On Discrete-Time Output Negative Imaginary Systems

Parijat Bhowmick, *Member, IEEE*, and Alexander Lanzon, *Senior Member, IEEE*

***Abstract*—This letter introduces the notion of linear Discrete-time Output Negative Imaginary (D-ONI) systems. The D-ONI class is defined in the z-domain and it includes the systems having poles on the unit circle. The proposed definition involves a real parameter $\delta \geq 0$, which indicates the strictness properties. $\delta > 0$ specifies the strict subset, Discrete-time Output Strictly Negative Imaginary (D-OSNI), within the stable D-ONI class. Interestingly, the new D-ONI class captures the existing D-NI systems while restricted to discrete-time LTI systems having a real, rational and proper transfer function. However, the D-OSNI systems are not identical to the existing *strictly* D-NI (D-SNI) subset. Instead, these two subsets intersect each other. An LMI-based state-space characterisation is derived to check the strict/non-strict D-ONI properties of a given system relying on the value of δ. The paper also establishes the connections between the discrete-time Passive and discrete-time NI systems. Finally, a closed-loop stability result is proposed for a positive feedback interconnection of two D-ONI systems without poles at $z = -1$ and $z = +1$.**

***Index Terms*—Discrete-time output negative imaginary systems, discrete-time passive systems, Bilinear transformation, DC-gain, z-domain stability.**

I. Introduction

NEGATIVE Imaginary (NI) systems theory has already established its worth due to its potential applications in real-world engineering problems, such as, in vibration control of lightly-damped mechanical systems [1]–[6], cantilever beam [7], large space structures [8] and robotic manipulators [8]; nano-positioning applications [9]; vehicle platooning; etc. NI theory was introduced in [1] and was primarily inspired by the positive position feedback control of highly resonant mechanical structures with colocated position sensors and force actuators. Recently, in [10] and [11], NI property has been extended to capture improper and non-rational systems. Of late, the Output Negative Imaginary (ONI) systems theory [2], [12]–[15] has drawn the attention of the control community due to establishing the missing links amongst the NI theory, classical dissipativity and passivity [14].

However, most of the developments on analysis and synthesis of NI systems have been done in the continuous-time setting. References [16] and [17] did the first works in extending the continuous-time NI (C-NI) theory to discrete-time LTI systems exploiting the well-established Bilinear Transformation ($s = \frac{z-1}{z+1}$) adopted in [18] for defining Discrete-time Positive Real (D-PR) systems. Reference [16] also defined two strict subsets within the D-NI[1] class, *viz.* Discrete-time Strictly Negative Imaginary (D-SNI) and Discrete-time Strongly Strict Negative Imaginary (D-SSNI), along the lines of the continuous-time versions of the SNI and SSNI systems, as introduced in [10] and [11]. Recently, [17] and [19] also have defined and characterised the D-NI and D-SNI systems for discrete-time real, rational and proper transfer functions without using the symmetry assumption. However, the discrete-time extension of the Output Negative Imaginary systems has not yet been explored in the literature.

This letter introduces the notion of real, rational, proper D-ONI systems and provides a complete theoretical analysis of such systems, including the connections amongst the discrete-time Passive [20], D-PR [18], [21] and D-ONI classes. A strict subset within the D-ONI class, referred to as the D-OSNI subclass, is defined only for the stable discrete-time systems (i.e., having no poles in $\{z \in \mathbb{C} : |z| \geq 1\}$) and attributed by a parameter $\delta > 0$. An in-depth study reveals that the D-OSNI subset is not identical to the existing D-SNI subset [16], [17]

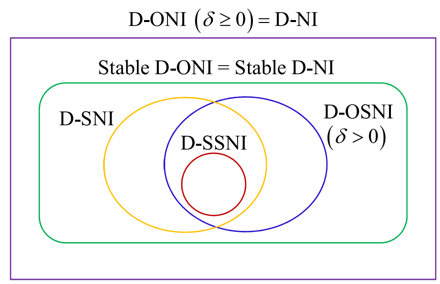

Fig. 1. Relationship among the strict and non-strict subsets within the class of real, rational and proper D-ONI systems.

[1] In [16], D-NI systems and the strict subsets D-SNI and D-SSNI were defined for discrete-time LTI systems having real and symmetric transfer functions. However, when restricted to real, rational transfer functions, the symmetry assumption in [16] can be easily removed.

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The authors are with the Control Systems Centre, Department of Electrical and Electronic Engineering, School of Engineering, The University of Manchester, Manchester M13 9PL, U.K. (e-mail: parijat.bhowmick@manchester.ac.uk; alexander.lanzon@manchester.ac.uk).

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– they rather intersect. D-OSNI systems may contain blocking zeros on the unit circle ($|z| = 1$), in contrast to the D-SNI systems. The relationship among the major subclasses within the D-ONI class is shown in the Venn diagram in Fig. 1, which indicates that the D-SSNI [16] systems belong to the intersection of D-OSNI and D-SNI subsets. This letter also derives an LMI-based state-space characterisation that can be conveniently used to test the strict/non-strict D-ONI properties of a given system depending on the parameter δ. In this context, note that the D-SNI property cannot be captured by the state-space characterisation of D-NI systems [16, Th. 3.2] and [17, Lemma 11], since a D-SNI system loses its strictness when transformed into the D-PR (or D-passive) domain. However, D-OSNI systems overcomes this difficulty because they are defined analogously as the discrete-time output passive (D-OP) systems [20] and hence, retains the strictness after getting transformed into the D-PR/D-passive domain. Finally, we propose a set of necessary and sufficient conditions to guarantee the internal stability of a positive feedback loop containing a D-ONI system without any poles at $z = -1, +1$ and a D-OSNI system.

