where $\| \cdot \|_S$ denotes the spectral norm, i.e., the largest singular value of a matrix.

The next lemma "almost" gets us to the desired result.

Lemma 4: Suppose there exists a matrix $Y \in \mathbb{R}^{n_r \times n_c}$ such that $E_\Omega \circ Y = Y$, and constants $k_1, k_2 \in (0,1), k_3 > 0$ such that

$$\|\mathcal{P}_{\mathcal{T}^\perp}(Y)\|_S \le k_1, \tag{27}$$

$$\|(1/\alpha)\mathcal{P}_{\mathcal{T}} E_\Omega \circ Z - Z\|_F \le k_2 \|Z\|_F, \ \forall Z \in \mathcal{T}, \tag{28}$$

$$\|UV^T - \mathcal{P}_{\mathcal{T}}(Y)\|_F \le k_3, \tag{29}$$

and

$$k_3 < (1-k_1)(\alpha(1-k_2))^{1/2}. \tag{30}$$

Define the constant

$$c := (1-k_1) - (\alpha(1-k_2))^{-1/2} k_3. \tag{31}$$

Then every solution $\hat{X}$ of (2) satisfies the bound

$$\|\hat{X} - X\|_{v,\infty} \le 2\frac{n_r\sqrt{d_r}}{c}\left(1 + \frac{1}{\alpha(1-k_2)}\right)^{1/2}\epsilon. \tag{32}$$

Proof: Define $H = \hat{X} - X$, so that $\hat{X} = X + H$. Also, it can be assumed without loss of generality that $\mathcal{W}$ is supported on Ω, so that $E_\Omega \circ \mathcal{W} = \mathcal{W}$ and $E_{\Omega^c} \circ \mathcal{W} = 0$. Now define $H_\Omega = E_\Omega \circ H$ and we can write

$$\begin{aligned}\|H_\Omega\|_{v,\infty} &= \|E_\Omega \circ (\hat{X} - X - \mathcal{W}) + E_\Omega \circ \mathcal{W}\|_{v,\infty} \\ &= \|E_\Omega \circ (\hat{X} - X_{\mathcal{W}})\|_{v,\infty} + \|E_\Omega \circ \mathcal{W}\|_{v,\infty} \\ &\le 2\epsilon,\end{aligned} \tag{33}$$

because (i) $\hat{X}$ is feasible for (2), and thus $\|E_\Omega \circ (\hat{X} - X_{\mathcal{W}})\|_{v,\infty} \le \epsilon$ (see (2)), and (ii) $\|\mathcal{W}\|_{v,\infty} \le \epsilon$. Define $H_{\Omega^c} = E_{\Omega^c} \circ H$. Next,

$$\begin{aligned}\|H\|_{v,\infty} &= \|E_\Omega \circ H + E_{\Omega^c} \circ H\|_{v,\infty} \\ &= \max\{\|H_\Omega\|_{v,\infty}, \|H_{\Omega^c}\|_{v,\infty}\} \\ &= \max\{2\epsilon, \|H_{\Omega^c}\|_{v,\infty}\}\end{aligned} \tag{34}$$

Therefore, once we are able to find an upper bound for $\|H_{\Omega^c}\|_{v,\infty}$, the above relationship leads to an upper bound for $\|H\|_{v,\infty} = \|\hat{X} - X\|_{v,\infty}$. Next

$$\begin{aligned}\|\hat{X}\|_N &= \|X + H\|_N \\ &= \|X + H_{\Omega^c} + H_\Omega\|_N \\ &\ge \|X + H_{\Omega^c}\|_N - \|H_\Omega\|_N\end{aligned} \tag{35}$$

Next, write $H_{\Omega^c} = \mathcal{P}_{\mathcal{T}}(H_{\Omega^c}) + \mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})$, and note that $\langle Y, H_{\Omega^c}\rangle_F = 0$ because Y is supported on Ω and H_{Ω^c} is supported on Ω^c. Now observe that, for any matrix $B \in \mathbb{R}^{n_r \times n_c}$, we have that

$$\|B\|_N = \max_{\|A\|_S \le 1} \langle A, B\rangle_F, |\langle A, B\rangle_F| \le \|A\|_S \|B\|_N. \tag{36}$$

