Connectivity and Synchronization in Bounded Confidence Kuramoto Oscillators

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***Abstract*—Frequency synchronization of bounded confidence Kuramoto oscillators is analyzed. The dynamics of each oscillator is defined by the average of the phase differences with its neighbors, where any two oscillators are considered neighbors if their geodesic distance is less than a certain confidence threshold. A phase-dependent graph is defined whose nodes and edges represent the oscillators and their connections, respectively. It is studied how the connectivity of the graph influences steady-state behaviors of the oscillators. It is proved that the oscillators synchronize asymptotically if the subgraph of each partition, possibly not complete, eventually remains constant over time. Simulation results show the application of the theoretical findings also in the presence of oscillators having different natural frequencies.**

***Index Terms*—Kuramoto oscillators, networks, bounded confidence opinion dynamics, synchronization, clustering.**

I. Introduction

SYNCHRONIZATION is one of the most interesting collective behaviors observed in many physical, biological, chemical, and social systems. The emergent behaviors of many of these systems depend on how the individuals in the system interact with each other over time. Some of these behaviors can be studied if each individual unit is modeled as a phase oscillator. The Kuramoto model [1], [2] consists of a set of phase oscillators where the intensity of interactions depends on pairwise phase differences. For this model synchronization means that the phase difference between any two oscillators remains constant over time.

Manuscript received 1 March 2024; revised 30 April 2024; accepted 15 May 2024. Date of publication 24 May 2024; date of current version 10 June 2024. This work was supported in part by the "Higher Order Interactions in Social Dynamics with Application to Monetary Networks" Project funded by European Union-Next Generation EU within the PRIN 2022 Program (D.D. 104 02/02/2022 MUR), and in part by the "IDA—Information Disorder Awareness" Project funded by the European Union-Next Generation EU within the SERICS Program through the MUR National Recovery and Resilience Plan under Grant PE00000014. Recommended by Senior Editor J. Daafouz. *(Corresponding author: Trisha Srivastava.)*

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Digital Object Identifier 10.1109/LCSYS.2024.3405379

On the basis of how the oscillators are selected to explain the evolution, different variations of the Kuramoto model can be obtained [3]. A mean-field model is obtained if there is an all-to-all coupling among the oscillators. In many practical scenarios individuals may have only a limited range of interaction with others. This is taken into account in the nearest-neighbor model where the time-evolution of an oscillator is affected only by the closest neighbor [4]. This model does not cover all aspects of the limited range interaction because the nearest neighbor may be at a large phase distance. This motivates the bounded confidence Kuramoto oscillators (BCKO) model, which is discussed in this letter, where the neighbors of an oscillator are those at geodesic distances less than a confidence bound.

The analysis of the BCKO model is still at its infancy. The idea of BCKO was introduced for the first time in [5] where different types of synchronized states with typical phenomena of opinion dynamics, such as consensus, polarization, and fragmentation [6], [7], were discussed. A notion of BCKO can be found in [8], [9] which discusses about oscillators arranged in a ring structure and only the oscillators in proximity can influence each other. Bounded confidence opinion dynamics is the basic inspiration also for the BCKO in [10] where a bimodal distribution of natural frequencies is considered and the influence of the amplitude of the confidence bound on the clusters achieved at steady-steady is studied. The BCKO model analyzed in [5], [10] considers the time-evolution of an oscillator as given by taking the sines of the phase differences with its neighbors. The summation of these sines is averaged over the total number of oscillators in the system.

In this letter a discrete-time BCKO model is proposed which differs from the ones mentioned in [5], [8], [9], [10] in two aspects: first, as considered in many bounded confidence opinion dynamics models [11], the average is taken over the number of neighbors, which makes the analysis nontrivial because at each time-instant the number of neighbors may vary depending on the phases of the oscillators; and second, motivated by the fact that bounded confidence implies small distances between interacting oscillators, it is shown that a suitable selection of the coupling function allows one to reformulate the Kuramoto model dynamics in terms of phase differences, differently from [5], [10]. The resulting model appears to be similar to Hegselmann–Krause opinion dynamics [11], [12] but it differs from it for three major reasons: *i*) in the BCKO a cyclic behavior is utilized, *ii*) differently from the opinions, the range of phases is not always decreasing over time, *iii*) there are exogenous inputs. By leveraging the row-stochastic and type-symmetric properties of the adjacency matrices of the phase-dependent graphs of BCKO, sufficient conditions for the asymptotic convergence to frequency synchronization are provided.

The rest of this letter is organized as follows. In Section II the structure of the proposed BCKO model is presented. Discussions on frequency synchronization with respect to clustering and connected subgraphs are presented in Section III. The asymptotic convergence to frequency synchronization under suitable conditions is proved in Section IV and corresponding numerical results are discussed in Section V. Section VI summarizes conclusions and future work.

