Iterative Greedy LMI for Sparse Control

Masaaki Nagahara, *Senior Member, IEEE*, Masaki Ogura, *Member, IEEE*, and Yutaka Yamamoto, *Life Fellow, IEEE*

***Abstract*—In this letter, we propose a novel method to find matrices that satisfy sparsity and LMI (linear matrix inequality) constraints at the same time. This problem appears in sparse control design such as sparse representation of the state feedback gain, sparse graph representation with fastest mixing, and sparse FIR (finite impulse response) filter design, to name a few. We propose an efficient algorithm for the solution based on Dykstra's projection algorithm. We then prove a convergence theorem of the proposed algorithm, and show some control examples to illustrate merits and demerits of the proposed method.**

***Index Terms*—Greedy algorithms, linear matrix inequalities, sparse control, network theory.**

I. Introduction

LMIS (linear matrix inequalities) are a fundamental and powerful tool for representing many types of constraints in control systems, such as stability conditions, H^2 and H^∞ norm constraints, and dissipativity [1], [2]. Since the set described by LMIs is convex, many related optimal control problems, such as H^2 and H^∞ control problems, are described as convex optimization problems (or sequences of convex optimization problems), which can be effectively solved by using modern optimization solvers such as Sedumi [3] and SDPT3 [4]. Coding LMIs is also an easy task if we use modeling languages such as Yalmip [5] or CVX [6] on MATLAB, or CVXPY [7] on Python.

More recently, resource-aware and energy-saving design of control systems is becoming more and more important for a sustainable world, which we call *green control* [8]. To realize green control, event-triggered control (or self-triggered control) [9], [10] is an effective control scheme. This is aperiodic control where sensing and actuation are performed when and only when they are needed, and hence, can reduce the number of updates of control values [11]. On the other hand, *sparse control* is another practical scheme for green control. Maximum hands-off control [12] is sparse control that minimizes the time duration on which the control values are nonzero by minimizing the L^1 norm of the control, relaxation of L^0 norm. This idea has been extended to distributed control [13], time-optimal control [14], and discrete-valued control [15]. Also, sparse control has been proposed for controller complexity reduction [16], [17], [18], [19] by promoting sparsity of the feedback gain minimizing the ℓ^1 norm or the sum-of-logs instead of its ℓ^0 norm.

The convex relaxation of the ℓ^0 norm by the ℓ^1 norm is widely used and proved to be effective in signal and image processing in particular, which is also known as *compressed sensing* [20], [21]. Although ℓ^1 convex optimization problems can be efficiently solved, it is very difficult to check their optimal solution is equivalent, or even close to ℓ^0-optimal solutions in general. In fact, checking the equivalence by using the well-known restricted isometry property (RIP) is as hard as to solve the ℓ^0 optimization itself [22]. Therefore, an alternative method called the *greedy method* has also been proposed to solve ℓ^0 problems [23], [24], [25]. Although the greedy method does not guarantee to return the global optimizer in general, it is shown to be effective in the convergence speed that is much faster than algorithms for ℓ^1-based optimization [25].

In this letter, we propose a novel greedy algorithm for sparse control with LMI constraints, which appears, for example, in sparse feedback gain design mentioned above. Namely, we consider a feasibility problem to find a matrix (or a vector) that is k-sparse (i.e., its ℓ^0 norm is less than k) and satisfies given LMIs at the same time. To solve this problem, we adapt Dykstra's projection algorithm [26], [27] that computes the projection onto the intersection of two (or more) convex sets. Although the set described by LMIs is convex, the set of k-sparse matrices (or vectors) is not convex, and Dykstra's algorithm cannot be applied to this case. However, we propose to naively use Dykstra's algorithm to obtain a matrix that is k-sparse and satisfies given LMIs. We call this the *iterative greedy LMI*.

