Recurrent Neural Network-Based MPC for Systems With Input and Incremental Input Constraints

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***Abstract*—This letter proposes a stabilizing Model Predictive Control algorithm, specifically designed to handle systems learned by Incrementally Input-to-State Stable Recurrent Neural Networks, in presence of input and incremental input constraints. Closed-loop stability is proven by relying on the Incremental Input-to-State Stability property of the model, and on a terminal equality constraint involving the control sequence only. The Incremental Input-to-State Stability is also used to derive a suitable formulation of the Model Predictive Control terminal cost. The proposed control algorithm can be readily applied to a wide range of Recurrent Neural Networks, including Gated Recurrent Units, Echo State Networks, and Neural Nonlinear Autoregressive eXogenous models. Furthermore, this letter specializes the approach to handle the particular case of Long Short-Term Memory Networks, and showcases its effectiveness on a four tanks process benchmark.**

***Index Terms*—Predictive control for nonlinear systems, constrained control, neural networks.**

I. Introduction

MODEL Predictive Control (MPC) [1] is an optimization-based control method that relies on a model of the system under control to predict the future dynamics, and optimize the closed-loop performance. Moreover, the possibility to explicitly consider constraints in the control problem formulation makes MPC extremely suitable to face real-world problems. While input, state and output constraints are commonly considered, the control input *variation* is often just penalized in the cost, and is not included in the hard constraints. As a matter of fact, the satisfaction of incremental input constraints can be very important to guarantee the safety of the system. For example, in hydraulic networks, the speed of control valves is usually limited. Fast valve operations results in strong hydraulic transients, which could stress and eventually damage the structure of the network [2]. Similarly, in mechanical systems, it is desirable to avoid large accelerations and jerks that could wear out the system [3]. In the control of assisted an driverless vehicles, accelerations and jerks are limited to improve passengers comfort [4]. In [5], incremental input constraints are introduced in the control of a mobile robot, in the consideration of the safety and comfort needs in real life.

The use of nonlinear, black-box models in MPC gained increasing popularity in recent years, thanks to the availability of large amount of data and to the increase in the available computing power. In particular, the class of Recurrent Neural Network (RNN) models is particularly effective in learning nonlinear plant dynamics, and only requires input-output data for the training [6]. In order to guarantee that the effect of the state initialization vanishes, and that a small input variation does not lead to a large state variation, it is useful to consider RNN models that satisfy an Incremental Input-to-State Stability (δISS) property [7]. Sufficient conditions for δISS have been derived for the most common classes of RNN, such as for Long Short-Term Memory (LSTM) in [8], for Gated Recurrent Units (GRU) in [9] and other particular classes of RNN including Echo State Networks (ESN) and Neural Nonlinear Autoregressive eXogenous models (NNARX) in [10]. For all these networks, a straightforward modification of the training loss can lead to a final model that satisfies the desired δISS condition. Several examples of MPC algorithms based on δISS RNN models are available in the literature: e.g., δISS LSTM models are used in MPC with input constraints for regulation in [8], and in MPC with input and output constraints for offset-free tracking in [11]. In [12], δISS GRU models are employed in robust MPC with input and output constraints. In [13], a stabilizing MPC based on a sufficiently long prediction horizon is designed for general δISS systems, and is applied to GRU models. In [14], the practical aspects of the implementation of MPC using LSTM models are analyzed, and the computation time of different optimization tools is compared. In [15], a stabilizing MPC is designed for Recurrent Equilibrium Networks (REN), that are a particular class of RNN for which the δISS property can be achieved by means of a proper parametrization.

This letter considers both aspects, and proposes a stabilizing MPC for a generic δISS RNN model with input

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and incremental input constraints. To take into account the incremental constraint on the input, the input variation is considered as control input, and an integrator is introduced to obtain the actual system input. Closed-loop stability is guaranteed by means of a suitable terminal cost and a terminal equality constraint only on the state of the integrator, that is the input of the system. Since state/output constraints are not considered, and in view of the stability property of the RNN model, no terminal constraint on the model state is required. This is a significant advance because it is not necessary to introduce conservative contracting constraints to cope with the observer estimation and modeling errors. In fact the equality constraint on the state of the integrator is not affected by uncertainties. The proposed algorithm is then applied to control a four tanks process benchmark [16] modeled with an LSTM network. In order to satisfy all the assumptions introduced in the general algorithm, the input constraint is tightened with respect to the set considered for stability.

A. Preliminaries

Notation: Considering a vector v, $\|v\|$ is its 2-norm, $\|v\|_\infty$ is its infinity-norm, and $\|v\|_A^2 = v^\top A v$ is its squared norm weighted with matrix A. $\lambda_{max}(M)$ and $\lambda_{min}(M)$ are the maximum and minimum eigenvalues of the symmetric matrix M. $\mathbf{I}_n$ is the $n \times n$ identity matrix. $\mathrm{int}(\mathcal{X})$ denotes the interior of the set $\mathcal{X}$. Given two vectors v and w, $v \circ w$ is their Hadamard product (element-wise product). $\sigma(z) = \frac{1}{1+e^{-z}}$ is the sigmoid activation function. The activation functions $\sigma(\cdot)$ and $\tanh(\cdot)$ are applied to vectors element by element. The standard definitions of functions of classes $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ are considered (see [1]).