A. Notation and Preliminaries

The following notation and definitions are adopted throughout this letter: $\mathbb{R}$ ($\mathbb{R}_+$) is the set of real (positive) numbers; $\mathbb{N}$ is the set of positive natural numbers; $\mathcal{N} \subset \mathbb{N}$ is used to indicate the set of consecutive natural numbers from 1 to N, i.e., $\mathcal{N} = \{1, 2, \dots, N\}$; for any $x \in \mathbb{R}$ the absolute value of x is expressed by $|x|$; for any pair of phases $\theta_i \in \mathbb{R}$ and $\theta_j \in \mathbb{R}$ the *geodesic distance* between θ_i and θ_j is the angular length of the minimum path between θ_i and θ_j on the unit circle, i.e., $\|\theta_j - \theta_i\| = \min\{|\hat{\theta}_j - \hat{\theta}_i|, 2\pi - |\hat{\theta}_j - \hat{\theta}_i|\}$, where $\hat{\theta}_i$ and $\hat{\theta}_j$ are the phases reported to the interval $[0, 2\pi)$; for any finite set $\mathcal{S}$ the notation $|\mathcal{S}|$ stands for the size of the set $\mathcal{S}$. Given a finite set $\mathcal{S}$, a *partition* of $\mathcal{S}$ is a finite family $\{\mathcal{P}_\ell\}_{\ell=1}^{L}$, $L \geq 1$, of its disjoint subsets $\mathcal{P}_\ell \subseteq \mathcal{S}$, such that $\bigcup_{\ell=1}^{L} \mathcal{P}_\ell = \mathcal{S}$. A matrix A is *type-symmetric* if there exists $\alpha \geq 1$ such that $\alpha^{-1}[A]_{ji} \leq [A]_{ij} \leq \alpha[A]_{ji}$ for all i, j, where $[A]_{ij}$ is the i-th row and j-th column entry of the matrix A; a matrix A is *row-stochastic* (column-stochastic) if $A\mathbf{1} = \mathbf{1}$ ($\mathbf{1}^\top A = \mathbf{1}^\top$), where $\mathbf{1}$ represents a column vector of all ones of suitable dimension; a matrix A is *entrywise nonnegative* if $[A]_{ij} \geq 0$ for all i, j. The following result is proved in [11, Corollary 7]: *given a sequence of row-stochastic and type-symmetric matrices* $A_0, A_1, \dots, A_k, \dots$, *if all matrices have strictly positive diagonal entries, then the product* $\prod_{i=k}^{0} A_i = A_k A_{k-1} \cdot \ldots \cdot A_0$ *converges to a constant matrix when* k *goes to infinity*. A graph is a pair of sets of nodes and edges (or arcs) between the nodes; a subgraph is a subset of nodes of a graph with the corresponding edges; an (undirected) graph is *connected* if there exists a path between any pair of nodes; a graph is *complete* if there exists an edge between any pair of nodes of the graph.

II. Bounded Confidence Kuramoto Model

In this section the proposed BCKO model is introduced starting from the bounded confidence concept applied to classical Kuramoto models.

A. Kuramoto Oscillators

The Kuramoto model describes the coupling of N oscillators, each having a constant natural frequency $\omega_i \in \mathbb{R}$, $i \in \mathcal{N}$. The dynamics of the i-th oscillator can be represented by the following scalar continuous-time differential equation

$$\dot{\theta}_i(t) = \omega_i + \sum_{j=1}^{N} \gamma_{ij}(\theta) \sin\big(\theta_j(t) - \theta_i(t)\big), \tag{1}$$

where $\theta_i(t)$ is the phase of the i-th oscillator at time t and $\gamma_{ij}(\theta)$ is the coupling function between the i-th and j-th oscillators, $i, j \in \mathcal{N}$, see [3], [13], [14]. By discretizing (1) with the forward Euler discretization technique one obtains

$$\theta_i^+ = \theta_i + h\omega_i + h\sum_{j=1}^{N} \gamma_{ij}(\theta) \sin\big(\theta_j - \theta_i\big), \tag{2}$$

where $h \in \mathbb{R}_+$ is the sampling period; for simplicity of notation, θ_i without any argument indicates the phase of the i-th oscillator at the discrete time-step $k \in \mathbb{N}$ and θ_i^+ stands for the phase at the next step $k+1$, $i \in \mathcal{N}$, $k \in \mathbb{N}$.

Different choices of the coupling functions $\gamma_{ij}(\theta)$, $i, j \in \mathcal{N}$, lead to variants of the Kuramoto model. In the classical Kuramoto oscillator model a constant all-to-all coupling of the oscillators is considered, i.e., $\gamma_{ij}(\theta) = \gamma/N$ for some positive γ, for all θ and for all $i, j \in \mathcal{N}$, see [2], [3]. Note that the coefficient $1/N$ highlights a sort of averaging operation.