This problem also appears in sparse control over networks. There exist several important networked dynamical processes taking place in sociotechnical networks including epidemic spreading processes over social networks [28], synchronization

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Masaaki Nagahara is with the Institute of Environmental Science and Technology, The University of Kitakyushu, Fukuoka 808-0139, Japan (e-mail: nagahara@ieee.org).

Masaki Ogura is with the Graduate School of Information Science and Technology, Osaka University, Suita 565-0871, Japan (e-mail: m-ogura@ist.osaka-u.ac.jp).

Yutaka Yamamoto is with the Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan (e-mail: yy@i.kyoto-u.ac.jp).

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dynamics in biological networks [29], and opinion formation in online social networks [30]. Conventional control strategies developed for small- or medium-scale systems are not necessarily scalable for the control of such networked dynamical systems, as the conventional strategies can result in the intervention on most of the nodes (or, edges) in the network, which can be challenging to implement when the network is large (see, e.g., [31]). For this reason, we can find in the literature several attempts to realize a sparse control via ℓ^1-relaxations for networked systems such as power systems [32], [33] and consensus dynamics [34], [35]. In contrast with these works, in this letter we propose a novel framework for the sparse control without resorting to the ℓ^1-relaxation, and illustrate the effectiveness of the framework by its application to the problem of finding fastest mixing Markov processes [36], [37].

The iterative greedy LMI is also effective in sparse FIR (finite impulse response) filter design [38], [39], which is important for implementing digital filters in small embedded systems.

This letter is organized as follows. Section II shows control problems that can be formulated in terms of sparse LMIs. Section III shows the projection operators of the set described by LMIs and the set of k-sparse matrices, and proposes a novel algorithm to find a k-sparse matrix that satisfies given LMIs based on Dykstra's projection algorithm. Section V shows numerical examples to illustrate the effectiveness and merits of the proposed method. Section VI makes a conclusion.

Notation: $\mathbb{N}$ is the set of natural numbers, namely, $\mathbb{N} = \{1, 2, 3, \ldots\}$. $\mathbb{R}^l$ and $\mathbb{R}^{m\times n}$ are respectively the set of l-dimensional real vectors, and $m \times n$ real matrices. For a matrix (or a vector) $X \in \mathbb{R}^{m\times n}$, the Frobenius norm $\|X\|_F$ is defined by $\|X\|_F \triangleq \sqrt{\mathrm{trace}(X^\top X)}$, where $\mathrm{trace}(X)$ is the trace of X. The Frobenius inner product of two matrices X and Y is defined by $\langle X, Y\rangle_F \triangleq \mathrm{trace}(X^\top Y)$. Also, for $X \in \mathbb{R}^{m\times n}$, the ℓ^0 norm of X is defined by $\|X\|_0 \triangleq |\mathrm{supp}(X)|$, where $|\mathrm{supp}(X)|$ is the number of elements of the support set $\mathrm{supp}(X) \triangleq \{(i,j) : X_{ij} \neq 0\}$. In other words, $\|X\|_0$ is the number of nonzero elements in X. By $\mathbf{1}$ we denote the all-one vector, that is, $\mathbf{1} = [1, 1, \ldots, 1]^\top$. For a symmetric matrix X, $X \succ 0$, $X \succeq 0$, $X \prec 0$, and $X \preceq 0$, respectively mean that X is positive definite, positive semi-definite, negative definite, and negative semi-definite. Also, for a matrix (or a vector) X, $X > 0$, $X \geq 0$, $X < 0$, and $X \leq 0$ are element-wise inequalities.

II. Sparse Control Problems

In this section, we introduce some sparse control problems described by LMIs with a sparsity constraint.

