Indirect Adaptive Control of Piecewise Affine Systems Without Common Lyapunov Functions

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***Abstract*—In this letter, we propose a novel indirect model reference adaptive control approach for uncertain piecewise affine systems. This approach exploits a barrier Lyapunov function to construct novel adaptation laws and average dwell time constraints for switching. Compared to the previous research, where closed-loop stability and asymptotic tracking can only be established with a common Lyapunov function, the current approach allows a multiple Lyapunov function setting and enables broader applications. Furthermore, the estimation errors of control gains and subsystem parameters are proved to converge to zero asymptotically if a persistent excitation condition is satisfied. A simulation example of the pitch control of a helicopter system shows the effectiveness of the proposed approach.**

***Index Terms*—Indirect adaptive control, multiple Lyapunov function, parameter estimation, switched systems.**

I. Introduction

CONVENTIONALLY, indirect adaptive control approaches for piecewise affine (PWA) systems are limited to be applied in the presence of common Lyapunov functions [1], [2]. This motivates us to explore the indirect adaptive control of PWA systems in a multiple Lyapunov function setting with provable stability, which allows more design freedom and broader applications.

For the stability analysis of non-adaptive switched systems with multiple Lyapunov functions, dwell time constraints are derived such that the instantaneous jumps of the Lyapunov function at switching instants can be compensated by its exponential decrease in between successive switches [3], [4]. This approach, however, cannot be directly applied to adaptive switched systems unless extra conditions are involved. The adaptive scheme for switched systems proposed in [5, Th. 1] introduces projections to establish the exponential decrease of the Lyapunov function in between sufficiently slow switches with a prerequisite that the bounds of the estimated parameters are known a-priori. Exponential decrease of the Lyapunov function in between switches can also be achieved by imposing persistently exciting (PE) conditions [5, Th. 2], [1, Th. 2]. This requires the input signal to possess rich frequencies, which is usually unwanted for a pure tracking task.

The aforementioned adaptive control approaches are categorized as direct adaptive control, where the controller gains are adapted directly based on tracking errors without identifying system parameters. Another important aspect of adaptive control is indirect adaptive control. Compared to direct adaptive control, it achieves the parameter identification in addition to the tracking task and is ideal for monitoring purposes. In [1, Th. 4], [2], indirect adaptive controllers are proposed for PWA systems and estimated parameters converge to their real values if some excitation conditions are fulfilled. Unlike direct adaptive control, the exponential decrease of the Lyapunov function in between switches cannot be established even when the PE condition is imposed. Therefore, these approaches are currently restricted to the cases with common Lyapunov functions, which indicates limited choices of the reference models and narrowed applications.

In this letter, we exploit a novel barrier function concept to develop an indirect adaptive control approach for uncertain PWA systems allowing a multiple Lyapunov function setting. The proposed multiple Lyapunov function is endowed with a key feature that it is non-increasing at each switching instant. The contributions of the current paper can be summarized as follows: first, compared with the previous approaches [1, Th. 3], [2], the novel indirect adaptive controller does not require the existence of a common Lyapunov function and achieves a more general setting. Second, in contrast with other multiple Lyapunov function-based direct adaptive control approaches of PWA systems, asymptotic state tracking is achieved without the need of projections [5] or PE conditions [1, Th. 2]. This enables the application of the proposed method with fewer prior knowledge requirements and less excitation. An average dwell time constraint is derived such that the boundedness of all the closed-loop signals is established. Finally, to show the capability of joint tracking and identification of the proposed indirect adaptive controller, we prove that the estimated system parameters converge to their real values in the presence of a PE input. Overall, this letter fills the theoretical gap of the indirect adaptive control of PWA systems in the multiple Lyapunov function setting with provable stability guarantee, while preserving the asymptotic tracking and parameter convergence properties of these previous works.

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II. Preliminaries and Problem Formulation

Suppose that the state space of the PWA system is partitioned into s convex regions $\{\Omega_i\}_{i\in\mathcal{I}}$ with $\mathcal{I} = \{1, 2, \ldots, s\}$. In each region, the PWA system is governed by the subsystem dynamics

$$\dot{x} = A_i x + B_i u + f_i, \quad \text{for } x \in \Omega_i \tag{1}$$

with $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times p}, f_i \in \mathbb{R}^n$ denoting the unknown system matrices of i-th subsystem. $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ denote the state and input, respectively. To express in which convex region the state vector lies, or in other words, which subsystem is activated, we define the indicator function $\chi_i(t) = 1$ for $x \in \Omega_i$ and $\chi_i(t) = 0$ otherwise. In PWA systems, the regions $\{\Omega_i\}_{i\in\mathcal{I}}$ are not overlapped, i.e., $\sum_{i=1}^s \chi_i = 1$ and $\prod_{i=1}^s \chi_i = 0$. Based on these two properties, the overall dynamics of the PWA system can be written as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t) \tag{2}$$

with $A(t) = \sum_{i=1}^s \chi_i(t)A_i$, $B(t) = \sum_{i=1}^s \chi_i(t)B_i$ and $f(t) = \sum_{i=1}^s \chi_i(t)f_i$.

Further, we let t_0 be the initial time instant and the set $\{t_1, t_2, \ldots, t_k, \ldots, |k \in \mathbb{N}^+\}$ be the set of switching time instants, i.e., the time instants at which x evolves across the boundary of two adjacent regions. For a given time period (τ, t), the number of switching instants over this interval is denoted by $N(t, \tau)$.