B. Coupled Oscillators With Bounded Confidence

A useful representation of the model (2), when a specific set of connections among the oscillators is assumed, consists of considering a graph whose nodes are the oscillators. When the coupling function $\gamma_{ij}(\theta)$ is nonzero, an edge from the node i to the node j is assumed to exist and the edge weight is given by $\gamma_{ij}(\theta)$. In this case the node j is said to belong to the set of neighbors of the node i, which is indicated by $\mathcal{N}_i(\theta) \subseteq \mathcal{N}$. Otherwise, if $\gamma_{ij}(\theta) = 0$ then there is no edge from the node i to the node j.

A particular class of (2) is obtained if only some of the oscillators are taken into account to determine the dynamics which leads to the so-called BCKO model [10]. In this model each oscillator has a "confidence threshold", say Δ, which defines the range within which it will consider other oscillators as neighbors [15]. In this sense, the model utilizes the same behavior described by the Hegselmann–Krause model of opinion dynamics [16], [17].

The BCKO considered in [5], [10] corresponds to a specific choice of the function $\gamma_{ij}(\theta)$ which can be expressed as

$$\gamma_{ij}(\theta) = \frac{1}{N}\phi\big(\theta_i, \theta_j\big), \tag{3}$$

with

$$\phi\big(\theta_i, \theta_j\big) = \begin{cases} 1, & \text{if } \|\theta_j - \theta_i\| \leq \Delta, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

where $\|\theta_j - \theta_i\| \in [0, \pi)$ is the geodesic distance between θ_j and θ_i, for all $i, j \in \mathcal{N}$. By using (3)–(4) the model (2) can be written in the following form

$$\theta_i^+ = \theta_i + h\omega_i + \frac{h}{N}\sum_{j \in \mathcal{N}_i(\theta)} \phi\big(\theta_i, \theta_j\big) \sin\big(\theta_j - \theta_i\big) \tag{5}$$

for all $i \in \mathcal{N}$. Notably in the summation on the right hand side of (5) there are at maximum $|\mathcal{N}_i(\theta)|$ nonzero terms, but the division is made with the coefficient $1/N$, i.e., it is similar of taking an average on the total number of oscillators N. It is straightforward to see that in the case $\Delta \geq \pi$ the classical Kuramoto model with all-to-all coupling is recovered.

C. Proposed BCKO

In order to define the BCKO model analyzed in this letter, let us consider

$$\gamma_{ij}(\theta) = \frac{\eta}{|\mathcal{N}_i(\theta)|}\phi\big(\theta_i, \theta_j\big)\frac{\theta_j - \theta_i}{\sin\big(\theta_j - \theta_i\big)}, \tag{6}$$

where $\eta > 0$ is the coupling gain, $\phi(\theta_i, \theta_j)$ is given by (4) and

$$j \in \mathcal{N}_i(\theta) \iff \phi\big(\theta_i, \theta_j\big) = 1 \tag{7}$$

for all $i, j \in \mathcal{N}$. Note that from (6) it follows that $\lim_{\theta_j \to \theta_i} \gamma_{ij}(\theta) = \eta/|\mathcal{N}_i(\theta)|$ for all $i, j \in \mathcal{N}$. Since bounded confidence implies small geodesic distances between interacting oscillators, i.e., less than Δ, the choice (6) motivates the analysis of the model represented in terms of phase differences. In particular, by substituting (6) in (2) and by considering (7) one obtains

$$\theta_i^+ = \theta_i + h\omega_i + \frac{h\eta}{|\mathcal{N}_i(\theta)|}\sum_{j \in \mathcal{N}_i(\theta)}\big(\theta_j - \theta_i\big), \tag{8}$$

where $\mathcal{N}_i(\theta)$ is defined by (7) and (4) which implies that $|\mathcal{N}_i(\theta)| = \sum_{j=1}^{N}\phi_i(\theta_i, \theta_j)$, $i \in \mathcal{N}$. The equation (8) highlights that the evolution of each oscillator's phase is driven by the deviations between its own and the neighbors' phases, similarly to what happens in opinion dynamics [11], [17], [18]. Since by construction it is $i \in \mathcal{N}_i(\theta)$ for all $i \in \mathcal{N}$ and for all θ, the model (8) can be rewritten as

$$\theta_i^+ = (1 - h\eta)\theta_i + h\omega_i + \frac{h\eta}{|\mathcal{N}_i(\theta)|}\sum_{j \in \mathcal{N}_i(\theta)}\theta_j \tag{9}$$

or in matrix form

$$\theta^+ = (I + h\eta(A(\theta) - I))\theta + h\omega, \tag{10}$$

where $\omega \in \mathbb{R}^N$ is the vector of all oscillators frequencies and the entry of the i-th row and j-th column of the matrix function $A(\theta)$ is given by

$$[A(\theta)]_{ij} = \begin{cases} \frac{1}{|\mathcal{N}_i(\theta)|}, & \text{if } \|\theta_j - \theta_i\| \leq \Delta, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

with $i, j \in \mathcal{N}$. From (11) it follows that for any θ the matrix $A(\theta)$ is entrywise nonnegative, in particular $[A(\theta)]_{ij} \in [0, 1]$, $i, j \in \mathcal{N}$, type-symmetric and row-stochastic.