A. Sparse Representation of Stabilizing Feedback Gain

We first introduce the sparse controller design proposed in [17]. Let us consider a linear time-invariant system with state feedback: $\dot{x} = Ax + Bu$ with $u = Kx$. We assume (A, B) is controllable. From the Lyapunov stability theorem, the feedback system is asymptotically stable if and only if there exists $Q \succ 0$ such that $(A + BK)^\top Q + Q(A + BK) \prec 0$. Now, introducing new variables $P \triangleq Q^{-1}$ and $Y \triangleq KP$, we have the following LMIs:

$$AP + PA^\top + BY + Y^\top B^\top \prec 0, \quad P \succ 0. \tag{1}$$

The sparse feedback gain problem is to find a sparse Y (i.e., $\|Y\|_0$ is small) that satisfies (1). If Y is sparse, then choosing the output as $y = P^{-1}x$, one can implement a sparse feedback gain $u = Yy$.

B. Sparse Graph Representation With Fastest Mixing

We state the problem of designing the fastest mixing Markov process [36]. In order to state the problem, we introduce the following notations and conventions. Let $G = (V, E)$ be an undirected and weighted network with N vertices $V = \{1, \ldots, N\}$ and the edge set E. We let $w_{ij} \geq 0$ denote the weight of an undirected edge $\{i, j\}$. Let L be the weighted Laplacian matrix $L \in \mathbb{R}^{N\times N}$ of the graph G [36]. It is well-known [36] that, if G is connected, then L has a zero eigenvalue with multiplicity one, and the other eigenvalues of L are all positive. Therefore, we can order the eigenvalues of L as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. The second-largest eigenvalue λ_2 is called the algebraic connectivity of G and is known to characterize the mixing rate of the Markov process associated with G.

The problem of designing the fastest mixing Markov process concerns finding a set of weights that maximizes λ_2. Due to the homogeneity of λ_2, it is customary [36] to impose the constraint $\mathbf{1}^\top w = 1$ for the column vector w containing all the edge weights of G. It is known that the resulting design problem can be solved by the following optimization problem with LMI constraints:

$$\begin{aligned} &\text{maximize} \;\; \gamma \\ &\text{subject to} \;\; \gamma I \succeq L + \frac{1}{n}\mathbf{1}\mathbf{1}^\top, \;\; w \geq 0, \;\; \mathbf{1}^\top w = 1, \end{aligned} \tag{2}$$

where L is the weighted Laplacian matrix.

In this letter, we aim to find a k-sparse network (i.e., a network with $\|w\|_0 \leq k$) achieving the fastest mixing, based on our empirical observation that the solution of (2) is not necessarily unique. We specifically formulate the problem as follows: find a k-sparse vector w that satisfies the constraints on w in (2) with $\gamma = \gamma_{\mathrm{opt}}$, which is the optimal value of (2). We note that this problem cannot be relaxed by the ℓ^1 norm, since the inequality constraint $\mathbf{1}^\top w = 1$ implies that the ℓ^1 norm of w is fixed to one. We also remark that, although the authors in [34] present a procedure for designing sparse and fastest mixing Markov processes without a relaxation, the procedure is combinatorial and, therefore, can suffer from the curse of dimensionality.

C. Sparse FIR Filter Implementation

For a given analog filter (or controller) $K_c(s)$, we often need to implement it on a digital system. For this, we need to discretize the analog filter to obtain a digital filter. We often adopt the bilinear transform or the zero-order hold discretization [40], which sometimes cause large discretization errors if the input signal contains high frequency components. For more precise discretization, we can instead use sampled-data H^∞ discretization [39]. In this design, we discretize a given analog filter into an FIR filter. We assume that the signal subspace to which the input signals belong is $\Omega_F \triangleq \{Fw : w \in L^2[0,\infty)\}$, with an analog low-pass filter $F(s)$ that models the frequency characteristic of the analog input signals. By the fast sample-hold approximation and the KYP (Kalman-Yukubovich-Popov) lemma, the design problem is reduced to the following optimization problem [39]:

$$\begin{aligned} &\text{minimize} && \gamma + \lambda\|a\|_0 \\ &\text{subject to} && \begin{bmatrix} A^\top XA - X & A^\top XB & C(a)^\top \\ B^\top XA & B^\top XB - \gamma I & D(a)^\top \\ C(a) & D(a) & -\gamma I \end{bmatrix} \prec 0, \\ &&& X \succ 0, \end{aligned} \tag{3}$$

where a is the coefficient vector and $\lambda > 0$ is the regularization parameter. If we choose larger λ, the solution may become more sparse, but we cannot specify the sparsity before solving (3). Instead, we can consider a feasibility problem to find a k-sparse coefficient vector that satisfies the LMIs in (3) with given $\gamma > 0$ that is larger than the optimal γ.