Notation: $\mathbb{R}$ and $\mathbb{C}$ denote respectively the sets of all real and all complex numbers. $\Re(\cdot)$ and $\Im(\cdot)$ express the real and the imaginary parts respectively. $A^\top$, A^* and $\bar{A}$ denote respectively the transpose, complex conjugate transpose and complex conjugate of a matrix A. A^{-*} and $A^{-\top}$ represent shorthand for $(A^{-1})^*$ and $(A^{-1})^\top$ respectively. $\lambda_{\max}(A)$ denotes the maximum eigenvalue of a matrix A that has only real eigenvalues. Let $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$ represent a state-space realisation of a discrete-time, real, rational and proper transfer function matrix $M(z) = D_d + C_d(zI - A_d)^{-1}B_d$.

In this letter, we will use a specific Bilinear Transformation

$$s = \frac{z-1}{z+1} \tag{1}$$

for transforming the s-domain transfer functions into the z-domain following the same philosophy adopted in [18] and [21]. Note here that the relationship $s = \frac{z-1}{z+1}$ is obtained from the standard Bilinear Transformation, when restricted to have a specific sample time $T_s = 2$s.

II. Technical Preliminaries

In this section, we present essential technical preliminaries, definitions and lemmas which underpin the proofs of the main results of this letter. We first set the notations and definitions for discrete-time NI and SNI systems in the z-domain with respect to the unit disc.

Definition 1 (D-NI Systems [16], [17], [19], [22]): Let $M(z)$ be the discrete-time, real, rational and proper transfer function matrix of a finite-dimensional, causal and square system M. Then, $M(z)$ is said to be a Discrete-time Negative Imaginary (D-NI) system if

- $M(z)$ has no poles in $\{z \in \mathbb{C} : |z| > 1\}$;
- $j[M(e^{j\theta}) - M(e^{j\theta})^*] \geq 0$ for all $\theta \in (0, \pi)$ except the values of θ for which $z = e^{j\theta}$ is a pole of $M(z)$;
- if $z_0 = e^{j\theta_0}$ with $\theta_0 \in (0, \pi)$ is a pole of $M(z)$, then it is a simple pole and the normalised residue matrix $K_0 = \frac{1}{z_0}\lim_{z\to z_0}(z - z_0)jM(z)$ is Hermitian and positive semidefinite;
- if $z_0 = 1$ is a pole of $M(z)$, then $\lim_{z\to 1}(z-1)^k M(z) = 0$ for all integer $k \geq 3$ and $\lim_{z\to 1}(z-1)^2 M(z)$ is Hermitian and positive semidefinite.

Definition 2 (D-SNI Systems [16], [22]): Let $M(z)$ be the discrete-time, real, rational and proper transfer function matrix of a finite-dimensional, causal and square system M. Then, $M(z)$ is said to be a Discrete-time Strictly Negative Imaginary (D-SNI) system if $M(z)$ has no poles in $\{z \in \mathbb{C} : |z| \geq 1\}$ and $j[M(e^{j\theta}) - M(e^{j\theta})^*] > 0$ for all $\theta \in (0, \pi)$.

The following lemma gives a state-space characterisation for D-NI systems having no poles at $z = -1$ and $z = +1$. This is considered as the discrete-time counterpart of the continuous-time NI lemma [1], [23].

Lemma 1 (D-NI Lemma [16], [22]): Let M be a finite-dimensional, causal and square system with the discrete-time, real, rational and proper transfer function matrix $M(z)$ and a minimal state-space realisation $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$. Suppose $\det[I + A_d] \neq 0$ and $\det[I - A_d] \neq 0$. Then, $M(z)$ is D-NI if and only if there exists $P = P^\top > 0$ such that

$$A_d^\top P A_d - P \leq 0 \quad \text{and} \quad C_d + B_d^\top (A_d^\top - I)^{-1} P (A_d + I) = 0. \tag{2}$$

We will now derive a version of the discrete-time output (strictly) passive[2] lemma taking the inspiration from [20, Lemma 3] and the discrete-time Positive Real lemma [18], [21]. This lemma will be exploited in Section III for proving the main results of this letter.

Lemma 2 (D-OP Lemma): Let $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$ be a minimal state-space realisation of a finite-dimensional, causal and square system M having a discrete-time, real, rational transfer function matrix $M(z)$. Suppose $M(z)$ has no poles in $\{z \in \mathbb{C} : |z| > 1 \text{ and } z = -1\}$. Then, M is Discrete-time Output Passive (D-OP) if and only if there exist a real scalar $\delta_p \geq 0$ and a real matrix $P = P^\top > 0$ such that

$$\begin{bmatrix} \left(P - A_d^\top P A_d - \delta_p C_d^\top C_d\right) & \left(C_d^\top - A_d^\top P B_d - \delta_p C_d^\top D_d\right) \\ \left(C_d^\top - A_d^\top P B_d - \delta_p C_d^\top D_d\right)^\top & \left(D_d + D_d^\top - A_d^\top P B_d - \delta_p D_d^\top D_d\right) \end{bmatrix} \geq 0. \tag{3}$$

Moreover, M is a Discrete-time Output Strictly Passive (D-OSP) system if and only if $M(z)$ has no poles in $\{z \in \mathbb{C} : |z| \geq 1\}$ and (3) holds for $\delta_p > 0$.

Proof (Sufficiency): Let M be an discrete-time, LTI output passive system, as described in the statement of Lemma 2, with $\delta_p \geq 0$. Then, there exists a C^1 storage function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that $2y_k^\top u_k - \delta_p y_k^\top y_k \geq \dot{V}(x_k)$ for all $k \in \{0, 1, 2, \ldots\}$. This inequality is equivalent to (3) when $V(x_k) = x_k^\top P x_k\ \forall x_k \in \mathbb{R}^n$ with $P = P^\top > 0$.