Therefore

$$\begin{aligned}\|X + H_{\Omega^c}\|_N &\ge \langle UV^T + U_\perp V_\perp^\top, X + H_{\Omega^c}\rangle_F \\ &\overset{(a)}{=} \|X\|_N + \langle UV^T + U_\perp V_\perp^\top, H_{\Omega^c}\rangle_F - \langle Y, H_{\Omega^c}\rangle_F \\ &= \|X\|_N + \langle UV^T - \mathcal{P}_{\mathcal{T}}(Y), \mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\rangle_F \\ &\quad + \langle U_\perp V_\perp^\top - \mathcal{P}_{\mathcal{T}^\perp}(Y), \mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\rangle_F \\ &\overset{(b)}{\ge} \|X\|_N - \|UV^T - \mathcal{P}_{\mathcal{T}}(Y)\|_F \|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F \\ &\quad + (1 - \|\mathcal{P}_{\mathcal{T}^\perp}(Y)\|_S)\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N\end{aligned} \tag{37}$$

where (a) follows from $\langle Y, H_{\Omega^c}\rangle_F = 0$, and (b) follows from (36). Using (35) and (37) together we get

$$\begin{aligned}\|\hat{X}\|_N &\ge \|X\|_N - \|UV^T - \mathcal{P}_{\mathcal{T}}(Y)\|_F \|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F \\ &\quad + (1 - \|\mathcal{P}_{\mathcal{T}^\perp}(Y)\|_S)\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N \\ &\quad - \|H_\Omega\|_N.\end{aligned} \tag{38}$$

On the other hand, $\|\hat{X}\|_N \le \|X\|_N$ because X is feasible for the problem in (2), and $\hat{X}$ is a solution of (2). Substituting this into (38), cancelling $\|X\|_N$ from both sides, and rearranging terms, gives

$$\begin{aligned}\|H_\Omega\|_N &\ge (1 - \|\mathcal{P}_{\mathcal{T}^\perp}(Y)\|_S)\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N \\ &\quad - \|UV^T - \mathcal{P}_{\mathcal{T}}(Y)\|_F \|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F\end{aligned} \tag{39}$$

Now,

$$\begin{aligned}&\|E_\Omega \circ \mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F^2 \\ &= \langle E_\Omega \circ \mathcal{P}_{\mathcal{T}}(H_{\Omega^c}), \mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\rangle_F \\ &= \langle \mathcal{P}_{\mathcal{T}} E_\Omega \circ \mathcal{P}_{\mathcal{T}}(H_{\Omega^c}) - \alpha\mathcal{P}_{\mathcal{T}}(H_{\Omega^c}), \mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\rangle_F \\ &\quad + \alpha\langle \mathcal{P}_{\mathcal{T}}(H_{\Omega^c}), \mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\rangle_F \\ &\overset{(a)}{\ge} \alpha\|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F^2 \\ &\quad - \|\mathcal{P}_{\mathcal{T}}(E_\Omega \circ \mathcal{P}_{\mathcal{T}}(H_{\Omega^c}) - \alpha\mathcal{P}_{\mathcal{T}}(H_{\Omega^c}))\|_F \|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F \\ &\overset{(b)}{\ge} \alpha\|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F^2 - \alpha k_2 \|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F^2 \\ &= \alpha(1-k_2)\|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F^2,\end{aligned} \tag{40}$$

where (a) follows from $\mathcal{P}_{\mathcal{T}}^2 = \mathcal{P}_{\mathcal{T}}$, and (b) follows from assumption (28). Next, note that $E_\Omega \circ H_{\Omega^c} = 0$, which in turn implies that

$$E_\Omega \circ \mathcal{P}_{\mathcal{T}} H_{\Omega^c} = -E_\Omega \circ \mathcal{P}_{\mathcal{T}^\perp} H_{\Omega^c},$$

so that

$$\|E_\Omega \mathcal{P}_{\mathcal{T}} H_{\Omega^c}\|_F = \|E_\Omega \mathcal{P}_{\mathcal{T}^\perp} H_{\Omega^c}\|_F,$$

Therefore

$$\begin{aligned}\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N &\ge \|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_F \\ &\ge \|E_\Omega \circ \mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_F \\ &= \|E_\Omega \circ \mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F \\ &\ge (\alpha(1-k_2))^{1/2}\|\mathcal{P}_{\mathcal{T}}(H_{\Omega^c})\|_F\end{aligned} \tag{41}$$

where the last step follows from (40). Now using (41) in (39) gives us

$$\begin{aligned}&\|H_\Omega\|_N \ge (1 - \|\mathcal{P}_{\mathcal{T}^\perp}(Y)\|_S)\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N \\ &\quad - (\alpha(1-k_2))^{-1/2}\|UV^T - \mathcal{P}_{\mathcal{T}}(Y)\|_F \|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N \\ &\ge [(1-k_1) - (\alpha(1-k_2))^{-1/2}k_3]\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N \\ &= c\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N,\end{aligned} \tag{42}$$

where we use the assumptions (27) and (30), together with (41). Next

$$\begin{aligned}\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_F &\le \|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_N \overset{(a)}{\le} (1/c)\|H_\Omega\|_N \\ &\overset{(b)}{\le} (\sqrt{n_r}/c)\|H_\Omega\|_F \overset{(c)}{\le} (n_r\sqrt{d_r}/c)\|H_\Omega\|_{v,\infty} \\ &\overset{(d)}{\le} 2(n_r\sqrt{d_r}/c)\epsilon,\end{aligned} \tag{43}$$