III. Projection Operators and Dykstra's Algorithm

In this section, we propose an algorithm to solve the problems discussed in the previous section, based on Dykstra's projection algorithm.

A. Problem Formulation

Let Φ be a subset of $\mathbb{R}^{m\times n}$ that is described by some LMIs. For example, in the sparse feedback gain problem, Φ is given by

$$\Phi = \{X \in \mathbb{R}^{m\times n} : \exists P \succ 0, AP + PA^\top + BX + X^\top B \prec 0\}. \tag{4}$$

Then, the problems discussed in the previous section can be described as the following problem:

Problem 1 (Sparse LMI): Given $k \in \mathbb{N}$, find $X \in \mathbb{R}^{m\times n}$ such that $\|X\|_0 \le k$ and $X \in \Phi$.

B. Projections

Let Σ_k be the set of k-sparse matrices, that is, $\Sigma_k \triangleq \{X \in \mathbb{R}^{m\times n} : \|X\|_0 \le k\}$. Then, Problem 1 is to find a matrix X such that $X \in \Sigma_k \cap \Phi$. The idea of finding X that satisfies this is to alternatively apply projections onto Σ_k and Φ, which is known as *alternating projection* [27]. For a subset $C \in \mathbb{R}^{m\times n}$, the projection operator Π_C onto C is defined by

$$\Pi_C(X) \triangleq \operatorname*{arg\,min}_{Z\in C} \|Z - X\|_F.$$

It is well known that if C is a non-empty, closed, and convex set, then $\Pi_C(X)$ is uniquely determined, otherwise $\Pi_C(X)$ may be empty or contain multiple elements. Actually, the set Σ_k is non-convex, and $\Pi_{\Sigma_k}(X)$ may have multiple elements for some X, while it is non-empty for any $X \in \mathbb{R}^{m\times n}$.

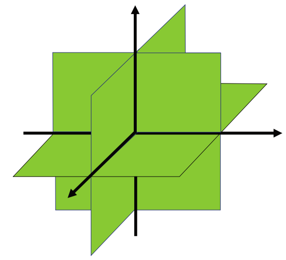


Fig. 1. Σ_2 in $\mathbb{R}^3$ is the union of 3 linear subspaces.

1) Projection Onto Φ: Since Φ is described by LMIs, this set is convex. However, if we take the strict inequality, as in (4) for example, the set Φ is not closed, and $\Pi_\Phi(X)$ may not exist. To avoid this, we adopt an approximation of the LMIs using a small number $\epsilon > 0$ as

$$P \succeq \epsilon I, \quad AP + PA^\top + BX + X^\top B \preceq -\epsilon I.$$

Then the resulting set Φ is closed and convex. We also check if Φ is non-empty or not, which depends on (A, B) and ϵ. More precisely, (A, B) should be controllable, and ϵ should be sufficiently small so that Φ is non-empty. Hereafter, we assume Φ is closed, convex, and non-empty.

Now we give an algorithm to obtain the projection $\Pi_\Phi(X)$ for a given $X \in \mathbb{R}^{m\times n}$. For this, we use the following lemma [1, Sec. 2.1].