(Necessity): Given a feasible solution $P = P^\top > 0$, $\delta_p \geq 0$ of (3) for a discrete-time LTI system M with a

[2]Time-domain and frequency-domain definitions of discrete-time output (strictly) passive systems are given in [20, Definition 3 and Lemma 1].

minimal state-space realisation, it can be established following [20, Lemma 1] and [24, Th. 1] that there always exists a C^1 storage function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that M satisfies $2y_k^\top u_k - \delta_p y_k^\top y_k \geq \dot{V}(x_k)$ with the same δ_p.

This completes the proof. ■

To this end, we present two algebraic lemmas that will be utilised in Section III for proving the D-ONI lemma.

Lemma 3: Let $A \in \mathbb{R}^{n\times n}$. Suppose $\det[A - I] \neq 0$ and $\det[A + I] \neq 0$. Then, $(A - I)^{-1}(A + I) = (A + I)(A - I)^{-1}$.

Proof: The proof is carried out as $(A+I)(A-I)^{-1} = A(A-I)^{-1} + (A-I)^{-1} = A[(I-A^{-1})A]^{-1} + (A-I) = (I-A^{-1})^{-1} + (A-I) = [A^{-1}(A-I)]^{-1} + (A-I)^{-1} = (A-I)^{-1}(A+I)$. ■

Lemma 4: Let $A \in \mathbb{R}^{n\times n}$. Suppose $\det[A - I] \neq 0$. Then, $A(A - I)^{-1} = I + (A - I)^{-1}$.

Proof: We have $I+(A-I)^{-1} = (A-I)(A-I)^{-1}+(A-I)^{-1} = [(A - I) + I](A - I)^{-1} = A(A - I)^{-1}$. ■

III. D-ONI Systems: Definition and Properties

This section caters the main contributions of this letter. We begin with the definitions, properties and examples of D-ONI and D-OSNI systems. After that, we will establish the connections between D-ONI and D-OP systems in Section III-A. Subsequently, Section III-B derives an LMI-based state-space characterisation for the entire class of D-ONI systems.

Definition 3 (D-ONI Systems): Let $M(z)$ be the discrete-time, real, rational and proper transfer function matrix of a finite-dimensional, causal and square system M. Define $F(z) = \left(\frac{z-1}{z+1}\right)[M(z)-M(-1)]$. Then, $M(z)$ is said to be a Discrete-time Output Negative Imaginary (D-ONI) system if $M(z)$ has no poles in $\{z \in \mathbb{C} : |z| > 1 \text{ and } z = -1\}$ and there exists a real scalar $\delta \geq 0$ such that

$$F(z) + F(z)^* - \delta F(z)^* F(z) \geq 0 \tag{4}$$

for all z such that $|z| > 1$ and $\Im[z] > 0$.

$M(z)$ is called a Discrete-time Output Strictly Negative Imaginary (D-OSNI) system if $M(z)$ has no poles in $\{z \in \mathbb{C} : |z| \geq 1\}$ and (4) holds for $\delta > 0$.

It is worth noting that for D-ONI systems without any poles at $z = -1$, $M(-1)$ exists and $M(-1) = M(-1)^\top$ (see [16, Remark 3.3]). Definition 3 can also be expressed with respect to the unit circle in the z-domain (i.e., $|z| = 1$).

Definition 4: Let $M(z)$ be the discrete-time, real, rational and proper transfer function matrix of a finite-dimensional, causal and square system M. Define $F(e^{j\theta}) = (\frac{e^{j\theta}-1}{e^{j\theta}+1})[M(e^{j\theta}) - M(e^{j\pi})]$. Then, $M(z)$ is D-ONI if

- $M(z)$ has no poles in $\{z \in \mathbb{C} : |z| > 1 \text{ and } z = -1\}$;
- there exists a real scalar $\delta \geq 0$ such that
$$F(e^{j\theta}) + F(e^{j\theta})^* - \delta F(e^{j\theta})^* F(e^{j\theta}) \geq 0 \tag{5}$$
for all $\theta \in [0, \pi]$ except the values of θ for which $z = e^{j\theta}$ is a pole of $M(z)$;
- if $z_0 = e^{j\theta_0}$ with $\theta_0 \in (0, \pi)$ is a pole of $M(z)$, then it is a simple pole and the normalised residue matrix $K_0 = \frac{1}{z_0}\lim_{z\to z_0}(z - z_0)jM(z)$ is Hermitian and positive semidefinite;
- if $z_0 = +1$ is a pole of $M(z)$, then $\lim_{z\to 1}(z-1)^k M(z) = 0$ for all integer $k \geq 3$ and $\lim_{z\to 1}(z-1)^2 M(z)$ is Hermitian and positive semidefinite.

$M(z)$ is D-OSNI if there exists no $\theta \in [0, \pi]$ such that $z = e^{j\theta}$ is a pole of $M(z)$ and (5) holds for $\delta > 0$.

Note that inequality (5) can equivalently be expressed in terms of $M(e^{j\theta})$, as mentioned below:

$$\frac{\sin\theta}{1+\cos\theta} j\big[M(e^{j\theta}) - M(e^{j\theta})^*\big] - \delta\left(\frac{\sin\theta}{1+\cos\theta}\right)^2 \bar{M}(e^{j\theta})^* \bar{M}(e^{j\theta}) \geq 0, \tag{6}$$

utilising the relationship $\frac{z-1}{z+1} = \frac{e^{j\theta}-1}{e^{j\theta}+1} = j\frac{\sin\theta}{1+\cos\theta}$ and on noting that $\bar{M}(e^{j\theta}) = M(e^{j\theta}) - M(e^{j\pi})$.