In the particular case that $\omega_i = 0$ for all $i \in \mathcal{N}$, by choosing h such that $h\eta < 1$ the model (9) coincides with the so-called self-belief bounded confidence opinion dynamics [19], [20], [21], [22]. Furthermore, by choosing $h\eta = 1$ the model (9) becomes the so-called symmetric homogeneous bounded confidence opinion dynamics [11], [23]. The analysis of opinion dynamics provides interesting insights for studying the more general case of BCKO in the form (9) with nonzero oscillators frequencies, which is the scenario considered in this letter.

III. Synchronization and Clustering

An interesting behavior in Kuramoto oscillators is the so-called frequency synchronization. The oscillators are said to converge to *frequency synchronization* if $\lim_{k\to\infty}(\hat{\omega}_i - \hat{\omega}_j) = 0$ for all $i, j \in \mathcal{N}$, with $\hat{\omega}_i = (\theta_i^+ - \theta_i)/h$, $i \in \mathcal{N}$, i.e., when the oscillators adjust their phases over time by converging to a common frequency, say ω_s. Then, any frequency synchronization implies the existence of a steady-state solution of (8), say $\bar{\theta}$, such that $\bar{\theta}_i^+ - \bar{\theta}_i = h\omega_s$ for some $\omega_s \in \mathbb{R}$, all $i \in \mathcal{N}$ and $k \in \mathbb{N}_0$. From (10), by imposing the condition $\bar{\theta}^+ - \bar{\theta} = \mathbf{1}h\omega_s$ it follows that any $\bar{\theta}$ must satisfy

$$\eta\big(A(\bar{\theta}) - I\big)\bar{\theta} + \omega = \mathbf{1}\,\omega_s, \tag{12}$$

where $\omega \in \mathbb{R}^N$ is the column vector with the natural frequencies of all oscillators.

Another useful concept is clustering. A solution of (8) is called a *clustering* if there exists a constant partition of $\mathcal{N}$, say $\{\mathcal{P}_\ell\}_{\ell=1}^{L}$, $L \geq 1$, such that the subgraph consisting of the oscillators in $\mathcal{P}_\ell$ is complete for all $\ell = 1, \ldots, L$. The following result shows that any clustering will asymptotically converge to frequency synchronization.

Theorem 1: Consider any solution of (8) which is a clustering. Then the conditions $h\eta < 2$ and

$$|\omega_j - \omega_i| \leq \eta\Delta \tag{13}$$

for all $\{i, j\} \in \mathcal{P}_\ell$, $\ell = 1, \ldots, L$, hold. Moreover, at steady-state all oscillators will rotate with the same frequency (independent of ℓ) given by

$$\omega_s = \frac{1}{|\mathcal{P}_\ell|}\sum_{i \in \mathcal{P}_\ell}\omega_i, \tag{14}$$

with $\ell = 1, \ldots, L$.

Proof: Consider any clustering solution of (8). For any pair of oscillators in the same partition, i.e., $\{i, j\} \in \mathcal{P}_\ell$, since the corresponding subgraph is complete it is $\mathcal{N}_i = \mathcal{P}_\ell$ for all $i \in \mathcal{P}_\ell$, $\ell = 1, \ldots, L$. Then by considering (9) for i and j and by taking their difference one can write

$$\theta_j^+ - \theta_i^+ = (1 - h\eta)\big(\theta_j - \theta_i\big) + h\big(\omega_j - \omega_i\big) \tag{15}$$

for all $\{i, j\} \in \mathcal{P}_\ell$, $\ell = 1, \ldots, L$. Taking the solution of (15) one has

$$\theta_j - \theta_i = (1 - h\eta)^k\big(\theta_j(0) - \theta_i(0)\big) + h\big(\omega_j - \omega_i\big)\sum_{m=0}^{k-1}(1 - h\eta)^m \tag{16}$$

then $h\eta < 2$ holds otherwise a time-step will exist such that the geodesic distance between the phases is larger than Δ which contradicts the assumption of a clustering solution. In order to prove (13), let us consider the steady-state of (16). Since $\lim_{k\to\infty}\sum_{m=0}^{k-1}(1 - h\eta)^m = 1/(h\eta)$, one can write

$$\bar{\theta}_j - \bar{\theta}_i = \frac{\omega_j - \omega_i}{\eta} \tag{17}$$

for all $\{i, j\} \in \mathcal{P}_\ell$, $\ell = 1, \ldots, L$, where $\bar{\theta}_j - \bar{\theta}_i$ is the phase difference at steady-state. Note that $1/\eta$ has the dimension of time. Since a bounded confidence model is considered and a clustering solution is assumed, from (4) and (17) it follows that (13) holds.