Lemma 1: For $M \in \mathbb{R}^{m\times n}$ and $\gamma > 0$, inequality $\|M\|_F < \gamma$ holds if and only if

$$\operatorname{trace}(S) < \gamma, \quad \begin{bmatrix} S & M^\top \\ M & I \end{bmatrix} \succ 0.$$

By this lemma, we can compute the projection $\Pi_\Phi(X)$ by the following optimization problem.

$$\begin{aligned} &\text{minimize} && \operatorname{trace}(S) \\ &\text{subject to} && \begin{bmatrix} S & (Z-X)^\top \\ (Z-X) & I \end{bmatrix} \succ 0, \\ &&& Z \in \Phi. \end{aligned} \tag{5}$$

Since Φ is described by LMIs, this is a convex optimization problem with LMIs, which can be efficiently solved by numerical optimization.

2) Projection Onto Σ_k: The k-sparse subspace Σ_k in $\mathbb{R}^{m\times n}$ is obviously non-convex, but it is the union of linear subspaces.

For example, Σ_1 in $\mathbb{R}^{2\times 2}$ is the union of four linear subspaces E_1, E_2, E_3, and E_4 respectively spanned by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

See also Fig. 1 that illustrates Σ_2 in $\mathbb{R}^3$, the union of 3 linear subspaces. From this property, a projection of $X \in \mathbb{R}^{m\times n}$ onto Σ_k always exists. However, it is not unique. The projection operator $\Pi_{\Sigma_k}(X)$ characterized by

$$\operatorname*{arg\,min}_{Z} \|Z - X\|_F \text{ subject to } \|Z\|_0 \le k,$$

is given as the *k-sparse operator* $\mathcal{H}_k(X)$, which sets all but the k largest (in magnitude) elements of X to 0 [23], [41]. Fig. 2 illustrates this operation with $k = 3$.

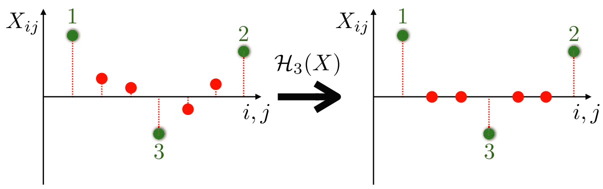


Fig. 2. Illustration of k-sparse operator $\mathcal{H}_k$ with $k = 3$: the 3 largest elements in magnitude are unchanged and the other elements are set to 0. The numbers 1, 2, 3 indicate the rank of the absolute values of the elements.

Algorithm 1 Iterative Greedy LMI

Require: sparsity parameter k, set Φ described by LMIs, and initial guess X_0
Ensure: a matrix in $\Phi \cap \Sigma_k$

$X[0] = X_0$, $P[0] = Q[0] = 0$
for $i = 0, 1, 2, \ldots, N$ **do**
  $Y[i] = \mathcal{H}_k(X[i] + P[i])$ ▷ k-sparse operator
  $P[i+1] = X[i] + P[i] - Y[i]$
  $X[i+1] = \Pi_\Phi(Y[i] + Q[i])$ ▷ LMI (5)
  $Q[i+1] = Y[i] + Q[i] - X[i+1]$
end for
return $X[N]$

C. Iterative Greedy LMI by Dykstra's Algorithm

Dykstra's projection algorithm is an efficient algorithm to find the projection onto the intersection of two or more convex sets. Let C and D be two non-empty, closed, and convex sets in a Hilbert space H. For $x_0 \in H$, Dykstra's algorithm to find the projection $\Pi_{C \cap D}(x_0)$ is given by

$$y[i] = \Pi_D(x[i] + p[i]), \;\; p[i+1] = x[i] + p[i] - y[i]$$
$$x[i+1] = \Pi_C(y[i] + q[i]), \;\; q[i+1] = y[i] + q[i] - x[i+1]$$

for $i = 0, 1, 2, \ldots$ with initial values $x[0] = x_0$ and $p[0] = q[0] = 0$.