A reference PWA system used to generate the desired behavior shares the same indicator function as the controlled PWA system (2). Its dynamics is given as

$$\dot{x}_m(t) = A_m(t)x_m(t) + B_m(t)r(t) + f_m(t), \tag{3}$$

where $x_m \in \mathbb{R}^n$ and $r \in \mathbb{R}^p$ denote the state and input of the reference system, $A_m(t) = \sum_{i=1}^s \chi_i(t)A_{mi}$, $B_m(t) = \sum_{i=1}^s \chi_i(t)B_{mi}$, $f_m(t) = \sum_{i=1}^s \chi_i(t)f_{mi}$ with $A_{mi} \in \mathbb{R}^{n\times n}, B_{mi} \in \mathbb{R}^{n\times p}, f_{mi} \in \mathbb{R}^n$, $i \in \mathcal{I}$ being known parameters of the i-th reference subsystem. A_{mi} are Hurwitz matrices and (A_{mi}, B_{mi}) are controllable $\forall i \in \mathcal{I}$. Furthermore, there exist a positive constant $h \in \mathbb{R}^+$ and a set of positive definite matrices $P_i, Q_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{I}$ such that

$$A_{mi}^T P_i + P_i A_{mi} + 2hP_i = -Q_i, \quad \forall i \in \mathcal{I} \tag{4}$$

In order to enforce the controlled PWA system (2) to behave as the reference system (3), we define the nominal controller

$$u(t) = K_x^* x(t) + K_r^* r(t) + K_f^*, \tag{5}$$

where the controller and the plant (2) switch synchronously. Therefore, $K_x^*(t) = \sum_{i=1}^s \chi_i(t)K_{xi}^*$, $K_r^*(t) = \sum_{i=1}^s \chi_i(t)K_{ri}^*$, $K_f^*(t) = \sum_{i=1}^s \chi_i(t)K_{fi}^*$ with $K_{xi}^*, K_{ri}^*, K_{fi}^*$ being the nominal controller gains for i-th subsystem. Taking (5) into (2) gives the closed-loop system, which should be equivalent to the reference system. That gives the matching equations

$$A_{mi} = A_i + B_i K_{xi}^*, \quad B_{mi} = B_i K_{ri}^*,$$
$$f_{mi} = f_i + B_i K_{fi}^*. \tag{6}$$

Since the matrices A_i, B_i, f_i are unknown, $K_{xi}^*, K_{ri}^*, K_{fi}^*$ cannot be obtained from these matching equations. The adaptive controller is designed based on the estimated gains $K_{xi}(t), K_{ri}(t), K_{fi}(t)$ and takes the form

$$u(t) = K_x(t)x(t) + K_r(t)r(t) + K_f(t) \tag{7}$$

with $K_x(t) = \sum_{i=1}^s \chi_i(t)K_{xi}$, $K_r(t) = \sum_{i=1}^s \chi_i(t)K_{ri}$ and $K_f(t) = \sum_{i=1}^s \chi_i(t)K_{fi}$.

Thus, the problem we would like to solve is formulated as follows.

Problem 1: Given a reference system (3) satisfying (4), a PWA system (2) with known region partitions Ω_i and unknown subsystem parameters A_i, B_i and f_i, develop an indirect adaptive controller $u(t)$ such that the state $x(t)$ tracks $x_m(t)$ and the unknown system parameters are identified.

III. Methodology

In indirect adaptive control, the unknown system parameters need to be identified while tackling the tracking task. So define $\hat{A}_i, \hat{B}_i, \hat{f}_i$ to be the estimated values of A_i, B_i and f_i. The estimated parameters are updated based on the state information x and the predicted state, denoted by $\hat{x} \in \mathbb{R}^n$, whose dynamics can be written as

$$\dot{\hat{x}} = A_m\hat{x} + \sum_{i=1}^s ((\hat{A}_i - A_{mi})x + \hat{B}_i u + \hat{f}_i)\chi_i. \tag{8}$$

Define $\tilde{A}_i = \hat{A}_i - A_i$, $\tilde{B}_i = \hat{B}_i - B_i$, and $\tilde{f}_i = \hat{f}_i - f_i$ to be the parameter estimation errors. By (2) and (8) we obtain

$$\dot{\tilde{x}} = A_m\tilde{x} + \sum_{i=1}^s (\tilde{A}_i x + \tilde{B}_i u + \tilde{f}_i)\chi_i \tag{9}$$

with $\tilde{x} = \hat{x} - x$ representing the state prediction error. Equation (9) relates $\tilde{x}$ with the parameter estimation errors $\tilde{A}_i, \tilde{B}_i, \tilde{f}_i$.