In order to prove the second part of the theorem, let us consider (8) at steady-state. Note that this situation does not correspond to θ_i being constant in time, $i \in \mathcal{N}$, because we are considering nonzero frequencies. By using (17) one can write

$$\begin{aligned}\frac{\bar{\theta}_i^+ - \bar{\theta}_i}{h} &= \omega_i + \frac{\eta}{|\mathcal{P}_\ell|}\sum_{j \in \mathcal{P}_\ell}\big(\bar{\theta}_j - \bar{\theta}_i\big) \\ &= \omega_i + \frac{1}{|\mathcal{P}_\ell|}\sum_{j \in \mathcal{P}_\ell}\big(\omega_j - \omega_i\big) = \frac{1}{|\mathcal{P}_\ell|}\sum_{j \in \mathcal{P}_\ell}\omega_j = \omega_\ell \end{aligned} \tag{18}$$

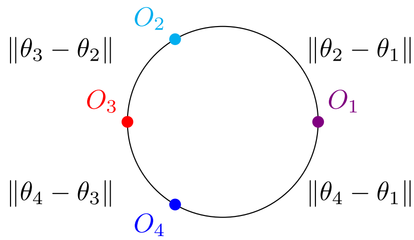

Fig. 1. BCKO model with four oscillators.

for $\ell = 1, \ldots, L$. Then all oscillators belonging to the same partition at steady-state will rotate with the same average frequency ω_ℓ. It is logically evident that all partitions must have the same average frequency ω_ℓ as if it is not the case, the clusters will eventually meet at some time-step which contradicts our assumption of clustering as a solution of (8). Then (14) holds and the proof is complete. ■

A natural question arises whether convergence to frequency synchronization is possible only under a clustering or other scenarios are possible. The following example shows that clustering is not a necessary condition for frequency synchronization.

Example 1: Consider four oscillators, say O_i, having phases θ_i, $i = 1, \ldots, 4$, distributed as in Fig. 1. Choose $2.8\Delta = 2\pi$ and $\theta_1(0) = 0$, $\theta_2(0) = 0.8\Delta$, $\theta_3(0) = 1.4\Delta$, and $\theta_4(0) = 2.0\Delta$. By considering (7) with (4) it is easy to verify that each oscillator has three neighbors, i.e., itself, the preceding one, and the succeeding one. Then, this situation is not a clustering because the graph is connected but not complete. By using (12) with $\omega_1 = \omega_s - \frac{2.8}{3}\eta\Delta$, $\omega_2 = \omega_s + \frac{0.2}{3}\eta\Delta$, $\omega_3 = \omega_s$, and $\omega_4 = \omega_s + \frac{2.6}{3}\eta\Delta$, it follows that the oscillators are synchronized at ω_s.

IV. Synchronization and Connectivity

In this section relationships between connectivity of the graph and frequency synchronization are discussed. The following result shows that if the oscillators constitute a constant partition then frequency synchronization is achieved.

Theorem 2: Consider (8) with $h\eta \in (0, 1)$ and assume there exists a time-step $\bar{k}$ such that a partition $\{\mathcal{P}_\ell\}_{\ell=1}^{L}$ is constant for any $k \geq \bar{k}$, i.e., the oscillators in $\mathcal{P}_\ell$ keep the same connections over time, for all $\ell = 1, \ldots, L$. Then the solution converges asymptotically to frequency synchronization with frequency

$$\omega_s = \frac{1}{\sum_{i\in\mathcal{P}_\ell} |\bar{\mathcal{N}}_i|} \sum_{i\in\mathcal{P}_\ell} |\bar{\mathcal{N}}_i|\omega_i \tag{19}$$

for all $\ell = 1, \ldots, L$, where $\bar{\mathcal{N}}_i$ is the set of neighbors of the oscillator i, $i \in \mathcal{N}$, for $k \geq \bar{k}$.

Proof: Consider (8) and its matrix form (10). With some abuse of notation, let us define the matrix function $M_k = h\eta(A(\theta) - I)$ (remind that with the notation adopted it is $\theta = \theta(k)$). Then by recursively applying (10) from $\theta(0)$ at the time-step k the following expression

$$\theta = \prod_{n=k-1}^{0} (I + M_n)\theta(0) + \left(\sum_{n=1}^{k-1} \prod_{q=k-1}^{k-n} \left(I + M_q\right) + I \right) h\omega \tag{20}$$

holds. Note that the right hand side of (20) depends on samples of the oscillators phases $\theta(q)$, $q = 0, \ldots, k-1$, because the matrix M_q depends on the phases $\theta(q)$.