Then we naively apply Dykstra's algorithm for $C = \Phi$ and $D = \Sigma_k$ (which is non-convex), with the projection operators Π_Φ and Π_{Σ_k}. Algorithm 1 is the proposed iteration algorithm, called the iterative greedy LMI.

Since the two projections Π_{Σ_k} and Π_Φ can be easily computed, Algorithm 1 is an efficient algorithm. However, the convergence cannot be guaranteed for every initial value X_0. In the next section, we prove the *local* convergence when the initial guess is sufficiently close to a point in $\Sigma_k \cap \Phi$.

IV. Convergence Analysis

To prove the convergence, the following property (local Kolmogorov criterion) is crucial.

Lemma 2: Let $X_0 \in \mathbb{R}^{m\times n}$ and Y be any matrix in $\Pi_{\Sigma_k}(X_0)$. Then there exists $\epsilon > 0$ and an open ball $\mathcal{B}(Y, \epsilon) \triangleq \{X \in \mathbb{R}^{m\times n} : \|X - Y\|_F \le \epsilon\}$ such that

$$\langle X_0 - Y, X - Y\rangle_F \le 0, \quad \forall X \in \mathcal{B}(Y, \epsilon) \cap \Sigma_k \tag{6}$$

holds.

Proof: If $\|X_0\|_0 \le k$ then $Y = \Pi_{\Sigma_k}(X_0) = X_0$, and (6) clearly holds. Hence, we assume $\|X_0\|_0 > k$. Then, by the definition of k-sparse operator, we have $\|Y\|_0 = k$, and hence $Y \in \partial\Sigma_k \triangleq \{X \in \mathbb{R}^{m\times n} : \|X\|_0 = k\}$. Then, k elements of Y are nonzero, and by perturbing these we can choose $\epsilon > 0$ such that $\mathcal{B}(Y, \epsilon) \cap \Sigma_k \subset \partial\Sigma_k$. It follows from this inclusion that $\mathcal{B}(Y, \epsilon) \cap \Sigma_k$ is contained in exactly one of these subspaces, since Σ_k is the union of linear subspaces of $\mathbb{R}^{m\times n}$. Therefore, Y is the projection onto this subspace, and from the (global) Kolmogorov criterion (see, e.g., [27, Sec. 5.1] [42, Th. 1, Sec. 3.12]) it follows that (6) holds. ■

From this lemma, we can prove the following theorem.

Theorem 1: Suppose that X_0 is chosen sufficiently close to a point $X^* \in \Sigma_k \cap \Phi$ such that $Y[i]$ in Algorithm 1 is in the same linear subspace $S \in \Sigma_k$ for any $i = 1, 2, \ldots$ Then, the algorithm converges to the projection of X_0 onto the intersection of Φ and S.

Proof: By the assumption on $Y[i]$ and the local Kolmogorov criterion (Lemma 2), the convergence proof is reduced to that for the conventional Dykstra's algorithm [27] for Φ and S. Since Φ and S are non-empty, closed, and convex sets, the convergence follows from the convergence result of the conventional Dykstra's algorithm. ■

From this theorem, the choice of initial guess X_0 is quite important. A suggestion of this is to use an ℓ^1 solution of minimizing $\|X\|_1$ subject to $X \in \Phi$, which is a relaxed convex problem of Problem 1. See numerical examples in Section V. On the other hand, if the initial guess X_0 is not sufficiently close to $\Sigma_k \cap \Phi$, then the convergence is not guaranteed. A typical behavior of the greedy algorithm is oscillatory between Φ and a linear subspace $\tilde{S}$ in Σ_k such that $\Phi \cap \tilde{S} = \emptyset$. See also the example shown in Section V-C.