Stability definitions: The notions of Input-to-State Stability (ISS) and δISS are now introduced for the generic discrete-time dynamical system $x_{k+1} = f(x_k, u_k)$, with $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^m$. The definitions are stated in the sets $\mathcal{X}^{ISS} \subseteq \mathbb{R}^{n_x}$ and $\mathcal{U}^{ISS} \subseteq \mathbb{R}^m$, with the set $\mathcal{X}^{ISS}$ assumed to be positive invariant, i.e., $u \in \mathcal{U}^{ISS}$ and $x \in \mathcal{X}^{ISS} \implies f(x, u) \in \mathcal{X}^{ISS}$.

Definition 1 (ISS, [17]): The system $x_{k+1} = f(x_k, u_k)$ is ISS in the sets $\mathcal{X}^{ISS}$ and $\mathcal{U}^{ISS}$ if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any $k \geq 0$, any initial condition $x_0 \in \mathcal{X}^{ISS}$ and any input sequence $u_0, u_1, \ldots, u_{k-1}$ with $u_h \in \mathcal{U}^{ISS}$ for $h = 0, \ldots, k-1$, it holds that

$$\|x_k\| \leq \beta(\|x_0\|, k) \; + \gamma\left(\max_{0 \leq h < k} \|u_h\|\right)$$

Definition 2 (ISS-Lyapunov Function, [17]): A function $V : \mathcal{X}^{ISS} \to \mathbb{R}_+$ is an ISS-Lyapunov function for the system $x_{k+1} = f(x_k, u_k)$ if there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that for all $x \in \mathcal{X}^{ISS}$, $u \in \mathcal{U}^{ISS}$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$
$$V(f(x, u)) - V(x) \leq -\alpha_3(\|x\|) + \gamma(\|u\|)$$

Lemma 1 [17]: If the system $x_{k+1} = f(x_k, u_k)$ admits an ISS-Lyapunov function in the sets $\mathcal{X}^{ISS}$ and $\mathcal{U}^{ISS}$, then it is ISS in $\mathcal{X}^{ISS}$ and $\mathcal{U}^{ISS}$.

Definition 3 (δISS, [7]): The system $x_{k+1} = f(x_k, u_k)$ is δISS in the sets $\mathcal{X}^{ISS}$ and $\mathcal{U}^{ISS}$ if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for any $k \geq 0$, any pair of initial conditions $x_{a,0} \in \mathcal{X}^{ISS}$ and $x_{b,0} \in \mathcal{X}^{ISS}$, any pair of input sequences $u_{a,0}, u_{a,1}, \ldots, u_{a,k-1}$ and $u_{b,0}, u_{b,1}, \ldots, u_{b,k-1}$, with $u_{a,h}, u_{b,h} \in \mathcal{U}^{ISS}$ for all $h = 0, .., k-1$, it holds that

$$\|x_{a,k} - x_{b,k}\| \leq \beta\big(\|x_{a,0} - x_{b,0}\|, k\big) + \gamma\Big(\max_{0 \leq h < k} \|u_{a,h} - u_{b,h}\|\Big)$$

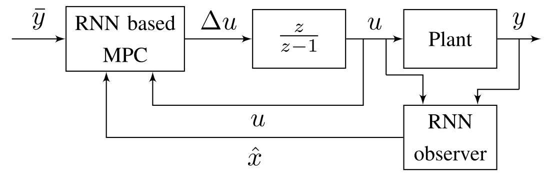

Fig. 1. Block diagram of the control scheme.

II. Problem Formulation

The objective of this letter is to design an MPC algorithm for a nonlinear system, given its RNN model. In particular, the problem of regulation to a set point $\bar{y}$ is considered, while respecting constraint on the input and on the maximum input variation, i.e.,

$$u \in \mathcal{U}, \quad \Delta u \in \mathcal{V} \tag{1}$$

where $\Delta u_k = u_k - u_{k-1}$, $\mathcal{U}$ is a bounded closed set and $\mathcal{V}$ includes the origin in its interior. In the following, the RNN model will be denoted in compact form as

$$x_{k+1} = f(x_k, u_k) \tag{2a}$$
$$y_k = g(x_k) \tag{2b}$$

where $u \in \mathbb{R}^m$ is the input of the model, $y \in \mathbb{R}^p$ is the predicted output, $x \in \mathbb{R}^{n_x}$ is the RNN state, and the pedix k denotes the time dependence. In order to ensure that the effect of the initialization asymptotically vanishes, and that bounded inputs produce bounded outputs, δISS RNN models are considered.