A. Relationship Between D-ONI and D-OP Systems

This subsection establishes the relationship between D-ONI and D-OP systems when they do not have any poles at $z = -1$ and $z = +1$. The proof of Lemma 5 relies on the definition and state-space characterisation of C-ONI systems [2], [12], [13] and exploits the Bilinear Transformation (1) to switch back and forth between the discrete-time and continuous-time settings.

Lemma 5: Let $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$ be a minimal state-space realisation of a finite-dimensional, causal and square system M having a discrete-time, real, rational and proper transfer function matrix $M(z)$. Let $M(z)$ have no poles in $\{z \in \mathbb{C} : |z| > 1,\ z = +1 \text{ and } z = -1\}$. Then, M is a D-ONI system if and only if $M(-1) = M(-1)^\top$ and $F(z) = \left(\frac{z-1}{z+1}\right)[M(z) - M(-1)]$ is D-OP.

Proof: We begin the proof on noting that for a D-ONI transfer function $M(z)$ without any poles at $z = -1$, $M(-1)$ exists and $M(-1) = M(-1)^\top$ according to [16, Lemma 3.8]. Now, we have the following set of equivalent statements:

$M(z)$ is D-ONI without any poles at $z = +1$

$\Leftrightarrow$ $\bar{M}(z) = M(z) - M(-1) = \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & 0 \end{array}\right]$ is D-ONI without any poles at $z = +1$

$\Leftrightarrow$ $\bar{M}_c(s) = M_c(s) - M_c(\infty) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$ is C-ONI without any poles at $s = 0$ and $D_c = M_c(\infty) = D_c^\top$ holds automatically; $M_c(s) = M(z)|_{z=\frac{1+s}{1-s}}$ and the triplet (A_c, B_c, C_c) is minimal via [18, Lemma 3]
[The equivalance between the D-ONI and C-ONI versions holds due to the Bilinear Transformation, along the lines of the proof of [16, Lemma 3.2] relying on [10, Lemma 3.1]]

$\Leftrightarrow$ $F_c(s) = s\bar{M}_c(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_cA_c & C_cB_c \end{array}\right]$ is continuous-time output passive without any poles at $s = 0$ and $D_c = D_c^\top$ [following [13, Theorem 2] and [12, Lemma 1]]

$\Leftrightarrow$ $F(z) = \left(\frac{z-1}{z+1}\right)\bar{M}(z) = \left(\frac{z-1}{z+1}\right)[M(z) - M(-1)]$ is D-OP without any poles at $z = +1$ and $M(-1) = M(-1)^\top$, where $M(z) = M_c(s)|_{s=\frac{z-1}{z+1}}$.

This completed the proof. ■

B. State-Space Characterisation of D-ONI Systems

The lemma given next, termed as the D-ONI lemma, provides a couple of LMI conditions that can be conveniently used to test the strict/non-strict D-ONI properties of a discrete-time LTI system based on its minimal state-space realisation.

Lemma 6 (D-ONI Lemma): Let $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$ be a minimal state-space realisation of a finite-dimensional, causal and square system M having a discrete-time, real, rational and proper transfer function matrix $M(z)$. Let $\det[A_d + I] \neq 0$ and $\det[A_d - I] \neq 0$. Define $\Sigma = (A_d - I)(A_d + I)^{-1}$. Then, M is a D-ONI system without poles at $z = +1$ if and only if there exist $\delta \geq 0$ and $P = P^\top > 0$ such that

$$P - A_d^\top P A_d - \delta\,(C_d\Sigma)^\top C_d\Sigma \geq 0 \quad \text{and} \tag{7a}$$

$$C_d + B_d^\top (A_d - I)^{-\top} P (A_d + I) = 0. \tag{7b}$$

Proof: We begin the proof on noting that for a D-ONI transfer function $M(z)$ without any poles at $z = -1$, $M(-1)$ exists and $M(-1) = M(-1)^\top$ according to [16, Lemma 3.8]. Furthermore, $F(z) = \left(\frac{z-1}{z+1}\right)[M(z) - M(-1)]$ has a minimal state-space realisation $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d(A_d - I)(A_d+I)^{-1} & C_d(A_d+I)^{-1}B_d \end{array}\right]$ when (A_d, B_d, C_d, D_d) is minimal and $M(z)$ has no poles at $z = +1$. We now have the following set of equivalent statements.

$M(z)$ is D-ONI without any poles at $z = +1$

$\Leftrightarrow$ $\bar{M}(z) = M(z) - M(-1)$ is D-ONI without any poles at $z = +1$

$\Leftrightarrow$ $F(z) = \left(\frac{z-1}{z+1}\right)\bar{M}(z) = \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d\Sigma & C_d(A_d+I)^{-1}B_d \end{array}\right]$ is D-OP without any poles at $z = +1$ [via Lemma 5]

$\Leftrightarrow$ there exist $\delta \geq 0$ and the matrix $P = P^\top > 0$ such that

$$\begin{bmatrix} \left(P - A_d^\top P A_d - \delta\,(C_d\Sigma)^\top C_d\Sigma\right) & \left(\Sigma^\top C_d^\top - A_d^\top P B_d - \delta\,\Sigma^\top C_d^\top C_d \times (A_d+I)^{-1}B_d\right) \\ \left(\Sigma^\top C_d^\top - A_d^\top P B_d - \delta\,\Sigma^\top C_d^\top C_d \times (A_d+I)^{-1}B_d\right)^\top & \left(C_d(A_d+I)^{-1}B_d + B_d^\top (A_d+I)^{-\top} C_d^\top - \delta\, B_d^\top (A_d+I)^{-\top} \times C_d^\top C_d (A_d+I)^{-1}B_d\right) \end{bmatrix} \geq 0$$

[via applying Lemma 2]