V. Numerical Examples

A. Sparse Feedback Gain

We here show a numerical example of the sparse feedback gain discussed in Section II-A. Let us consider the following transfer function:

$$P(s) = \frac{s - 10}{(s-1)^9(s-2)(s+3)},$$

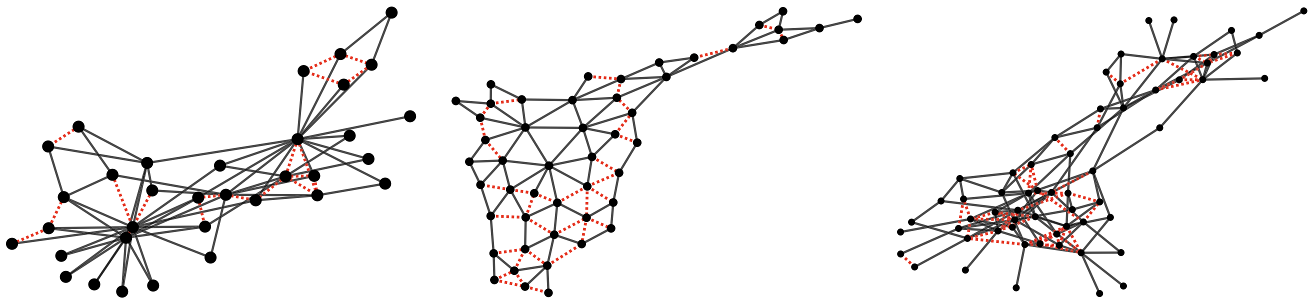
and compute the matrices A and B by using `ssdata` in MATLAB. Now $A \in \mathbb{R}^{11\times 11}$ and $B \in \mathbb{R}^{11\times 1}$. Then we compute the sparse feedback matrix $Y \in \mathbb{R}^{1\times 11}$ that satisfies (1), by the ℓ^1 relaxation proposed in [43] and the proposed iterative greedy LMI with sparsity $k = 1$. The solutions Y_1 by ℓ^1 optimization and Y_0 by the proposed method are obtained as

$$Y_1 = \begin{bmatrix} -0.0027 & 0.0003 & -0.0001 & \cdots & 1.12 \times 10^{-10} \end{bmatrix}$$
$$Y_0 = \begin{bmatrix} -0.00498 & 0 & \cdots & 0 \end{bmatrix}$$

Note that we used Y_1 as the initial guess of the iterative greedy LMI, and iterated 10 times. The proposed greedy algorithm returned exactly a 1-sparse Y_0, while the ℓ^1 optimization gave just an approximately sparse matrix.

B. Sparse Representation of a Graph

In this section, we illustrate the effectiveness of the proposed approach for sparse control with numerical simulations on the design of fastest mixing Markov processes described in

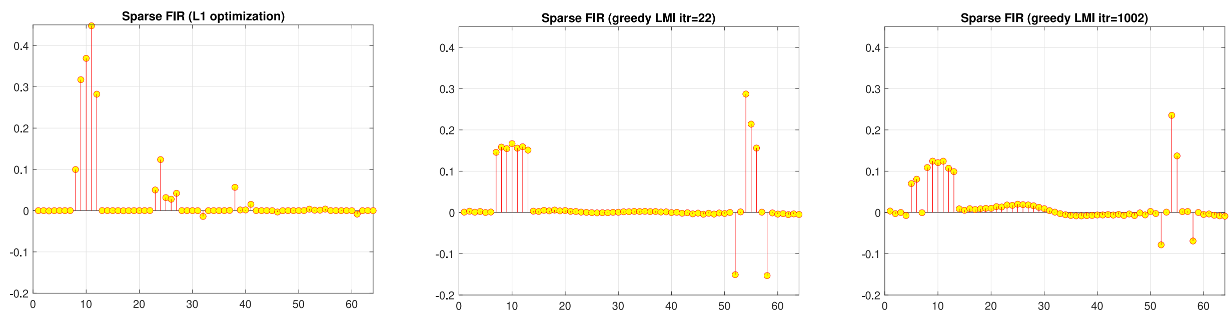


(a) Karate network. Optimal network has 78 edges. Sparse optimal network has 59 edges.