Since the matrix $A(\theta)$ is row-stochastic for any θ, the matrix $M_q = h\eta(A(\theta(q)) - I)$ is zero row sum for any $\theta(q)$, $q \in \mathbb{N}_0$. In particular, since all entries of $A(\theta)$ belong to the interval $[0, 1]$ and $h\eta \in (0, 1)$ by hypothesis, all entries of M_q except for those on the main diagonal belong to the interval $[0, 1]$. The diagonal entries of M_q belong to the interval $(-1, 0]$ because $[M_q]_{ii} = h\eta([A(\theta(q))]_{ii} - 1)$, $h\eta \in (0, 1)$, and $[A(\theta(q))]_{ii} = 1/|\mathcal{N}_i(\theta(q))|$ which is strictly positive and less than or equal to 1, for all $i \in \mathcal{N}$. Then the matrix $I + M_q$ is row-stochastic, type-symmetric, entrywise nonnegative with all entries in the interval $[0, 1]$ and all diagonal entries strictly positive, for all $q \in \mathbb{N}_0$.

Let us now consider (20) at the next time-step which allows one to write

$$\theta^+ - \theta = \left(\prod_{n=k}^{0} (I + M_n) - \prod_{n=k-1}^{0} (I + M_n) \right) \theta(0) + (Q_k - Q_{k-1}) h\omega, \tag{21}$$

where

$$Q_{k-1} = \sum_{n=1}^{k-1} \prod_{q=k-1}^{k-n} \left(I + M_q\right) + I. \tag{22}$$

By assumption there exists a time-step, say $\bar{k}$, such that the oscillators can be grouped in a constant partition which does not change in the future. Then it is $M_k = \bar{M}$ with $\bar{M}$ constant matrix, for any $k \geq \bar{k}$. Clearly $I + \bar{M}$ is type-symmetric, row-stochastic, entrywise nonnegative with all entries in the interval $[0, 1]$ and all diagonal entries strictly positive. By considering the first term on the right hand side of (21) one can write

$$\begin{aligned} &\prod_{n=k}^{0} (I + M_n) - \prod_{n=k-1}^{0} (I + M_n) \\ &\quad = \left(I + \bar{M}\right)^{k-\bar{k}} \prod_{n=\bar{k}}^{0} (I + M_n) - \left(I + \bar{M}\right)^{k-1-\bar{k}} \prod_{n=\bar{k}}^{0} (I + M_n) \\ &\quad = \left(I + \bar{M}\right)^{k-1-\bar{k}} \bar{M} \prod_{n=\bar{k}}^{0} (I + M_n). \end{aligned} \tag{23}$$

By using (23) it follows that the properties of the matrix $I + \bar{M}$ allows one to conclude that as k goes to infinity (23) converges to the constant matrix

$$\begin{aligned} Q^\infty &= \lim_{k\to\infty} \left(\prod_{n=k}^{0} (I + M_n) - \prod_{n=k-1}^{0} (I + M_n) \right) \\ &= \mathcal{M}^\infty \bar{M} \prod_{n=\bar{k}}^{0} (I + M_n), \end{aligned} \tag{24}$$

where $\mathcal{M}^\infty = \lim_{k\to\infty} (I + \bar{M})^{k-\bar{k}-1}$.

Let us now consider the second term on the right hand side of (20). From (22) one can write

$$\begin{aligned} Q_{k-1} &= \sum_{n=1}^{k-1} \prod_{q=k-1}^{k-n} \left(I + M_q\right) + I \\ &= \mathcal{Q}_{k-1} + \left(I + \bar{M}\right)^{k-\bar{k}-2} + \cdots + \left(I + \bar{M}\right) + I, \end{aligned} \tag{25}$$

with

$$\begin{aligned}\mathcal{Q}_{k-1} &= \left(I+\bar{M}\right)^{k-\bar{k}-1}\left(I+M_{\bar{k}-1}\right)\cdot\ldots\cdot\left(I+M_1\right)\\ &\quad+\left(I+\bar{M}\right)^{k-\bar{k}-1}\left(I+M_{\bar{k}-1}\right)\cdot\ldots\cdot\left(I+M_2\right)\\ &\quad+\cdots+\left(I+\bar{M}\right)^{k-\bar{k}-1}\left(I+M_{\bar{k}-1}\right)\end{aligned}\tag{26}$$

for all $k \geq \bar{k}$. From (26) it follows

$$\mathcal{Q}_k = \left(I+\bar{M}\right)\,\mathcal{Q}_{k-1}\tag{27}$$

for all $k \geq \bar{k}$ and then $\mathcal{Q}_k = (I+\bar{M})^{k-\bar{k}+1}\mathcal{Q}_{\bar{k}-1}$ for all $k \geq \bar{k}$. Then the matrix function $\mathcal{Q}_k$ converges to the constant matrix $\mathcal{Q}^{\infty} = \mathcal{M}^{\infty}\mathcal{Q}_{\bar{k}-1}$ when k goes to infinity. From (25) and (27) one can write

$$\begin{aligned}Q_k - Q_{k-1} &= \mathcal{Q}_k + \left(I+\bar{M}\right)^{k-\bar{k}-1} - \mathcal{Q}_{k-1}\\ &= \bar{M}\mathcal{Q}_{k-1} + \left(I+\bar{M}\right)^{k-\bar{k}-1}\end{aligned}\tag{28}$$

for all $k \geq \bar{k}$. By substituting (23) and (28) in (21) and by taking the limit when k goes to infinity it follows