$\Leftrightarrow$ there exist $\delta \geq 0$ and the matrices $L \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{m\times m}$ and $P = P^\top > 0$ such that
$P - A_d^\top P A_d - \delta\,(C_d\Sigma)^\top C_d\Sigma = L^\top L$,
$\Sigma^\top C_d^\top - A_d^\top P B_d - \delta\,\Sigma^\top C_d^\top C_d (A_d+I)^{-1}B_d = L^\top W$,
$C_d(A_d+I)^{-1}B_d + B_d^\top (A_d+I)^{-\top}C_d^\top - B_d^\top P B_d - \delta\, B_d^\top (A_d+I)^{-\top} C_d^\top C_d (A_d+I)^{-1}B_d = W^\top W$

$\Leftrightarrow$ there exist $\delta \geq 0$ and the matrices $L \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{m\times m}$ and $P = P^\top > 0$ such that
$P - A_d^\top P A_d - \delta\,(C_d\Sigma)^\top C_d\Sigma = L^\top L$,
$C_d = \left[B_d^\top P A_d + W^\top L\right](A_d - I)^{-1}(A_d + I) + \delta\, B_d^\top \times (A_d+I)^{-\top} C_d^\top C_d$ [using Lemma 3 and Lemma 4]
and $C_d(A_d+I)^{-1}B_d + B_d^\top (A_d+I)^{-\top}C_d^\top - B_d^\top P B_d - \delta\, B_d^\top (A_d+I)^{-\top} C_d^\top C_d (A_d+I)^{-1}B_d = W^\top W$

$\Leftrightarrow$ there exist $\delta \geq 0$ and the matrices $L \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{m\times m}$ and $P = P^\top > 0$ such that
$P - A_d^\top P A_d - \delta\,(C_d\Sigma)^\top C_d\Sigma = L^\top L$,
$C_d = \left[B_d^\top P A_d + W^\top L\right](A_d - I)^{-1}(A_d + I) + \delta\, B_d^\top (A_d+I)^{-\top} C_d^\top C_d$ and
$\left[B_d^\top P B_d + B_d^\top P (A_d - I)^{-1}B_d + B_d^\top (A_d - I)^{-\top} P B_d - \delta\, B_d^\top (A_d+I)^{-\top} C_d^\top C_d (A_d+I)^{-1}B_d + B_d^\top (A_d - I)^{-\top} L^\top L (A_d - I)^{-1} B_d\right]$
$= \left[W - L(A_d - I)^{-1}B_d\right]^\top \left[W - L(A_d - I)^{-1}B_d\right]$

$\Leftrightarrow$ there exist $\delta \geq 0$ and the matrices $L \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{m\times m}$ and $P = P^\top > 0$ such that
$P - A_d^\top P A_d - \delta\,(C_d\Sigma)^\top C_d\Sigma = L^\top L$,
$C_d = \left[B_d^\top P A_d + W^\top L\right](A_d - I)^{-1}(A_d + I) + \delta\, B_d^\top (A_d+I)^{-\top} C_d^\top C_d$ and
$0 = \left[W - L(A_d - I)^{-1}B_d\right]^\top \left[W - L(A_d - I)^{-1}B_d\right]$

$\Leftrightarrow$ there exist $\delta \geq 0$ and the matrices $L \in \mathbb{R}^{m\times n}$ and $P = P^\top > 0$ such that
$P - A_d^\top P A_d - \delta\,(C_d\Sigma)^\top C_d\Sigma = L^\top L$ and
$C_d = \left[B_d^\top P A_d + B_d^\top (A_d - I)^{-\top} L^\top L\right](A_d - I)^{-1} \times (A_d + I) + \delta\, B_d^\top (A_d+I)^{-\top} C_d^\top C_d$

$\Leftrightarrow$ there exist $\delta \geq 0$ and $P = P^\top > 0$ such that
$P - A_d^\top P A_d - \delta\,(C_d\Sigma)^\top C_d\Sigma \geq 0$ and
$C_d + B_d^\top (A_d - I)^{-\top} P (A_d + I) = 0$.

Hence, the proof is done. ■

Note that for $\delta = 0$, the proposed D-ONI lemma implies the existing D-NI lemma derived in [16] and [17]. The following corollary is an important specialisation of Lemma 1 to D-OSNI systems (characterised by $\delta > 0$).

Corollary 1 (D-OSNI Lemma): Let $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$ be a minimal state-space realisation of a finite-dimensional, causal and square system M having a discrete-time, real, rational and proper transfer function matrix $M(z)$. Let $M(z)$ have no poles in $\{z \in \mathbb{C} : |z| \geq 1\}$. Define $\Sigma = (A_d - I)(A_d + I)^{-1}$. Then, M is D-OSNI if and only if there exist $\delta > 0$ and $Y = Y^\top > 0$ such that

$$\begin{bmatrix} Y & (C_d\Sigma Y)^\top & Y A_d^\top \\ C_d\Sigma Y & \frac{1}{\delta} I_m & 0 \\ A_d Y & 0 & Y \end{bmatrix} \geq 0 \quad \text{and} \tag{8a}$$

$$B_d + (A_d - I) Y (A_d + I)^{-\top} C_d^\top = 0. \tag{8b}$$

Proof: The proof is a straightforward specialisation of Theorem 1 to the cases where $M(z)$ has no poles in $\{z \in \mathbb{C} : |z| \geq 1\}$. The LMI condition (8a) is obtained from (7a) via applying first an appropriate congruence transformation and then taking a Schur complement, since $\delta > 0$ and on noting that $Y = P^{-1} > 0$. The equality condition (8b) is a simple rearrangement of (7b) with $Y = P^{-1}$. ■

The following lemma shows that a D-OSNI system $M(z)$ with a full-rank B_d matrix enjoys the property $M(1) - M(-1) > 0$.