Before we derive the adaptation laws for parameters and control gains, we first define the prediction error metric

$$\|\tilde{x}\|_P^2 = \tilde{x}^T(\sum_{i=1}^s \chi_i P_i)\tilde{x}. \tag{10}$$

$\|\tilde{x}\|_P$ is piecewise continuous and piecewise differentiable. Moreover, the following property holds at each switching instant t_k

$$\|\tilde{x}(t_k)\|_P^2 \leq \mu\|\tilde{x}(t_k^-)\|_P^2, \quad \text{with } \mu \triangleq \max_{i,j\in\mathcal{I}} \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)}, \tag{11}$$

where $\tilde{x}(t_k^-) \triangleq \lim_{\tau\uparrow t_k} \tilde{x}(\tau)$. This definition also applies for other signals at t_k^-. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denotes the maximum and minimum eigenvalue of a symmetric matrix, respectively. Next, we define the following generalized restricted potential function (barrier Lyapunov function) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ on the domain $\mathcal{D}_\epsilon(t) = \{\tilde{x} \mid \|\tilde{x}\|_P \in [0, \epsilon(t))\}$

$$\phi(\|\tilde{x}\|_P) = \frac{\|\tilde{x}\|_P^2}{\epsilon^2(t) - \|\tilde{x}\|_P^2}, \tag{12}$$

where $\epsilon(t)$, as the "barrier" of ϕ, is a time-varying piecewise continuous signal generated by the dynamics

$$\dot{\epsilon}(t) = -h\epsilon(t) + g, \quad \epsilon(t_0) > \frac{g}{h}$$
$$\epsilon(t_k) = \sqrt{\mu}\epsilon(t_k^-), \tag{13}$$

with $g, h \in \mathbb{R}^+$ being some positive constants and h satisfying (4). At each switching instant t_k, ϵ is reset by the reset map $\epsilon(t_k) = \sqrt{\mu}\epsilon(t_k^-)$. Considering the jump of $\|\tilde{x}\|_P$ at t_k shown in (11), the reset map of ϵ guarantees $\|\tilde{x}(t_k)\|_P < \epsilon(t_k)$ if $\|\tilde{x}(t_k^-)\|_P < \epsilon(t_k^-)$, i.e., the increment of $\|\tilde{x}\|_P$ at switching instants will not lead to invalidity of $\phi(\|\tilde{x}\|_P)$.

In every interval between two successive switching instants $[t_k, t_{k+1}), k \in \mathbb{N}^+$, ϕ is differentiable and its partial derivative with respect to $\|\tilde{x}\|_P^2$ is $\phi_d \triangleq \partial\phi/\partial\|\tilde{x}\|_P^2 = \epsilon^2(t)/(\epsilon^2(t) - \|\tilde{x}\|_P^2)^2 > 0$. Furthermore, ϕ and ϕ_d have the relation that

$$2\phi_d(\|\tilde{x}\|_P)\|\tilde{x}\|_P^2 - \phi > 0. \tag{14}$$

Remark 1: The generalized restricted potential function ϕ is originally proposed by Arabi and Yucelen [6] for linear systems and extended to switched systems in [7] aiming to enforce the state tracking error in adaptive systems to satisfy a user-defined performance guarantee. In this letter, we show how this function can be exploited in indirect adaptive control of PWA systems in a multiple Lyapunov function setting to achieve asymptotic tracking and parameter estimation.

The indirect adaptation laws for estimated system parameters and control gains are based on dynamic gain adjustment technique [1], [8]. It introduces intermediate variables to adjust both control gains and system parameters. These intermediate variables are called closed-loop estimation errors, defined as

$$\begin{aligned} \varepsilon_{Ai} &= \hat{A}_i + \hat{B}_i K_{xi} - A_{mi}, \quad \varepsilon_{Bi} = \hat{B}_i K_{ri} - B_{mi}, \\ \varepsilon_{fi} &= \hat{f}_i + \hat{B}_i K_{fi} - f_{mi}. \end{aligned} \tag{15}$$

These can be interpreted as residual errors of matching conditions (see (6)). On the one hand, the adaptation of control gains is driven by closed-loop estimation errors

$$\begin{aligned} \dot{K}_{xi} &= -S_i^T B_{mi}^T \varepsilon_{Ai}, \quad \dot{K}_{ri} = -S_i^T B_{mi}^T \varepsilon_{Bi}, \\ \dot{K}_{fi} &= -S_i^T B_{mi}^T \varepsilon_{fi}, \end{aligned} \tag{16}$$

where $S_i, i \in \mathcal{I}$ are known matrices such that $K_{ri}^* S_i$ are symmetric and positive definite. The knowledge of S_i matrices is a common assumption in adaptive control [5], [9]. In scalar case, it represents that the control direction is known a priori.

On the other hand, the estimation of system parameters is adjusted based on closed-loop estimation errors, the state prediction error $\tilde{x}$, the state x, and input u, specifically,

$$\begin{aligned} \dot{\hat{A}}_i &= -\chi_i \phi_d P_i \tilde{x} x^T - \varepsilon_{Ai}, \\ \dot{\hat{B}}_i &= -\chi_i \phi_d P_i \tilde{x} u^T - \varepsilon_{Ai} K_{xi}^T - \varepsilon_{Bi} K_{ri}^T - \varepsilon_{fi} K_{fi}^T, \\ \dot{\hat{f}}_i &= -\chi_i \phi_d P_i \tilde{x} - \varepsilon_{fi}. \end{aligned} \tag{17}$$

In (17), the partial derivative ϕ_d is used as a time-varying gain of the first summand of each adaptation law, which differs from the classical adaptation laws (see [1, eq. (37)]). Such arrangement as well as the adaptation of control gains (16) are suggested by Lyapunov-based stability analysis, which is shown in the next section.

IV. Stability

We start by exploring under which condition the auxiliary "barrier" signal $\epsilon(t)$ is bounded, which is summarized in the following lemma.