$$\begin{aligned}\lim_{k\to\infty}\left(\theta^{+}-\theta\right) &= Q^{\infty}\theta(0)\\ &\quad+\lim_{k\to\infty}\left(\bar{M}\mathcal{Q}_{k-1}+\left(I+\bar{M}\right)^{k-\bar{k}-1}\right)h\omega\\ &= Q^{\infty}\theta(0)+\left(\bar{M}\mathcal{Q}^{\infty}+\lim_{k\to\infty}\left(I+\bar{M}\right)^{k-\bar{k}-1}\right)h\omega\\ &= Q^{\infty}\theta(0)+\left(\bar{M}\mathcal{Q}^{\infty}+\mathcal{M}^{\infty}\right)h\omega\end{aligned}\tag{29}$$

which means that the time-step variation of the phase vector will asymptotically converge to a constant.

At steady-state all oscillators belonging to the same partition must have the same rotational frequency, otherwise there will exist a time-step such that the partition changes which contradicts the hypothesis. Similarly all different partitions must have the same rotational frequency. Then the solution of (8) converges asymptotically to frequency synchronization. The expression (19) for each subset of the partition can be derived from (8) and by imposing that

$$\frac{\bar{\theta}_i^{+}-\bar{\theta}_i}{h} = \omega_s = \omega_i + \frac{\eta}{|\bar{\mathcal{N}}_i|}\sum_{j\in\bar{\mathcal{N}}_i}\left(\bar{\theta}_j-\bar{\theta}_i\right)\tag{30}$$

and then

$$|\bar{\mathcal{N}}_i|\omega_s = |\bar{\mathcal{N}}_i|\omega_i + \eta\sum_{j\in\bar{\mathcal{N}}_i}\left(\bar{\theta}_j-\bar{\theta}_i\right)\tag{31}$$

for all $i \in \mathcal{P}_\ell$, $\ell = 1,\ldots,L$. The symmetry of (11), i.e., if the oscillator i influences j also the opposite holds, allows one to say that for each term $\bar{\theta}_j - \bar{\theta}_i$ with $j \in \bar{\mathcal{N}}_i$, there will exist a term $\bar{\theta}_i - \bar{\theta}_j$ in the dynamic equation of the oscillator j with $i \in \bar{\mathcal{N}}_j$. Then by adding all equations (31) for $i \in \mathcal{P}_\ell$ one obtains (19) and the proof is complete. ■

The following example verifies the results in Theorem 2, i.e., a synchronization may exist even when the subgraphs of each partition are connected but not complete.

Example 2: Consider the scenario in Fig. 1 with $\omega_1 = \omega_3 = \omega_s$, $\omega_2 = \omega_s - \bar{\omega}$, and $\omega_4 = \omega_s + \bar{\omega}$. The natural frequencies satisfy (19) with $\mathcal{N}_1 = \{1\}$, $\mathcal{N}_2 = \{2, 3\}$, $\mathcal{N}_3 = \{2, 3, 4\}$, and $\mathcal{N}_4 = \{3, 4\}$ (which implies $\mathcal{P}_1 = \{1\}$ and $\mathcal{P}_2 = \{2, 3, 4\}$), for any $\bar{\omega} \in \mathbb{R}$. Let us look if there exists a steady-state with that partition which corresponds to a synchronization. From (30) the connection conditions $\|\bar{\theta}_3 - \bar{\theta}_2\| \leq \Delta$, $\|\bar{\theta}_4 - \bar{\theta}_3\| \leq \Delta$, and $\|\bar{\theta}_4 - \bar{\theta}_2\| > \Delta$ imply that it must be $\bar{\omega} \in (\frac{\eta\Delta}{4}, \frac{\eta\Delta}{2}]$ which can be easily satisfied by choosing a suitable coupling gain η. Then for any $\bar{\theta}_1$ such that $\|\bar{\theta}_4 - \bar{\theta}_1\| > \Delta$ and $\|\bar{\theta}_2 - \bar{\theta}_1\| > \Delta$, the corresponding trajectories $\bar{\theta}$ represent oscillators which are synchronized at ω_s with the desired partition. Numerical simulations show that under the hypotheses of Theorem 2 these steady-states are locally asymptotically stable.

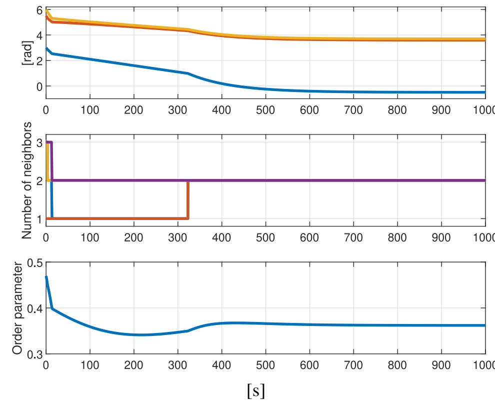

Fig. 2. Test 1: Time-evolution of phase differences of the oscillators with respect to those of the first one, i.e., $\theta_i - \theta_1$, $i = 2, 3, 4$ (top), number of neighbors (middle), and order parameter (bottom).