Lemma 7: Let $M(z)$ be the real, rational and proper transfer function matrix of a $(m \times m)$ D-OSNI system with a minimal state-space realisation $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$ where $A_d \in \mathbb{R}^{n\times n}$, $\text{rank}[B_d] = m$ and $m \le n$. Then, $M(1) - M(-1) > 0$.

Proof: Note that $\text{rank}[B_d] = m$ implies $\text{rank}[C_d] = m$ from (8b) since $Y > 0$, $\det[A_d - I] \ne 0$ and $\det[A_d + I] \ne 0$. We now obtain $M(1) - M(-1) = C_d(I - A_d)^{-1}B_d + D_d - C_d(-I - A_d)^{-1}B_d - D_d = C_d[(I - A_d)^{-1} + (I + A_d)^{-1}]B_d = 2C_d(I+A_d)^{-1}(I-A_d)^{-1}B_d = 2C_d(I+A_d)^{-1}(I-A_d)^{-1}[-(A_d-I)Y(A_d+I)^{-\top}C_d^\top] = 2C_d(I+A_d)^{-1}Y(A_d+I)^{-\top}C_d^\top$ with the help of (8b). It readily follows that $\text{rank}[C_d(I + A_d)^{-1}] = m$. Therefore, $\text{rank}[C_d(I+A_d)^{-1}Y(A_d+I)^{-\top}C_d^\top] = \text{rank}[(C_d(I+A_d)^{-1}Y^{\frac{1}{2}})(C_d(I+A_d)^{-1}Y^{\frac{1}{2}})^\top] = \text{rank}[(C_d(I+A_d)^{-1}Y^{\frac{1}{2}})] = m$ exploiting [25, Th. 2.4.3 and Corollary 2.5.1]. Hence, $M(1) - M(-1) > 0$. ■

Remark 1: An inquisitive reader may wonder that why a D-SNI system $N(z)$ automatically satisfies the properties $\text{rank}[B_d] = \text{rank}[C_d] = m$ and $N(1)-N(-1) > 0$ [16], [17], in contrast to a D-OSNI system. The reason is that the D-OSNI systems are not defined by the strict z-domain inequality $j[N(e^{j\theta})-N(e^{j\theta})^*] > 0 \ \forall\theta \in (0,\pi)$, unlike the D-SNI systems. For a D-OSNI system, $\det[N(e^{j\theta}) - N(e^{j\theta})^*] = 0$ may occur at a finite number of distinct $\theta \in (0,\pi)$.

C. Numerical Examples

We will now study several numerical examples that will show the usefulness of the D-ONI lemma to test the strict/non-strict D-ONI properties of a given system. The examples have been tested in the MATLAB software environment using the CVX (SeDuMi and SDPT3 solvers) toolbox [26].

Example 1: Consider the discrete-time real, rational, proper transfer function $M_1(z) = \frac{0.26316(z+1)(z+0.6)}{(z+0.7315)(z+0.2158)}$ $[M_{c_1}(s) = \frac{s+4}{s^2+8s+10}]$ having a minimal state-space realisation $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right] = \left[\begin{array}{cc|c} -0.90 & -0.26 & 0.21 \\ 0.42 & -0.053 & 0.84 \\ \hline 0.13 & 0.17 & 0.26 \end{array}\right]$. It satisfies the D-ONI lemma, that is the LMI conditions (7a) and (7b), with $P = \begin{bmatrix} 1.55 & 0.31 \\ 0.31 & 0.31 \end{bmatrix} > 0$ and $\delta = 0.7882 > 0$. Hence, $M_1(z)$ is a D-OSNI system. Note that $M_1(z)$ satisfies also the D-SNI property, which can be readily verified via Definition 2.

Example 2: Consider another discrete-time transfer function $M_2(z) = \frac{0.066667(z+1)^2(z^2+1.556z+1)}{(z^2+1.741z+0.9259)(z^2+1.296z+0.9361)}$ $[M_{c_2}(s) = \frac{s^2+8}{s^4+s^3+25s^2+8s+100}]$ with a minimal state-space realisation. Applying LMI conditions (7a) and (7b) to this system, we obtain a feasible solution

$$P = \begin{bmatrix} 2.00 & 0.00 & 2.00 & 0.00 \\ 0.00 & 2.13 & 0.00 & 1.56 \\ 2.00 & 0.00 & 3.13 & 0.00 \\ 0.00 & 1.56 & 0.00 & 1.56 \end{bmatrix} > 0 \text{ and } \delta = 1.1192 > 0.$$

This implies that $M_2(z)$ is D-OSNI. Note however that

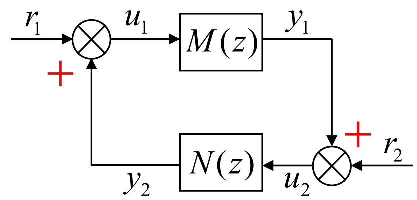

Fig. 2. A positive feedback interconnection of two D-ONI systems.

M_2 does not satisfy the strict frequency-domain condition $j[M_2(e^{j\theta}) - M_2(e^{j\theta})^*] > 0 \ \forall\theta \in (0,\pi)$. Hence, $M_2(z)$ is not D-SNI.

Example 3: Let $M_3(z) = \frac{12.195(z+1)(z+0.6)}{(z^2+1.512z+0.6098)}$ $[M_{c_3}(s) = \frac{100(s+4)}{s^2+8s+32}]$ represent a discrete-time LTI system with a minimal state-space realisation $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$. Similar to the previous examples, in order to test the strict/non-strict D-ONI property of M_3, we apply again the D-ONI lemma and obtain a feasible solution $P = \begin{bmatrix} 2.34 & 0.78 \\ 0.78 & 0.39 \end{bmatrix} > 0$ and $\delta = 0$, which indicates that $M_3(z)$ does not satisfy the D-OSNI property. However, it can be verified that $M_3(z)$ satisfies the D-SNI property since $j[M_3(e^{j\theta}) - M_3(e^{j\theta})^*] > 0 \ \forall\theta \in (0,\pi)$.