Assumption 1: The model (2) is δISS in the sets $\mathcal{X}^{ISS}$ and $\mathcal{U}^{ISS} \supseteq \mathcal{U}$, and there exists an incremental Lyapunov function $V_f(x_a, x_b)$ such that for any $x_a, x_b \in \mathcal{X}^{ISS}$, $u \in \mathcal{U}^{ISS}$

$$a\|x_a - x_b\|^2 \leq V_f(x_a, x_b) \leq b\|x_a - x_b\|^2 \tag{3a}$$
$$V_f(f(x_a, u), f(x_b, u)) - V_f(x_a, x_b) \leq -c\|x_a - x_b\|^2 \tag{3b}$$

for some $a, b, c > 0$. Moreover, $V_f(x_a, x_b)$ is locally Lipschitz continuous in any bounded set, i.e., given any bounded set $\bar{\mathcal{X}} \subseteq \mathcal{X}^{ISS}$, if $x_a, x_a', x_b \in \bar{\mathcal{X}}$ then there exist a finite constant $\bar{L}_V$ such that

$$\big|V_f(x_a, x_b) - V_f\big(x_a', x_b\big)\big| \leq \bar{L}_V \big\|x_a - x_a'\big\|$$

δISS conditions and incremental Lyapunov functions that satisfy (3) can be found for the main RNN architectures (LSTM [8], GRU [9], [12], REN [15], and other particular classes of RNN [10]). Moreover, the property of local Lipschitzianity is satisfied by quadratic functions, that are the most common types of Lyapunov functions used to study RNN stability.

III. Control Algorithm

In Fig. 1 the scheme of the proposed control algorithm is reported. It is composed by three main blocks. The first one is the observer, that is needed in order to properly initialize the predictions in the MPC. The observer is necessary because in general the state x of the RNN model is not measurable and has no physical meaning. The second block is the MPC. In order to take into account the constraint on the input variation, the MPC considers Δu as optimization variable. The value of the control variable u is obtained from Δu by means of a discrete time integrator.

A. Observer

In view of the δISS property of the model, it is always possible to design a converging observer. The most simple one is the open-loop observer.

Assumption 2: There exists an observer for the RNN model that can be written in the form

$$\hat{x}_{k+1} = f(\hat{x}_k, u_k) + \delta x(\hat{x}_k, u_k, y_k) \tag{4}$$

where

$$\|\delta x(\hat{x}_k, u_k, y_k)\| \leq L_{max}\|x_k - \hat{x}_k\| \tag{5}$$

for some $L_{max} \geq 0$. The observer satisfies the following properties:

1) there exist a finite $\bar{k} \geq 0$ and a set $\hat{\mathcal{X}}$ such that for any $\hat{x}_0 \in \hat{\mathcal{X}}$, and for any input sequence in $\mathcal{U}$, $\hat{x}_k \in \mathcal{X}^{ISS}$ $\forall k \geq \bar{k}$;
2) the observer estimation error converges to 0, i.e., for any input sequence in $\mathcal{U}$, $\|x_k - \hat{x}_k\| \to 0$ for $k \to \infty$.

Observers respecting Assumption 2 have been designed for the main RNN architectures (LSTM [8], GRU [13], REN [15] and other particular classes of RNN [10]).

B. MPC Design

The MPC solves at every sample time instant a Finite Horizon Optimal Control Problem (FHOCP), where the deviation from the state and input reference values $\bar{x}$ and $\bar{u}$ are penalized. The references values are computed from the RNN model (2) as the equilibrium state and input corresponding to the output $\bar{y}$. It is assumed that $\bar{u} \in \mathcal{U}$ and $\bar{x} \in \mathcal{X}^{ISS}$. The FHOCP is given by

$$\min_{\Delta u_{\cdot|k}} \sum_{i=0}^{N-1}\left(\|x_{i|k} - \bar{x}\|_Q^2 + \|u_{i-1|k} - \bar{u}\|_R^2 + \|\Delta u_{i|k}\|_{R_\Delta}^2\right) + sV_f(x_{N|k}, \bar{x}) \tag{6a}$$

$$\text{s.t.}\quad x_{0|k} = \hat{x}_k, \quad u_{-1|k} = u_{k-1} \tag{6b}$$

$$u_{N-1|k} = \bar{u} \tag{6c}$$

$$\text{for } i = 0, \dots, N-1$$

$$x_{i+1|k} = f(x_{i|k}, u_{i|k}) \tag{6d}$$

$$u_{i|k} = u_{i-1|k} + \Delta u_{i|k} \tag{6e}$$

$$u_{i|k} \in \mathcal{U} \tag{6f}$$

$$\Delta u_{i|k} \in \mathcal{V} \tag{6g}$$

In the cost function (6a), the matrices $Q = Q^\top > 0$, $R = R^\top > 0$, $R_\Delta = R_\Delta^\top > 0$ and

$$s \geq \frac{\lambda_{max}(Q)}{c} \tag{7}$$

are design choices. In order to guarantee closed-loop stability, the terminal equality constraint (6c) is introduced on the last element of the input sequence, while no terminal constraints are introduced on the system state. Denote by $\Delta u^*_{0|k}, \dots, \Delta u^*_{N-1|k}$ the optimal solution of the FHOCP. According to the Receding Horizon principle, the MPC control law is

$$\Delta u_k = k^{MPC}(\hat{x}_k, u_{k-1}) = \Delta u^*_{0|k} \tag{8}$$

Then, the system input is

$$u_k = u_{k-1} + k^{MPC}(\hat{x}_k, u_{k-1}) \tag{9}$$

C. Stability Analysis

In this subsection the stability properties of the closed-loop system are analyzed, under the assumption that the system behaves according to its RNN model.