Lemma 1: Given the piecewise continuous signal $\epsilon(t)$ defined in (13), if the switching satisfies $N(t,\tau) \le N_0 + \frac{t-\tau}{\tau_D}$ with $N_0 = \frac{2\alpha}{\ln\mu}$ and $\tau_D = \frac{\ln\mu}{2(h-l)}$ for any positive constants $\alpha \in \mathbb{R}^+, l \in (0, h)$, then $\epsilon(t)$ is bounded.

Proof: For an arbitrary time interval $[t_0, t)$ containing the number of switches $N(t, t_0) = k$, namely, $t_0 < t_1 < \cdots < t_k < t, k \in \mathbb{N}^+$, we have

$$\begin{aligned} \epsilon(t) &= \epsilon(t_k)\mathrm{e}^{-h(t-t_k)} + \int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau \\ &= \sqrt{\mu}\epsilon(t_k^-)\mathrm{e}^{-h(t-t_k)} + \int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau. \end{aligned} \tag{18}$$

Replacing $\epsilon(t_k^-)$ in (18) with

$$\epsilon(t_k^-) = \epsilon(t_{k-1})\mathrm{e}^{-h(t_k-t_{k-1})} + \int_{t_{k-1}}^{t_k^-} g\mathrm{e}^{-h(t_k-\tau)}\mathrm{d}\tau \tag{19}$$

leads to

$$\begin{aligned} \epsilon(t) = \mu^{\frac{1}{2}}\epsilon(t_{k-1})\mathrm{e}^{-h(t-t_{k-1})} + \mu^{\frac{1}{2}}\int_{t_{k-1}}^{t_k^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau \\ + \int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau. \end{aligned}$$

Recursively doing the same derivation shown above yields

$$\begin{aligned} \epsilon(t) &= \mu^{\frac{k}{2}}\epsilon(t_0)\mathrm{e}^{-h(t-t_0)} + \mu^{\frac{k}{2}}\int_{t_0}^{t_1^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau \\ &+ \mu^{\frac{k-1}{2}}\int_{t_1}^{t_2^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau + \cdots + \mu^{\frac{0}{2}}\int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau. \end{aligned}$$

Note that $k = N(t, t_0)$, $k-1 = N(t, t_1)$, etc. Therefore,

$$\begin{aligned} \epsilon(t) &= \mu^{\frac{N(t,t_0)}{2}}\epsilon(t_0)\mathrm{e}^{-h(t-t_0)} + \mu^{\frac{N(t,t_0)}{2}}\int_{t_0}^{t_1^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau \\ &+ \mu^{\frac{N(t,t_1)}{2}}\int_{t_1}^{t_2^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau + \cdots + \mu^{\frac{N(t,t_k)}{2}}\int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau. \end{aligned}$$

Since $N(t,\tau) = N(t, t_j)$ for $\tau \in [t_j, t_{j+1})$, the $\mu^{\frac{N(t,t_j)}{2}}, j \in \{0, 1, \ldots, k\}$ terms can be put into the integral operators, which yields

$$\begin{aligned} \epsilon(t) &= \mu^{\frac{N(t,t_0)}{2}}\epsilon(t_0)\mathrm{e}^{-h(t-t_0)} + \int_{t_0}^{t_1^-} g\mathrm{e}^{-h(t-\tau)}\mu^{\frac{N(t,\tau)}{2}}\mathrm{d}\tau \\ &+ \int_{t_1}^{t_2^-} g\mathrm{e}^{-h(t-\tau)}\mu^{\frac{N(t,\tau)}{2}}\mathrm{d}\tau + \cdots + \int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mu^{\frac{N(t,\tau)}{2}}\mathrm{d}\tau. \end{aligned}$$

Merging all the integral terms yields

$$\epsilon(t) = \epsilon(t_0)\mathrm{e}^{-h(t-t_0)+\frac{N(t,t_0)}{2}\ln\mu} + g\int_{t_0}^{t} \mathrm{e}^{-h(t-\tau)+\frac{N(t,\tau)}{2}\ln\mu}\mathrm{d}\tau.$$

To ensure the boundedness of $\epsilon(t)$, it suffices to let $-h(t-\tau) + \frac{N(t,\tau)}{2}\ln\mu \le \alpha - l(t-\tau)$ for some positive constants $\alpha \in \mathbb{R}^+$ and $l \in (0, h)$. This further leads to

$$N(t,\tau) \le N_0 + \frac{t-\tau}{\tau_D} \tag{20}$$

with $N_0 \triangleq \frac{2\alpha}{\ln\mu}$ and $\tau_D \triangleq \frac{\ln\mu}{2(h-l)}$. ■

The average dwell time means that the switches do not necessarily fulfill a fixed dwell time constraint but are constrained in an average sense. Despite having the same form as the one in [3], (20) is derived based on the nonautonomous switched system (13). Furthermore, applying [3, Th. 2], the reference model (3) is stable and x_m is bounded with the constraint (20).

Theorem 1: Consider the reference system (3) satisfying (4) and the PWA system (2) with known regions Ω_i and unknown subsystem parameters A_i, B_i, f_i. Let the PWA system (2) be controlled by the adaptive controller (7) with adaptation laws (15), (16), and (17). If the switch of the controlled PWA system satisfies the dwell time constraint (20), then the state tracking error $e \triangleq x - x_m \to 0$ as $t \to \infty$.