Following the same procedure adopted in Examples 1–3, we can easily verify that $M_4(z) = \frac{0.083333(z+1)^2(z^2-0.5z+0.5)}{z(z-0.3333)(z^2+z+0.5)}$ $[M_{c_4}(s) = \frac{2s^2+s+1}{(s+1)(2s+1)(s^2+2s+5)}]$ is a non-strict (i.e., $\delta = 0$) stable D-ONI system. Finally, $M_5(z) = \frac{0.8(z+1)^2}{(z^2+1.2z+1)}$ $[M_{c_5}(s) = \frac{4}{s^2+4}]$, $M_6(z) = \frac{z+1}{z-1}$ $[M_{c_6}(s) = \frac{1}{s}]$ and $M_7(z) = \frac{z^2+2z+1}{z^2-2z+1}$ $[M_{c_7}(s) = \frac{1}{s^2}]$ are examples of D-ONI systems having pole(s) on the unit circle ($|z| = 1$).

IV. Internal Stability of D-ONI Systems

This section deals with internal stability of a positive feedback interconnection of D-ONI (without any poles at $z = +1$) and D-OSNI systems, as shown in Fig. 2. The proof of this result builds on the following technical lemma, which gives the closed-loop stability conditions in the continuous-time setting, that is, for interconnected C-ONI and C-OSNI systems.

Lemma 8 [14]: Let $M(s)$ be a (not necessarily stable) C-NI system without poles at the origin and $N(s)$ be a C-OSNI system. Let $\Omega = \{\omega \in (0,\infty) : s = j\omega \text{ is not a pole of } M(s)\}$ and let $j[N(j\omega_0) - N(j\omega_0)^*] > 0 \ \forall\omega_0 \in (0,\infty)\backslash\Omega$. Suppose there exists no $\omega \in \Omega$ such that $\det[M(j\omega) - M(j\omega)^*] = 0$ and $\det[N(j\omega) - N(j\omega)^*] = 0$. Then, the positive feedback interconnection of $M(s)$ and $N(s)$ is internally stable if and only if

$$\begin{cases} \det[I - M(\infty)N(\infty)] \ne 0, \\ \lambda_{\max}\left[(I - M(\infty)N(\infty))^{-1}(M(\infty)N(0) - I)\right] < 0, \\ \lambda_{\max}\left[(I - N(0)M(\infty))^{-1}(N(0)M(0) - I)\right] < 0. \end{cases} \quad (9)$$

We will now present the internal stability theorem for discrete-time ONI systems.

Theorem 1 (D-ONI Stability Theorem): Let $M(z)$ be a D-ONI system without any poles at $z = +1$ and $N(z)$ be a D-OSNI system. Let $\Theta = \{\theta \in (0,\pi) : z = e^{j\theta} \text{ is not a pole of } M(z)\}$ and let $j[N(e^{j\theta_0}) - N(e^{j\theta_0})^*] > 0$ $\forall\theta_0 \in (0,\pi)\backslash\Theta$. Suppose there exists no $\theta \in \Theta$ such that

$\det[M(e^{j\theta_0}) - M(e^{j\theta_0})^*] = 0$ and $\det[N(e^{j\theta_0}) - N(e^{j\theta_0})^*] = 0$. Then, the positive feedback interconnection of $M(z)$ and $N(z)$, as shown in Fig. 2, is internally stable if and only if

$$\begin{cases} \det[I - M(-1)N(-1)] \neq 0, \\ \lambda_{\max}\big[[I - M(-1)N(-1)]^{-1}\{M(-1)N(1) - I\}\big] < 0, \\ \lambda_{\max}\big[[I - N(1)M(-1)]^{-1}\{N(1)M(1) - I\}\big] < 0. \end{cases} \quad (10)$$

Proof: Let $M_c(s) = M(z)|_{z=\frac{1+s}{1-s}}$ and $N_c(s) = N(z)|_{z=\frac{1+s}{1-s}}$ be respectively the C-ONI and C-OSNI versions of $M(z)$ and $N(z)$ in the spirit of Lemma 5. Now, the positive feedback interconnection of $M_c(s)$ and $N_c(s)$ is guaranteed to be internally stable, via Lemma 8, if and only if (9) holds. Note that the set of conditions in (10) is the discrete-time equivalent of (9). This completes the proof. ■

The following corollary is an immediate consequence of Theorem 1 under the additional assumptions $N(-1) \geq 0$ and $M(-1)N(-1) = 0$. It offers an appealing and more elegant 'DC loop gain' condition for checking the internal stability of a D-ONI interconnection.

Corollary 2: Suppose either $N(-1) \geq 0$ and $M(-1)N(-1) = 0$, or else $M(-1) = 0$, in addition to the suppositions of Theorem 1. Then, the positive feedback interconnection of $M(z)$ and $N(z)$ is internally stable if and only if $\lambda_{\max}[N(1)M(1)] < 1$.

Proof: The proof readily follows from Theorem 1 subject to the additional constraints $N(-1) \geq 0$ and $M(-1)N(-1) = 0$, or $M(-1) = 0$. ■

Remark 2: Note that the internal stability result of a positive feedback interconnection of D-NI and D-SNI systems, without considering poles at $z = +1$, is already available in the literature [22, Th. 3] and [17, Th. 1]. However, this result cannot capture the stability of D-NI and D-OSNI systems since the D-SNI and D-OSNI subsets are not identical. At the same time, the proposed Theorem 1 also cannot capture the stability of D-NI and D-SNI (but not D-OSNI) systems.