Firstly, note that the feasibility of the FHOCP only depends on the value of u_{k-1} and not on the initial state, since there are no state constraints. Hence, define

$$\mathcal{U}^{MPC}(\bar{u}) = \{u \in \mathcal{U} : \exists \text{ a sequence } \Delta u_{0|k}, \dots, \Delta u_{N-1|k} \text{ respecting (6c)-(6e)-(6f)-(6g) with } u_{-1|k} = u\}$$

The size of $\mathcal{U}^{MPC}$ depends on the incremental constraint set $\mathcal{V}$ and of the prediction horizon N. In presence of a small set $\mathcal{V}$ it may be necessary to enlarge N to obtain a sufficiently large feasibility set $\mathcal{U}^{MPC}(\bar{u})$.

Theorem 1: Let Assumptions 1–2 hold. Then for the closed-loop system composed by the RNN model (2), the observer (4) and the MPC (9), the FHOCP is recursively feasible and the closed-loop system converges to $x = \bar{x}$, $u = \bar{u}$, $\hat{x} = \bar{x}$ with domain of attraction $x \in \mathcal{X}^{ISS}$, $u \in \mathcal{U}^{MPC}(\bar{u})$, $\hat{x} \in \hat{\mathcal{X}}$.

Proof: Denote the optimal input sequence and state trajectory computed by the MPC at time step k respectively by $u^*_{i-1|k}$ and $x^*_{i|k}$ for $i = 0, \dots, N.$, where $x^*_{0|k} = \hat{x}_k$ and $u^*_{-1|k} = u_{k-1}$.

To prove recursive feasibility, assuming that the FHOCP is feasible at time step k, it is sufficient to consider as candidate solution at time step $k+1$ the sequence $\Delta\tilde{u}_{0|k+1}, \dots, \Delta\tilde{u}_{N-1|k+1}$, defined by $\Delta\tilde{u}_{i|k+1} = \Delta u^*_{i+1|k}$ for $i = 1, \dots, N-2$ and $\Delta\tilde{u}_{N-1|k+1} = 0$, that trivially satisfies constraints (6c)-(6f)-(6g).

In view of Assumption 2, there exists a finite value $\bar{k} \geq 0$ such that $\hat{x}_k \in \mathcal{X}^{ISS}$ for all $k \geq \bar{k}$. Moreover, for Assumption 1, the set $\mathcal{X}^{ISS}$ is positive invariant for the model state x. Hence, along the proof of convergence, it is assumed that $\hat{x} \in \mathcal{X}^{ISS}$ and $x \in \mathcal{X}^{ISS}$, which is true for all $k \geq \bar{k}$. In addition, in view of Assumption 2, $\|x_k - \hat{x}_k\|$ is bounded and converges to zero. Hence, the observer innovation δx is considered as a disturbance term, and it is shown that the closed-loop system is ISS with respect to it. Then, convergence follows in view of (5) and of the convergence of the observer estimation error.

In order to prove ISS, consider the optimal cost of the MPC as candidate ISS-Lyapunov function

$$V(x, u) = \sum_{i=0}^{N-1}\left(\|x^*_{i|k} - \bar{x}\|_Q^2 + \|u^*_{i-1|k} - \bar{u}\|_R^2 + \|\Delta u^*_{i|k}\|_{R_\Delta}^2\right) + sV_f(x^*_{N|k}, \bar{x})$$

where $x = x^*_{0|k}$ and $u = u^*_{-1|k}$. V is also a function of u, that is the state of the discrete time integrator at time step k.

Firstly, derive a lower bound for $V(x, u)$

$$V(x, u) \geq \left\|x^*_{0|k} - \bar{x}\right\|^2_Q + \left\|u^*_{-1|k} - \bar{u}\right\|^2_R \geq \lambda_{min}(Q)\|x - \bar{x}\|^2 + \lambda_{min}(R)\|u - \bar{u}\|^2 \quad (10)$$

To derive an upper bound for $V(x, u)$, consider as candidate solution at time k the sequence $\Delta\tilde{u}_{0|k}, \dots, \Delta\tilde{u}_{N-1|k}$, where $\Delta\tilde{u}_{0|k} = \bar{u} - u$ and $\Delta\tilde{u}_{i|k} = 0$ for $i = 1, \dots, N-1$. This solution is feasible for u in a neighborhood of $\bar{u}$. Denote by $\tilde{x}_{i|k}$, $i = 0, \dots, N$, the associate state trajectory. One has that

$$V(x, u) \leq \sum_{i=0}^{N-1}\left(\left\|\tilde{x}_{i|k} - \bar{x}\right\|^2_Q\right) + \|u - \bar{u}\|^2_R + \|\bar{u} - u\|^2_{R_\Delta} + sV_f(\tilde{x}_{N|k}, \bar{x})$$

In view of the δISS property of the model and of (3a), there exists $b_x > 0$ such that, for u in a neighborhood of $\bar{u}$,

$$V(x, u) \leq b_x\|x - \bar{x}\|^2 + (\lambda_{max}(R) + \lambda_{max}(R_\Delta))\|u - \bar{u}\|^2 \quad (11)$$

Given this local upper bound, the existence of an upper bound for any $u \in \mathcal{U}^{MPC}(\bar{u})$ follows from [18, Lemma 4].