Proof: Consider the following candidate Lyapunov function

$$V = \frac{1}{2}\phi(\|\tilde{x}\|_P) + V_K + V_\theta$$

where

$$V_K \triangleq \frac{1}{2}\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_{si}\tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_{si}\tilde{K}_{ri}) + \tilde{K}_{fi}^T M_{si}\tilde{K}_{fi})$$

$$V_\theta \triangleq \frac{1}{2}\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\tilde{A}_i) + \mathrm{tr}(\tilde{B}_i^T\tilde{B}_i) + \tilde{f}_i^T\tilde{f}_i))$$

with $M_{si} = (K_{ri}^* S_i)^{-1} \in \mathbb{R}^{p\times p}$. Suppose i-th subsystem is activated in the interval $[t_{k-1}, t_k)$ and the time-derivative of V in this interval is

$$\dot{V} = \frac{1}{2}\dot{\phi}(\|\tilde{x}\|_{P_i}) + \sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\dot{\tilde{A}}_i) + \mathrm{tr}(\tilde{B}_i^T\dot{\tilde{B}}_i) + \tilde{f}_i^T\dot{\tilde{f}}_i) + \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_i\dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i\dot{\tilde{K}}_{ri}) + \tilde{K}_{fi}^T M_i\dot{\tilde{K}}_{fi})}_{\triangleq v_k}$$

Considering (17), we expand the second summand of $\dot{V}$ as

$$\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\dot{\tilde{A}}_i) + \mathrm{tr}(\tilde{B}_i^T\dot{\tilde{B}}_i) + \tilde{f}_i^T\dot{\tilde{f}}_i)$$
$$= -\phi_d(\mathrm{tr}(\tilde{A}_i^T P_i\tilde{x}x^T) + \mathrm{tr}(\tilde{B}_i^T P_i\tilde{x}u^T) + \tilde{f}_i^T P_i\tilde{x}) - \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\varepsilon_{A_i} + \tilde{B}_i^T(\varepsilon_{Ai}K_{xi}^T + \varepsilon_{Bi}K_{ri}^T + \varepsilon_{fi}K_{fi}^T)) + \tilde{f}_i^T\varepsilon_{fi})}_{\triangleq v_\varepsilon}$$

Inserting (16) into v_k we obtain

$$v_k - v_\varepsilon = -\sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T\varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T\varepsilon_{Bi}) + \varepsilon_{fi}^T\varepsilon_{fi}).$$

Detailed derivations of this step can be found in [1, Sec. 4]. Thus, we have

$$\dot{V} = \frac{1}{2}\dot{\phi}(\|\tilde{x}\|_{P_i}) - \phi_d(\mathrm{tr}(\tilde{A}_i^T P_i\tilde{x}x^T) + \mathrm{tr}(\tilde{B}_i^T P_i\tilde{x}u^T) + \tilde{f}_i^T P_i\tilde{x}) - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T\varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T\varepsilon_{Bi}) + \varepsilon_{fi}^T\varepsilon_{fi}). \quad (21)$$

The time-derivative of ϕ can be further simplified as

$$\dot{\phi} = \frac{\partial\phi}{\partial\|\tilde{x}\|_{P_i}^2}\frac{\mathrm{d}\|\tilde{x}\|_{P_i}^2}{\mathrm{d}t} + \frac{\partial\phi}{\partial\epsilon}\dot{\epsilon} = 2\phi_d\tilde{x}^T P_i\dot{\tilde{x}} + \frac{\partial\phi}{\partial\epsilon}\dot{\epsilon} \quad (22)$$

Substituting $\dot{\tilde{x}}$ in (22) with (9) yields

$$\dot{\phi} = \phi_d\tilde{x}^T(A_{mi}^T P_i + P_i A_{mi})\tilde{x} + 2\phi_d\tilde{x}^T P_i(\tilde{A}_i x + \tilde{B}_i u + \tilde{f}_i) + \frac{\partial\phi}{\partial\epsilon}\dot{\epsilon}. \quad (23)$$

Inserting (23) into (21) we obtain after some cancellations

$$\dot{V} = \frac{1}{2}\phi_d\tilde{x}^T(A_{mi}^T P_i + P_i A_{mi})\tilde{x} + \frac{1}{2}\frac{\partial\phi}{\partial\epsilon}\dot{\epsilon} - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T\varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T\varepsilon_{Bi}) + \varepsilon_{fi}^T\varepsilon_{fi}).$$

We know that

$$\frac{\partial\phi}{\partial\epsilon}\dot{\epsilon} = \frac{-2\epsilon\|\tilde{x}\|_{P_i}^2}{(\epsilon^2 - \|\tilde{x}\|_{P_i}^2)^2}\dot{\epsilon} = -2\phi_d\|\tilde{x}\|_{P_i}^2\frac{\dot{\epsilon}}{\epsilon} \le 2\phi_d\|\tilde{x}\|_{P_i}^2\frac{|\dot{\epsilon}|}{\epsilon}.$$

From the dynamics of ϵ (13) we have $|\dot{\epsilon}|/\epsilon \le h$, which gives $\frac{\partial\phi}{\partial\epsilon}\dot{\epsilon} \le 2h\phi_d\|\tilde{x}\|_{P_i}^2 = \phi_d\tilde{x}^T(2hP_i)\tilde{x}$. This together with (4) leads to

$$\dot{V} \le -\frac{1}{2}\phi_d\tilde{x}^T Q_i\tilde{x} - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T\varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T\varepsilon_{Bi}) + \varepsilon_{fi}^T\varepsilon_{fi}).$$

Therefore, V is nonincreasing in between two successive switching instants $[t_{k-1}, t_k)$, $k \in \mathbb{N}^+$.