V. Conclusion

This letter defines and characterises the class of Discrete-time Output Negative Imaginary (D-ONI) systems, which captures the existing class of real, rational, proper D-NI systems [16], [17], [22]. An asymptotically stable, strict subset, designated as the D-OSNI systems, is also defined within the D-ONI class. Interestingly, the D-OSNI subset is not identical to the existing D-SNI subset. Instead, they form an intersection, as depicted in the Venn diagram in Fig. 1. In the sequel, an LMI-based state-space characterisation is developed for the D-ONI systems without poles at $z = +1$. Finally, an internal stability result is proposed for a D-ONI system when interconnected with a D-OSNI system via positive feedback.

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References

[1] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 1042–1046, May 2008.

[2] P. Bhowmick and S. Patra, "On LTI output strictly negative-imaginary systems," *Syst. Control Lett.*, vol. 100, pp. 32–42, Feb. 2017.

[3] P. Bhowmick and S. Patra, "An observer-based control scheme using negative-imaginary theory," *Automatica*, vol. 81, pp. 196–202, Jul. 2017.

[4] P. Bhowmick and S. Patra, "On decentralized integral controllability of stable negative-imaginary systems and some related extensions," *Automatica*, vol. 94, pp. 443–451, Aug. 2018.

[5] P. Bhowmick and S. Patra, "Solution to negative-imaginary control problem for uncertain LTI systems with multi-objective performance," *Automatica*, vol. 112, Feb. 2020, Art. no. 108735.

[6] P. Bhowmick and A. Lanzon, "Applying negative imaginary systems theory to non-square systems with polytopic uncertainty," *Automatica*, vol. 128, Jun. 2021, Art. no. 109570.

[7] B. Bhikkaji, S. O. R. Moheimani, and I. R. Petersen, "A negative imaginary approach to modeling and control of a collocated structure," *IEEE/ASME Trans. Mechatronics*, vol. 17, no. 4, pp. 717–727, Aug. 2012.

[8] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion," *IEEE Trans. Autom. Control*, vol. 59, no. 10, pp. 2692–2707, Oct. 2014.

[9] S. K. Das, H. R. Pota, and I. R. Petersen, "Resonant controller design for a piezoelectric tube scanner: A 'mixed' negative-imaginary and small-gain approach," *IEEE Trans. Control Syst. Technol.*, vol. 22, no. 5, pp. 1899–1906, Sep. 2014.

[10] A. Ferrante and L. Ntogramatzidis, "Some new results in the theory of negative imaginary systems with symmetric transfer matrix function," *Automatica*, vol. 49, no. 7, pp. 2138–2144, Jul. 2013.

[11] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Foundations of not necessarily rational negative imaginary systems theory: Relations between classes of negative imaginary and positive real systems," *IEEE Trans. Autom. Control*, vol. 61, no. 10, pp. 3052–3057, Oct. 2016.

[12] P. Bhowmick and A. Lanzon, "Output strictly negative imaginary systems and its connections to dissipativity theory," in *Proc. 58th IEEE Conf. Decis. Control*, 2019, pp. 6754–6759.

[13] P. Bhowmick and A. Lanzon, "Time-domain output negative imaginary systems and its connection to dynamic dissipativity," in *Proc. 59th IEEE Conf. Decis. Control*, 2020, pp. 5167–5172.

[14] A. Lanzon and P. Bhowmick, "Characterisation of input-output negative imaginary systems in a dissipative framework," *IEEE Trans. Autom. Control*, to be published.

[15] S. Kurawa, P. Bhowmick, and A. Lanzon, "Negative imaginary theory for a class of linear time-varying systems," *IEEE Control Syst. Lett.*, vol. 5, no. 3, pp. 1001–1006, Jul. 2021.

[16] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Discrete-time negative imaginary systems," *Automatica*, vol. 79, pp. 1–10, May 2017.

[17] M. Liu and J. Xiong, "Properties and stability analysis of discrete-time negative imaginary systems," *Automatica*, vol. 83, pp. 58–64, Sep. 2017.

[18] L. Hitz and B. Anderson, "Discrete positive-real functions and their application to system stability," *Proc. Inst. Electr. Eng.*, vol. 116, no. 1, pp. 153–155, Jan. 1969.

[19] M. Liu and J. Xiong, "Bilinear transformation for discrete-time positive real and negative imaginary systems," *IEEE Trans. Autom. Control*, vol. 63, no. 12, pp. 4264–4269, Dec. 2018.

[20] N. Kottenstette, M. J. McCourt, M. Xia, V. Gupta, and P. J. Antsaklis, "On relationships among passivity, positive realness, and dissipativity in linear systems," *Automatica*, vol. 50, no. 4, pp. 1003–1016, Apr. 2014.

[21] G. Tao and P. Ioannou, "Necessary and sufficient conditions for strictly positive real matrices," *IEE Proc. G Circuits Devices Syst.*, vol. 137, pp. 360–366, Oct. 1990.

[22] H.-J. Chen and A. Lanzon, "Closed-loop stability analysis of discrete-time negative imaginary systems," *Syst. Control Lett.*, vol. 114, pp. 52–58, Apr. 2018.

[23] A. Lanzon and H.-J. Chen, "Feedback stability of negative imaginary systems," *IEEE Trans. Autom. Control*, vol. 62, no. 11, pp. 5620–5633, Nov. 2017.

[24] D. Hill and P. Moylan, "The stability of nonlinear dissipative systems," *IEEE Trans. Autom. Control*, vol. 21, no. 5, pp. 708–711, Oct. 1976.

[25] D. S. Bernstein, *Matrix Mathematics: Theory, Facts and Formulas*, 2nd ed. Princeton, NJ, USA: Princeton Univ. Press, 2009.

[26] M. Grant and S. Boyd. (Mar. 2014). *CVX: MATLAB Software for Disciplined Convex Programming, Version 2.1*. [Online]. Available: http://cvxr.com/cvx