To prove the decreasing property of $V(x, u)$, denote $u^+ = u + k^{MPC}(x, u)$ and $x^+ = f(x, u^+) + \delta x$. The term δx due to the observer presence is regarded as an external disturbance. Consider now the same candidate solution introduced to prove recursive feasibility. Denote by $\tilde{u}_{-1|k+1}, \dots, \tilde{u}_{N-1|k+1}$ the associate input sequence, defined by $\tilde{u}_{-1|k+1} = u^+$ and $\tilde{u}_{i|k+1} = \tilde{u}_{i-1|k+1} + \Delta\tilde{u}_{i|k+1}$ for $i = 0, \dots, N-1$, and by $\tilde{x}_{0|k+1}, \dots, \tilde{x}_{N|k+1}$ the associate state trajectory, defined by $\tilde{x}_{0|k+1} = x^+$ and $\tilde{x}_{i+1|k+1} = f(\tilde{x}_{i|k+1}, \tilde{u}_{i|k+1})$ for $i = 0, \dots, N-1$.

Define $x^*_{N+1|k} = f(x^*_{N|k}, \bar{u})$, and denote for $i = 1, \dots, N$, $\varepsilon_{k+i} = \tilde{x}_{i-1|k+1} - x^*_{i|k}$. One has that

$$\begin{aligned}
V(x^+, u^+) - V(x, u) \leq\ & \sum_{i=0}^{N-1}\Big(\left\|\tilde{x}_{i|k+1} - \bar{x}\right\|^2_Q + \left\|\tilde{u}_{i-1|k+1} - \bar{u}\right\|^2_R \\
& + \left\|\Delta\tilde{u}_{i|k+1}\right\|^2_{R_\Delta}\Big) + sV_f(\tilde{x}_{N|k+1}, \bar{x}) \\
& - \sum_{i=0}^{N-1}\left(\left\|x^*_{i|k} - \bar{x}\right\|^2_Q + \left\|u^*_{i-1|k} - \bar{u}\right\|^2_R + \left\|\Delta u^*_{i|k}\right\|^2_{R_\Delta}\right) \\
& - sV_f\left(x^*_{N|k}, \bar{x}\right) \\
\leq\ & -\|x - \bar{x}\|^2_Q - \|u - \bar{u}\|^2_R - \left\|\Delta u^*_{0|k}\right\|^2_{R_\Delta} \\
& + \sum_{i=1}^{N-1}\left(\|\varepsilon_{k+i}\|^2_Q + \varepsilon^\top_{k+i}Q\left(x^*_{i|k} - \bar{x}\right)\right) \\
& + sV_f(\tilde{x}_{N|k+1}, \bar{x}) + \left\|\tilde{x}_{N-1|k+1} - \bar{x}\right\|^2_Q - sV_f\left(x^*_{N|k}, \bar{x}\right)
\end{aligned}$$

Consider now the terms related to the terminal cost. First, note that $\tilde{x}_{N|k+1}$, $x^*_{N+1|k}$ and $\bar{x}$ are bounded in view of the boundedness of $\mathcal{U}$ and of the δISS property of the system. Then, in view of Assumption 1, there exists a Lipschitz constant $\bar{L}_V$ such that

$$|V_f(\tilde{x}_{N|k+1}, \bar{x}) - V_f\left(x^*_{N+1|k}, \bar{x}\right)| \leq \bar{L}_V\|\varepsilon_{k+N+1}\|$$

Then

$$\begin{aligned}
& sV_f(\tilde{x}_{N|k+1}, \bar{x}) + \|\tilde{x}_{N-1|k+1} - \bar{x}\|^2_Q - sV_f\left(x^*_{N|k}, \bar{x}\right) \\
& \leq sV_f\left(x^*_{N+1|k}, \bar{x}\right) + \left\|x^*_{N|k} - \bar{x}\right\|^2_Q - sV_f\left(x^*_{N|k}, \bar{x}\right) \\
& + s\bar{L}_V\|\varepsilon_{k+N+1}\| + \|\varepsilon_{k+N}\|^2_Q + \varepsilon^\top_{k+N}Q\left(x^*_{N|k} - \bar{x}\right) \\
& \leq sV_f\left(f(x^*_{N|k}, \bar{u}), f(\bar{x}, \bar{u})\right) - sV_f\left(x^*_{N|k}, \bar{x}\right) \\
& + \lambda_{max}(Q)\left\|x^*_{N|k} - \bar{x}\right\|^2 \\
& + s\bar{L}_V\|\varepsilon_{k+N+1}\| + \|\varepsilon_{k+N}\|^2_Q + \varepsilon^\top_{k+N}Q\left(x^*_{N|k} - \bar{x}\right) \\
& \overset{(3b)}{\leq} -sc\left\|x^*_{N|k} - \bar{x}\right\|^2 + \lambda_{max}(Q)\left\|x^*_{N|k} - \bar{x}\right\|^2 \\
& + s\bar{L}_V\|\varepsilon_{k+N+1}\| + \|\varepsilon_{k+N}\|^2_Q + \varepsilon^\top_{k+N}Q\left(x^*_{N|k} - \bar{x}\right) \\
& \overset{(7)}{\leq} s\bar{L}_V\|\varepsilon_{k+N+1}\| + \|\varepsilon_{k+N}\|^2_Q + \varepsilon^\top_{k+N}Q\left(x^*_{N|k} - \bar{x}\right)
\end{aligned}$$