Now we analyse the behavior of V at switching instant t_k. Without loss of generality, we suppose that i-th subsystem is activated in $[t_{k-1}, t_k)$ and j-th subsystem is activated in $[t_k, t_{k+1})$, where $i, j \in \mathcal{I}$, $i \neq j$. From (11) we have

$$\phi(\|\tilde{x}(t_k)\|_P) = \frac{\|\tilde{x}(t_k)\|_{P_j}^2}{\epsilon^2(t_k) - \|\tilde{x}(t_k)\|_{P_j}^2} \le \frac{\mu\|\tilde{x}(t_k^-)\|_{P_i}^2}{\epsilon^2(t_k) - \mu\|\tilde{x}(t_k^-)\|_{P_i}^2}$$
$$= \frac{\mu\|\tilde{x}(t_k^-)\|_{P_i}^2}{\mu\epsilon^2(t_k^-) - \mu\|\tilde{x}(t_k^-)\|_{P_i}^2} = \phi(\|\tilde{x}(t_k^-)\|_P).$$

Since V_K and V_θ remain unchanged at t_k, we have $V(t_k) \le V(t_k^-)$, $k \in \mathbb{N}^+$.

According to the above analysis, $\dot{V}$ is negative semidefinite for $t \in [t_0, \infty)$. Thus, we have $\phi \in \mathcal{L}_\infty$, $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$, and $\hat{A}_i, \hat{B}_i, \hat{f}_i \in \mathcal{L}_\infty$, which, according to the definition (15), leads to $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty$. The boundedness of ϕ implies that $\|\tilde{x}(t)\|_P < \epsilon(t)$ for $\forall t \in [t_0, \infty)$. Since ϵ is bounded, we have $\tilde{x} \in \mathcal{L}_\infty$ and $\phi_d \in \mathcal{L}_\infty$. Integrating $\dot{V}$ over $[t_0, \infty)$, we obtain $\int_{t_0}^{\infty}\dot{V}\mathrm{d}t = V(\infty) - V(t_0) \le -\int_{t_0}^{\infty}(\frac{1}{2}\phi_d\tilde{x}^T Q_i\tilde{x} + \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T\varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T\varepsilon_{Bi}) + \varepsilon_{fi}^T\varepsilon_{fi}))\mathrm{d}t$. Because ϕ_d, $V(\infty)$, and $V(t_0)$ are bounded, we conclude $\tilde{x}, \varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_2$. Letting $\varepsilon_A = \sum_{i=1}^{s}\chi_i\varepsilon_{Ai}$, $\varepsilon_B = \sum_{i=1}^{s}\chi_i\varepsilon_{Bi}$, $\varepsilon_f = \sum_{i=1}^{s}\chi_i\varepsilon_{fi}$ and inserting (7) and (15) into (8) yields

$$\dot{\hat{x}} = (A_m + \varepsilon_A)\hat{x} - \varepsilon_A\tilde{x} + (B_m - \varepsilon_B)r + f_m - \varepsilon_f.$$

This equation together with stable A_m, $r \in \mathcal{L}_\infty$, $\tilde{x} \in \mathcal{L}_\infty$, $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ leads to $\hat{x}, x \in \mathcal{L}_\infty$. According to (7) we have $u \in \mathcal{L}_\infty$. Bounded x, u imply bounded $\dot{\hat{A}}_i, \dot{\hat{B}}_i, \dot{\hat{f}}_i, \dot{\tilde{x}}$ and further $\dot{\varepsilon}_{Ai}, \dot{\varepsilon}_{Bi}, \dot{\varepsilon}_{fi} \in \mathcal{L}_\infty$.

Therefore, $\tilde{x}, \varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \to 0$ as $t \to \infty$. This together with $\hat{x} \to x_m$ leads to $e \to 0$ as $t \to \infty$. ■

Remark 2: Exponential convergence of switched systems in non-adaptive cases can be obtained based on multiple

Lyapunov functions with finite jumps at switching instants compensated by exponential decaying behaviors in between switches [3], [4]. In direct adaptive control cases, extra conditions such as projections [5] or PE conditions [1, Th. 2] are required to establish the exponential decrease of V in between switches. Compared with [1], [5], our method achieves asymptotic tracking without introducing these extra conditions.

Remark 3: The conventional multiple Lyapunov function concept introduced in [3], [4] does not work for indirect adaptive control of PWA systems without common Lyapunov functions. Due to the presence of $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi}$ in $\dot{V}$, the exponential decaying property of V in between successive switches cannot be established. No dwell time constraint can be found to compensate the increment of V at switches and the closed-loop stability cannot be obtained (see [1, Th. 4]). Therefore, previous approaches [1], [2] require the existence of common Lyapunov functions ($P_i = P_j, \forall i \neq j$) to avoid jumps of V at each t_k. In contrast, the proposed Lyapunov function in this letter is non-increasing at each switch instant and provable closed-loop stability can be established without a common Lyapunov function.