where (a) follows from (42), (b) follows from the fact that $H_\Omega \in \mathbb{R}^{n_r \times n_c}$ and thus has rank no more than $\min\{n_r, n_c\} = n_r$, (c) follows from the inequality $\|H_\Omega\|_F \leq \sqrt{n_r d_r}\|H_\Omega\|_{v,\infty}$ and (d) follows from (5). Using (40) we get

$$\begin{aligned}\|\mathcal{P}_\mathcal{T}(H_{\Omega^c})\|_F &\leq (\alpha(1-k_2))^{-1/2}\|E_\Omega \circ \mathcal{P}_\mathcal{T}(H_{\Omega^c})\|_F \\ &\overset{(a)}{=} (\alpha(1-k_2))^{-1/2}\|E_\Omega \circ \mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_F \\ &\leq (\alpha(1-k_2))^{-1/2}\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_F \qquad (44)\end{aligned}$$

where, as above, (a) follows from the fact $E_\Omega \circ H_{\Omega^c} = 0$. Now $\|H_{\Omega^c}\|_{v,\infty}$ can be written as

$$\begin{aligned}\|H_{\Omega^c}\|_F^2 &= \|\mathcal{P}_\mathcal{T}(H_{\Omega^c})\|_F^2 + \|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_F^2 \\ &\overset{(a)}{\leq} \left(1 + \frac{1}{\alpha(1-k_2)}\right)\|\mathcal{P}_{\mathcal{T}^\perp}(H_{\Omega^c})\|_F^2 \\ &\overset{(b)}{\leq} \left(1 + \frac{1}{\alpha(1-k_2)}\right)\frac{4n_r^2 d_r}{c^2}\epsilon^2, \qquad (45)\end{aligned}$$

where (a) follows from (44) and (b) follows from (43). It now follows that

$$\|H_{\Omega^c}\|_{v,\infty} \leq \|H_{\Omega^c}\|_F \leq 2\frac{n_r\sqrt{d_r}}{c}\left(1 + \frac{1}{\alpha(1-k_2)}\right)^{1/2}\epsilon, \quad (46)$$

Eq. (46) along with (6) gives the required bound. ■

Proof: (Of Theorem 1) At last we come to the proof of Theorem 1. Suppose $\theta + \phi < 1$ (which automatically implies that $\phi < 1$), and define

$$k_1 = \phi, k_2 = \theta + \phi, k_3 = \sqrt{r(\theta^2 + \phi^2)}.$$

The proof consists of showing that, under the stated hypotheses, there exists a matrix $Y \in \mathbb{R}^{n_r \times n_c}$ that satisfies the hypotheses of Lemma 4.

We start with (28). Lemma 2 states the following: If $Z \in \mathcal{T}$ and $\bar{Z} := (1/\alpha)\mathcal{P}_\mathcal{T} E_\Omega \circ Z - Z$, then

$$\|\bar{Z}\|_F \leq (\theta + \phi)\|Z\|_F. \qquad (47)$$

where θ is defined in (7) and ϕ is defined in (10). Therefore (28) is satisfied with $k_2 = \theta + \phi$. Next, let $W_0 := UV^\top \in \mathcal{T}$, and define

$$Y = (1/\alpha)E_\Omega \circ W_0.$$

Then

$$\mathcal{P}_\mathcal{T}(Y) - UV^\top = (1/\alpha)\mathcal{P}_\mathcal{T} E_\Omega \circ W_0 - W_0.$$

Now we can write $W_0 = UB^\top$ where $B = V$, and apply Lemma 1. This gives

$$\|\mathcal{P}_\mathcal{T}(Y) - W_0\|_F \leq \sqrt{r(\theta^2 + \phi^2)} = k_3.$$

Finally, because $W_0 \in \mathcal{T}$, we can reason as follows:

$$\begin{aligned}\|\mathcal{P}_{\mathcal{T}^\perp} Y\|_S &= \|\mathcal{P}_{\mathcal{T}^\perp}(Y - W_0)\|_S \\ &\leq \|Y - W_0\|_S = \|M \circ W_0\|_S.\end{aligned}$$

Now we can apply Lemma 3, with

$$a = \sqrt{\frac{\mu_0 r}{n_r}}, b = \sqrt{\frac{\mu_0 r}{n_c}}, \|M\|_S = \frac{\sigma_2}{a}.$$

This gives

$$\|Y\|_S \leq \frac{\sigma_2}{\alpha}\frac{\mu_0 r}{\sqrt{n_r n_c}} = \phi$$

Hence we can choose $k_1 = \phi$. The theorem now follows from applying Lemma 4. ■

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