Combining the computation, one has that

$$\begin{aligned}
V(x^+, u^+) - V(x, u) \leq\ & -\|x - \bar{x}\|^2_Q - \|u - \bar{u}\|^2_R - \left\|\Delta u^*_{0|k}\right\|^2_{R_\Delta} \\
& + \sum_{i=1}^{N-1}\left(\|\varepsilon_{k+i}\|^2_Q + \varepsilon^\top_{k+i}Q\left(x^*_{i|k} - \bar{x}\right)\right) \\
& + s\bar{L}_V\|\varepsilon_{k+N+1}\| + \|\varepsilon_{k+N}\|^2_Q + \varepsilon^\top_{k+N}Q\left(x^*_{N|k} - \bar{x}\right)
\end{aligned}$$

In view of the boundedness of $\mathcal{U}$ and of the δISS property of the system, the values of $x^*_{i|k}$ are bounded for $i = 1, \dots, N$. Moreover, $\varepsilon_{k+1} = \delta x$ and, for the δISS property of the model, $\|\varepsilon_{k+i+1}\| \leq \beta(\|\delta x\|, i)$ for a $\mathcal{KL}$-function β. Then there exist a $\mathcal{K}$-function $\gamma(\cdot)$ such that

$$V(x^+, u^+) - V(x, u) \leq -\|x - \bar{x}\|^2_Q - \|u - \bar{u}\|^2_R + \gamma(\|\delta x\|) \quad (12)$$

In view of (10)-(11)-(12), the closed-loop system is ISS with respect to the disturbance δx. Convergence follows from the fact that $\delta x \to 0$ for $k \to \infty$. ■

IV. Simulation Example With LSTM Models

This section discusses how the control algorithm proposed in Section III for the generic RNN model can be applied to the particular case of LSTM networks, considering as system under control a simulated four tank benchmark process [16].

A. Algorithm for the LSTM Model

LSTM networks are an architecture in the family of RNN, known to achieve remarkable performances in predictions tasks, in view of their ability to learn both long and short term dependencies between data. The equations of the LSTM network are the following

$$c_{k+1} = \sigma(W_f u_k + U_f h_k + b_f) \circ c_k + \sigma(W_i u_k + U_i h_k + b_i) \circ \tanh(W_c u_k + U_c h_k + b_c) \quad (13a)$$

$$h_{k+1} = \sigma(W_o u_k + U_o h_k + b_o) \circ \tanh(c_{k+1}) \quad (13b)$$

$$y_k = W_y h_k + b_y \quad (13c)$$

where $c, h \in \mathbb{R}^n$ are two vector states, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^p$ is the output of the network. The overall state of the LSTM model is $x = [c^\top \ h^\top]^\top \in \mathbb{R}^{2n}$. Matrices $W_\mathrm{f}, W_\mathrm{i}, W_\mathrm{c}, W_\mathrm{o} \in \mathbb{R}^{n\times m}$, $U_\mathrm{f}, U_\mathrm{i}, U_\mathrm{c}, U_\mathrm{o} \in \mathbb{R}^{n\times n}$, $W_\mathrm{y} \in \mathbb{R}^{p\times n}$ and vectors $b_\mathrm{f}, b_\mathrm{i}, b_\mathrm{c}, b_\mathrm{o} \in \mathbb{R}^n$, $b_\mathrm{y} \in \mathbb{R}^p$ contain the trainable weights of the network. In [8], a condition on the weights of the LSTM model that guarantees δISS is derived. Note that the δISS property is guaranteed in the sets $\mathcal{X}^{ISS} = \mathcal{C} \times \mathcal{H}$ and $\mathcal{U}^{ISS}$, where

$$\mathcal{C} = \left\{c \in \mathbb{R}^n : \|c\|_\infty \leq \bar{c}\right\}$$
$$\mathcal{H} = \{h \in \mathbb{R}^n : \|h\|_\infty \leq 1\}$$
$$\mathcal{U}^{ISS} = \{u \in \mathbb{R}^m : \|u\|_\infty \leq u_{max}\}$$

and $\bar{c}$ is a function of the LSTM weigths. Under the δISS condition, the function

$$V_f(x_a, x_b) = \left\| \begin{bmatrix} \|c_a - c_b\| \\ \|h_a - h_b\| \end{bmatrix} \right\|_P^2 \tag{14}$$

is an incremental Lyapunov function for the LSTM model, where $P \in \mathbb{R}^{2\times 2}$ is the solution of a suitable Lyapunov equation,[1] $x_a = [c_a^\top \ h_a^\top]^\top$ and $x_b = [c_b^\top \ h_b^\top]^\top$. Note that condition (3) is fulfilled with $a = \lambda_{min}(P)$, $b = \lambda_{max}(P)$ and $c = 1$. Therefore, $V_f(x_a, x_b)$ can be used to define the terminal cost for the MPC.