V. Simulation Results

In this section results obtained by considering two numerical tests illustrate the theoretical analysis presented above.

Consider the model (8) with $N = 4$, $\eta = 0.01$, $h = 0.0628$ s, and $\Delta = 0.9817$ rad. Initial conditions (in radians) are taken as $\theta_1(0) = 0, \theta_2(0) = 3, \theta_3(0) = 5.5$, $\theta_4(0) = 6$ and natural frequencies (in rad/s) $\omega_1 = \omega_s + 0.005, \omega_2 = \omega_s, \omega_3 = \omega_s + 0.002$, and $\omega_4 = \omega_s + 0.003$, with $\omega_s = 0.1$. Time-evolution of the phase differences, number of neighbors, and order parameter are observed together in Fig. 2. At the initial conditions the oscillator O_2 is isolated while the other three are connected. Eventually, after 323 s, O_2 gets connected with O_1, both the partitions remain constant hereafter, and all oscillators have two neighbors each (including themselves). So as indicated by Theorem 1 and Theorem 2, the asymptotic convergence to frequency synchronization is observed. This is also confirmed by the constant order parameter which is given by $r = |\frac{1}{N}\sum_{j=1}^{N} e^{\iota\theta_j}|$, with ι the imaginary unit, and it represents a measure of the degree of frequency synchronization among oscillators.

For the second test the model (8) is considered with $N = 10$ oscillators and natural frequencies (in rad/s) taken as $\omega - \omega_s\mathbf{1} = [-\bar{\omega}_1, 0, \bar{\omega}_1, 0, -\bar{\omega}_2, \bar{\omega}_2, -\bar{\omega}_3, 0, \bar{\omega}_3, 0]^\top$, with $\bar{\omega}_1 = 0.0034, \bar{\omega}_2 = 0.0039$, and $\bar{\omega}_3 = 0.0008$. The other model parameters are the same as in the previous example. It is easy to verify that the natural frequencies satisfy (19) with different partitions, e.g., $\mathcal{P}_1 = \{1,\ldots,4\}$ and $\mathcal{P}_2 = \{5,\ldots,10\}$. We choose as initial phases (in radians) $\theta(0) = [0, 0.6, 0.9, 1.8, 3.6, 3.8, 3.9, 4.2, 4.4, 4.7]$ which corresponds to having the first four oscillators disconnected with the remaining six oscillators. Figure 3 shows that the order parameter is constant which validates the asymptotic convergence to frequency synchronization, coherently with Theorem 1 (the subgraphs corresponding to the two partitions are complete at steady-state) and Theorem 2. It was observed that for other

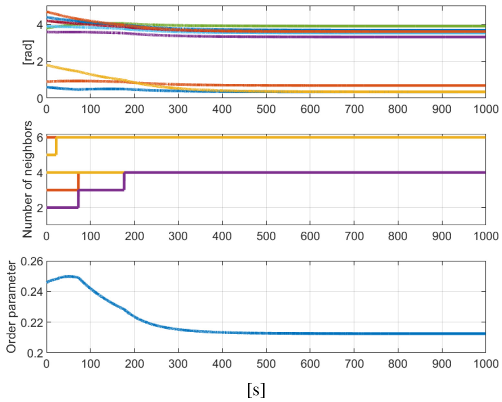

Fig. 3. Test 2: Time-evolution of phase differences of the oscillators with respect to those of the first one, i.e., $\theta_i - \theta_1$, $i = 2, 3, \ldots, 10$ (top), number of neighbors (middle), and order parameter (bottom).

initial conditions which are close enough to the chosen one, the convergence to frequency synchronization is achieved at steady-state.

VI. Conclusion

The bounded confidence Kuramoto oscillators model, where at each time-step the phase differences are averaged over the number of neighbors, which itself depends on the phase differences, has been analyzed with respect to frequency synchronization. It is proved that clustering, i.e., partitioning with complete subgraphs, implies asymptotic convergence to frequency synchronization. It is also shown that clustering is not a necessary condition for frequency synchronization. In particular, it is proved that frequency synchronization is ensured also under the weaker hypothesis that the partitions remain constant over time even if the subgraphs of the partitions are not complete. Simulations have verified steady-state frequency synchronization in some situations where the assumptions of the theoretical results hold. Future work is directed on the analysis of behaviors, recognized in the numerical campaign, where connections continuously change over time even though the order parameter reaches an almost constant value. Finding the initial conditions such that a specific steady-state clustering pattern is obtained is another interesting, although nontrivial, open problem. Furthermore, the analysis of the model where the geodesic distances are used in the state equation is another interesting direction for future research.

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Open Access funding provided by 'Università degli Studi del Sannio' within the CRUI CARE Agreement