Remark 4: Adaptation laws proposed in [10] exploits time-varying $P_i(t)$ matrices and achieves asymptotic tracking for switched systems without projections. $P_i(t)$ are obtained by interpolating a collection of $P_{i,k}$ matrices for each subsystem, which are pre-calculated to satisfy some linear matrix inequalities associated to a given fixed dwell time. In our method, the switching obeying average dwell time instead of fixed dwell time is allowed, which enables more flexibility.

V. Parameter Convergence

We explore here the parameter convergence property when applying the proposed approach.

Theorem 2: If the reference input r is sufficiently rich of order $n+1$ such that all subsystems are repeatedly activated, then the state tracking error $e \to 0$ and the estimated parameters $\hat{A}_i, \hat{B}_i, \hat{f}_i$ as well as the estimated gains K_{xi}, K_{ri}, K_{fi} converge to their nominal values as $t \to \infty$.

Proof: The stability and the asymptotic convergence of e has been proved in Theorem 1. In this proof, we remove the subscript i and let the following steps refer to the activated subsystem. Since all subsystems are activated intermittently, the convergence of estimated parameters of all the subsystems can be concluded.

Define $\tilde{\theta} = \text{vec}([\tilde{A}, \tilde{B}, \tilde{f}])$ and $\Psi = [x^T, u^T, 1]^T \otimes I_n$ with $I_n \in \mathbb{R}^{n\times n}$ being the identity matrix. From (9) and (17), we can write the dynamics of the prediction error $\tilde{x}$ and parameter estimation error $\tilde{\theta}$ in compact form as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A_m & \Psi^T \\ -\phi_d \Psi P & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \tag{24}$$

where $\varepsilon = -\text{vec}([\varepsilon_A, \varepsilon_A K_x^T + \varepsilon_B K_r^T + \varepsilon_f K_f^T, \varepsilon_f])$. Define $X \triangleq [\tilde{x}^T, \tilde{\theta}^T]^T$, we can rewrite (24) as

$$\dot{X} = \bar{A}X + L\tilde{x} + d, \quad \tilde{x} = CX \tag{25}$$

where

$$\bar{A} = \begin{bmatrix} A_m & \Psi^T \\ -\Psi P & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ (1-\phi_d)\Psi P \end{bmatrix} \quad C^T = \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

and $d = [0, \varepsilon^T]^T$. Equation (25) reveals that the dynamics of X can be decomposed into a homogeneous part $\bar{A}X$, an output injection part $L\tilde{x}$ and a disturbance term d. It is proved in Theorem 1 that $\varepsilon \to 0$ and $\tilde{x} \to 0$ as $t \to \infty$, so $L\tilde{x} \to 0, d \to 0$. We can focus on proving the convergence property of the homogeneous part of (25): $\dot{X} = \bar{A}X$. Let $\bar{P} = \text{diag}\{P, I_{n(n+p+1)}\}$. We construct the Lyapunov function $V = X^T\bar{P}X$, whose derivative along the solution $\dot{X} = \bar{A}X$ is given by

$$\dot{V} = X^T(\bar{A}^T\bar{P} + \bar{P}\bar{A})X = -\tilde{x}(Q + 2hP)\tilde{x} \leq -\nu\tilde{x}^T\tilde{x}$$

with $\nu = \lambda_{\min}(Q + 2hP) \in \mathbb{R}^+$. This further leads to $\dot{V} \leq -\nu X^T C^T C X$. Invoking [1, Lemma 2] and considering the conclusion of Theorem 1 that $e \to 0$, we have the signal vector $z = [x^T, u^T, 1]^T$ is PE if the reference signal r is sufficiently rich. Applying [11, Lemma 5.6.3] we obtain $\tilde{\theta} \to 0$ and therefore, $\hat{A} \to A, \hat{B} \to B, \hat{f} \to f$ as $t \to \infty$. This together with $\varepsilon_A, \varepsilon_B, \varepsilon_f \to 0$ gives $\tilde{K}_x, \tilde{K}_r, \tilde{K}_f \to 0$ as $t \to \infty$. ■

Remark 5: In the barrier function-based approaches for direct adaptive control of linear systems [6] and PWA systems [7], the tracking error is confined within a predefined performance bound, whereas we prove in this letter the asymptotic convergence of the tracking error. Built upon this result, we further prove the estimated parameter convergence, which has not been explored in [6], [7]. The challenge of parameter convergence analysis lies in the presence of the time-varying gain ϕ_d in the joint dynamics of $[\tilde{x}, \tilde{\theta}]$ in (24), where the conventional analysis [1, Th. 4] cannot be applied. To bypass this issue, we transform the joint dynamics of $[\tilde{x}, \tilde{\theta}]$ into an output injection form (25), where ϕ_d is shifted into the injected output $\tilde{x}$. Since $\tilde{x} \to 0$ is proved in Theorem 1, the effect of ϕ_d on the parameter convergence decays to zero.