The following observer was proposed in [8] for the LSTM model

$$\hat{c}_{k+1} = \sigma\Big(W_\mathrm{f} u_k + U_\mathrm{f}\hat{h}_k + b_\mathrm{f} + L_\mathrm{f}(y_k - \hat{y}_k)\Big) \circ \hat{c}_k + \sigma\Big(W_\mathrm{i} u_k + U_\mathrm{i}\hat{h}_k + b_\mathrm{i} + L_\mathrm{i}(y_k - \hat{y}_k)\Big) \circ \tanh\Big(W_\mathrm{c} u_k + U_\mathrm{c}\hat{h}_k + b_\mathrm{c}\Big) \tag{15a}$$

$$\hat{h}_{k+1} = \sigma\Big(W_\mathrm{o} u_k + U_\mathrm{o}\hat{h}_k + b_\mathrm{o} + L_\mathrm{o}(y_k - \hat{y}_k)\Big) \circ \tanh\big(\hat{c}_{k+1}\big) \tag{15b}$$

$$\hat{y}_k = W_\mathrm{y}\hat{h}_k + b_\mathrm{y} \tag{15c}$$

The state of the observer is $\hat{x} = [\hat{c}^\top \ \hat{h}^\top]^\top \in \mathbb{R}^{2n}$, and $L_\mathrm{f}, L_\mathrm{i}, L_\mathrm{o} \in \mathbb{R}^{n\times p}$ are the observer gains. Under a proper selection of L_f, L_i and L_o, the LSTM observer (15) provides a converging state estimation, i.e., $\|\hat{x}_k - x_k\| \to 0$ for $k \to \infty$, for any $\hat{x}_0 \in \mathbb{R}^{2n}$.

The observer structure satisfies (4)-(5). In particular, it is possible to prove (5) by following the proof of Theorem 2 of [11] with $d = \hat{d} = 0$. However, it is not guaranteed that $\mathcal{X}^{ISS}$ is an invariant set for the observer state $\hat{x}$, as required by Assumption 2. To circumvent this issue, a tightened input constraint set $\mathcal{U}$ such that $\mathcal{U} \subset \mathrm{int}(\mathcal{U}^{ISS})$ can be considered in the MPC formulation. In fact, if $\mathcal{U} \subset \mathrm{int}(\mathcal{U}^{ISS})$, then there exist a positive invariant set $\mathcal{X} \subset \mathrm{int}(\mathcal{X}^{ISS})$ for the LSTM state. The observer convergence implies that for any $\varepsilon > 0$ there exists a finite $\bar{k}$ such that $\|\hat{x}_k - x_k\| \leq \varepsilon$ for all $k \geq \bar{k}$. Hence, there exists $\bar{k}$ such that $\hat{x}_{\bar{k}}$ is sufficiently close to $x_{\bar{k}}$ to guarantee that $\hat{x}_{\bar{k}} \in \mathcal{X}^{ISS}$, in view of the fact that $x_k \in \mathcal{X}$.

In conclusion, in view of the existence of an incremental Lyapunov (14) for the LSTM model that satisfies Assumption 1, and of an observer that satisfies Assumption 2, the proposed control algorithm can be applied using an LSTM model, provided that the input constraint set is such that $\mathcal{U} \subset \mathrm{int}(\mathcal{U}^{ISS})$.

[1]The Lyapunov equation is $A_\delta^\top P A_\delta - P = -\mathbf{I}_2$, where $A_\delta \in \mathbb{R}^{2\times 2}$ is a Schur matrix that depends on the LSTM weights [8], [11].

B. Simulation

The considered four tanks process is described by the following differential equations

$$\dot{h}_1 = -\frac{a_1}{S}\sqrt{2gh_1} + \frac{a_3}{S}\sqrt{2gh_3} + \frac{\gamma_a}{S}q_a$$
$$\dot{h}_2 = -\frac{a_2}{S}\sqrt{2gh_2} + \frac{a_4}{S}\sqrt{2gh_4} + \frac{\gamma_b}{S}q_b$$
$$\dot{h}_3 = -\frac{a_3}{S}\sqrt{2gh_3} + \frac{1-\gamma_b}{S}q_b$$
$$\dot{h}_4 = -\frac{a_4}{S}\sqrt{2gh_4} + \frac{1-\gamma_a}{S}q_a$$

where the states h_1, h_2, h_3, h_4 are the water levels in the four tanks, and the inputs q_a and q_b represent the inlet flows in the two valves. The numerical values of the system parameters can be found in [16]. The objective is to control the water level in the two bottom tanks, h_1 and h_2, by acting on q_a and q_b. The system is subject to the operational and safety constraints $q_a \in [0.0, 9.05] \times 10^{-4} m^3/s$, $q_b \in [0.0, 11.1] \times 10^{-4} m^3/s$ and $\Delta q_a, \Delta q_b \in [-1.0, 1.0] \times 10^{-4} m^3/s$, where Δq_a and Δq_b are the maximum input variations between subsequent sampling instants. Note that the need for incremental input constraint is motivated by the fact that the system is actuated by pumps, whose acceleration is related to wear and electrical consumption [19].