(b) USA network. Optimal network has 107 edges. Sparse optimal network has 74 edges.

(c) Dolphin network. Optimal network has 159 edges. Sparse optimal network has 109 edges.

Fig. 3. Structures of optimized networks. Black lines: edges existing in networks optimized by both existing and proposed methods. Red lines: edges existing only in networks optimized by existing method.



(a) ℓ^1 optimization

(b) Iterative greedy LMI (22)

(c) Iterative greedy LMI (1002)

Fig. 4. Sparse FIR filter coefficients obtained by ℓ^1 optimization (left), iterative greedy LMI with 22 iterations (center), and 1002 iterations (right).

Section II-B. For the purpose of illustration, we consider the following three benchmark networks commonly used in the Network Science [44]; a social network of Zachary's Karate club ($N = 34$ nodes and $M = 78$ edges), the connectivity network of states in the USA ($N = 49$ and $M = 107$), and a social network of bottlenose dolphins ($N = 69$ and $M = 159$). The weights of these three networks are optimized for the maximum algebraic connectivity by the existing method [36] and by the proposed method. In the proposed method, we set the sparsity index k to be the maximum integer that does not exceed $(4/5)M$, aiming at the reduction of at least 20% of the total edges. The values of the algebraic connectivity optimized by the existing and proposed method differ only slightly; although the existing method achieves higher algebraic connectivities, the differences are all less than 10^{-6}. Then, the structures of the optimized networks are shown in Fig. 3. We see that the proposed method allows us to design sparser network structures without significantly altering the values the algebraic connectivity, confirming the effectiveness of the proposed method. Specifically, the networks optimized by the proposed method have 19 (Karate), 33 (USA), and 50 (dolphins) fewer edges than the networks optimized by the existing method.

C. Sparse FIR Filter Design

Finally, we show an example of sparse FIR filter design discussed in Section II-C. In this example, we will show not only the effectiveness of the proposed method but also a demerit, which greedy algorithms have in general.

We consider the same example as in [39, Sec. 5]. The target analog filter is

$$K_c(s) = \frac{0.02567s^2 + 0.2636}{s^3 + 0.594s^2 + 0.9388s + 0.2636},$$

which is a third-order elliptic filter. The frequency characteristic of input signals is given by $F(s) = 1/(s+1)$. We first solve (3) with $\lambda = 0.01$. Fig. 4(a) shows the obtained filter coefficients. We can see this is already sparse. Let γ^* be the optimal γ in (3) in this case, and we adopt $\hat{\gamma} \triangleq 2\gamma^*$ to obtain 12-sparse coefficients. That is, we solve the feasibility problem with $k = 12$ and the LMIs in (3) with $\gamma = \hat{\gamma}$. We adopt the ℓ^1-optimal coefficients in Fig. 4(a) as the initial guess for the iterative greedy LMI algorithm. Note that the LMI is infeasible over the support set of the 12 largest in magnitude elements in the ℓ^1-optimal coefficients.

Fig. 4(b) is the filter coefficients obtained after 22 iterations of our algorithm. This is another sparse filter with performance level $2\gamma^*$. Then, we continue the iterations, and observe the result after 1002 iterations. Fig. 4(c) shows the result. This is *not* sparse and even worse than the result after 22 iterations. As mentioned in Section IV, the algorithm converges if the initial guess is sufficiently close to a feasible solution, but otherwise it may not converge. The phenomenon shown in Fig. 4(c) well shows this property. This implies the difficulty of the choice of an initial guess for a greedy algorithm in general.

VI. Conclusion

In this letter, we have proposed a novel algorithm called the iterative greedy LMI to find a k-sparse matrix (or vector) that satisfies LMIs. The algorithm is based on Dykstra's projection algorithm, for which local convergence is guaranteed. Future work includes a more efficient method than the ℓ^1 solution to choose the initial guess X_0 for the proposed algorithm to meet the sufficient condition of Theorem 1.

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