VI. Numerical Experiments

The indirect adaptive control scheme is applied to the PWA model of the pitch control of a helicopter system [12]. This PWA model can be written in form of (2) with the state $x = [x_1, x_2]^T \in \mathbb{R}^2$ denoting the vector of pitch angle and pitch rate. The state space for $x_1 \in [-\frac{3\pi}{5}, \frac{3\pi}{5}]$ is divided into 3 regions, $\Omega_1 = \{x| -\frac{\pi}{5} \leq x_1 \leq \frac{\pi}{5}\}$, $\Omega_2 = \{x|\frac{\pi}{5} < x_1 < \frac{3\pi}{5}\}$ and $\Omega_3 = \{x| -\frac{3\pi}{5} < x_1 < -\frac{\pi}{5}\}$. The associated system parameters can be found in [12]. The reference PWA model has the subsystem form

$$A_{mi} = \begin{bmatrix} 0 & 1 \\ -a_{1i} & -a_{2i} \end{bmatrix}, \quad B_{mi} = \begin{bmatrix} 0 \\ b_i \end{bmatrix}, \quad f_{mi} = \begin{bmatrix} 0 \\ f_i \end{bmatrix}$$

with $a_{11} = b_1 = 25$, $a_{21} = 10$, $f_1 = 0$, $a_{12} = b_2 = 16$, $a_{22} = 8$, $f_2 = 5$, and $a_{13} = b_3 = 49$, $a_{23} = 14$, $f_3 = -10$. The auxiliary signal $\epsilon(t)$ is generated with $h = 0.12$, $g = 0.01$, and $\epsilon(0) = 9$. Numerical experiments show that larger h and smaller g may improve the tracking and parameter convergence rate but lead to smaller denominator of ϕ_d, which may

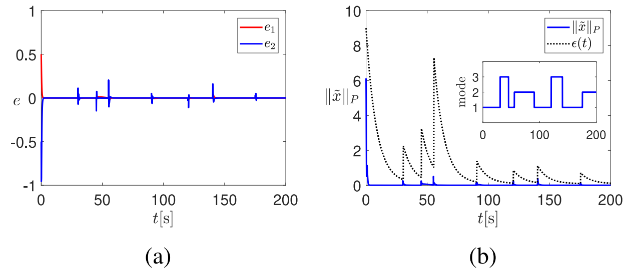

Fig. 1. Tracking performance with the proposed indirect adaptive controller.

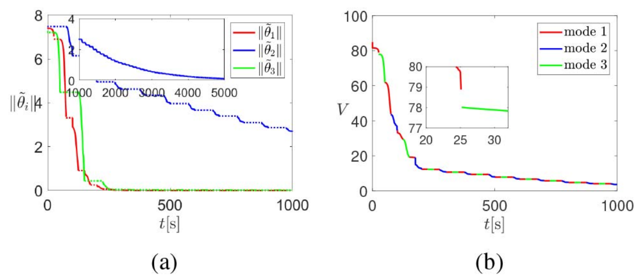

Fig. 2. Parameter convergence and the Lyapunov function.

cause ill-conditioned problems when solving it numerically. P_i matrices are determined by letting $Q_i = [100, 10; 10, 100]$ based on (4). Therefore, by (11), $\mu = 52.0045$ and $\tau_D > 16.5s$ for $l \in (0, h)$. Let $x(0) = [0.5, -0.5]^T$ and $x_m(0) = \hat{x}(0) = 0$. The initial values of the estimated system parameters are specified as 80% of their nominal values with zero initial gains. Given the reference input rectangular signal switching among $\{-1, 0, 1\}$, the tracking performance is shown in Fig. 1. One can observe that the tracking error $e \to 0$. Besides, each jump of $\epsilon(t)$ (marked by the black dashed line) indicates a switch instant. The overall switch (7 switches within $200s$) is slower than τ_D and the decrease of $\epsilon(t)$ together with the decaying $\|\tilde{x}\|_P$ validates Theorem 1.

To test the parameter convergence property, let the input signal be $r = \bar{r} + 0.4\sin(1.2t) + 0.2\sin(11t)$ for $\bar{r}$ switching among $\{-1, 0, 1\}$ with a fixed interval $25s$. It contains 2 distinct frequencies and is sufficiently rich of order $4 > n+1 = 3$. $\bar{r}(t)$ is exerted such that the state of the closed-loop system is driven through all the partitioned regions. The estimation errors of system parameters are shown in Fig. 2(a), where the parameters of different subsystems are distinguished with difference colors. The dashed sections represent the phase where the corresponding subsystem is inactive and the solid ones display the active phase. As we can see, $\|\tilde{\theta}_1\|$ and $\|\tilde{\theta}_3\|$ converge quite close to 0 within 300 s, while $\|\tilde{\theta}_2\|$ decreases relatively slower. The Lyapunov function V is displayed in Fig. 2(b). Different colors indicate which mode is active. Although no common Lyapunov matrix is applied (because $P_i \neq P_j, i \neq j$), V is non-increasing at each switching instant and decreasing in between every two consecutive switches. This indicates the stability of the closed-loop system with the proposed method and validates the theoretical results.

VII. Conclusion

In this letter, the indirect model reference adaptive control of piecewise affine systems with uncertain subsystem parameters is studied. The proposed approach exploits a novel barrier Lyapunov function to develop indirect adaptation laws. Average dwell time constraints are derived to ensure the boundedness of closed-loop signals. It relaxes the assumption of the existence of a common Lyapunov function, which is known as one key limitation of the previous study. In addition, given a persistently exciting input, we prove that the estimation of the controller gains and subsystem parameters converge to their nominal values.

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