In order to derive an LSTM model of the plant, a sampling time of $T_s = 25s$ is considered. The model is trained using an input-output dataset composed of 20000 time steps, that is split in a training set of 15000 time steps, a validation set of 2500 time steps, and a test set of 2500 time steps. The training set is then divided into subsequences of 1000 time steps each, and a training of 1000 epochs is performed using a learning rate of 0.001 and Adam optimizer. The loss function is the sum of the mean squared error and of a regularization term used to enforce the δISS property [6]. The model inputs and outputs are normalized within the range [-1, 1], using the maximum and minimum values of the input constraints to normalize the inputs, and the maximum and minimum values in the training dataset to normalize the outputs. The FIT index [12] is considered to assess the predictive performance of the model on test data. The resulting network has $n = 10$ neurons, has a FIT on the test dataset of 93.4% for h_1 and of 87.9% for h_2, and fulfills the δISS condition considering $u_{max} = 1.05$, that is larger that the range of feasible inputs in the normalized variables.

For the design of the control algorithm, the observer is tuned according to the optimization proposed in [8], whereas the cost matrices for the MPC are set to $Q = \mathbf{I}_{2n}$, $R = \mathbf{I}_m$, $R_\Delta = \mathbf{I}_m$ and s is selected according to (7) with the equality. The considered prediction horizon is $N = 10$. The optimization problem is solved numerically using CasADi.

The output, input and input variation of the closed-loop trajectories for the four tanks process are reported respectively in Fig. 2, 3 and 4. It can be seen that the control system is capable of correctly managing the plant and respecting the constraints on the input and on the input variation. Moreover,

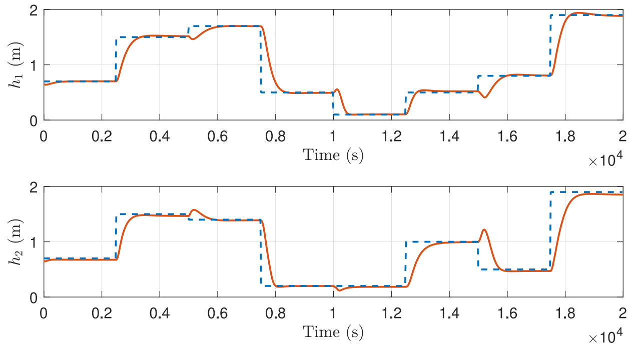

Fig. 2. Closed-loop trajectories: output (orange solid line) and reference (blue dashed line).

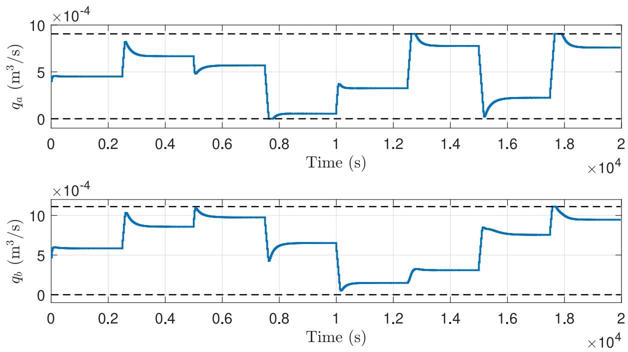

Fig. 3. Closed-loop trajectories: input (blue solid line) and constraints (black dashed lines).

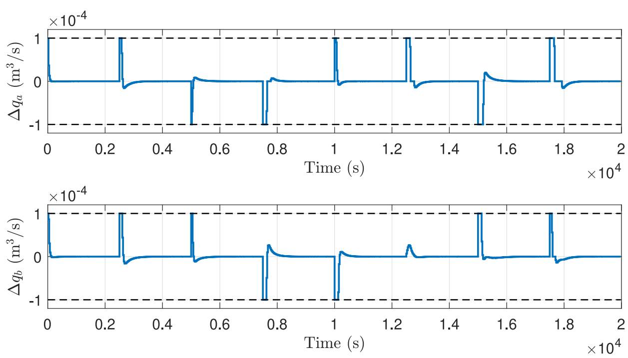

Fig. 4. Closed-loop trajectories: variation of the input (blue solid line) and constraints (black dashed lines).

the simulation shows that the proposed algorithm is able to handle also step-wise variations of the reference, and not only constant references as proven in Theorem 1. In fact the feasibility of the FHOCP only depends on the input, so it is maintained provided that the new reference $\bar{u}$ is such that the current value of the input is in $\mathcal{U}^{MPC}(\bar{u})$.

V. Conclusion

In this letter an MPC algorithm based on a δISS RNN model is proposed for systems with hard constraints on the input and on the input incremental variation. Closed-loop convergence is guaranteed by introducing a general formulation for the terminal cost, while a terminal equality constraint on the optimal input sequence takes into account the presence of the integrator. Remarkably, the proposed approach can be easily applied to different δISS RNN architectures. In the application to LSTM models it is shown how it is possible to manage the case where the δISS property is valid only in an invariant set.

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