Data Informativity for Distributed Positive Stabilization

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***Abstract*—This letter investigates data informativity for distributed positive stabilization. Under the situation that the system model is unavailable but the measurement data are available, we address the problem of finding a distributed controller such that the closed-loop system by state feedback is positive and stable. We clarify that a necessary and sufficient condition for the problem to be solvable is characterized by linear matrix inequalities (LMIs), and derive an LMI-based solution to the problem. To demonstrate our results, a numerical example is provided. Moreover, an application to the vehicle formation is given.**

***Index Terms*—Data informativity, positive systems, distributed control, stabilization, LMI.**

I. Introduction

MODEL-BASED control theory, which relies on the system model, is traditional and many methods on it have been developed. On the other hand, data-driven control theory, which is based on the measurement data of system, is relatively new [1]. On this theory, iterative feedback tuning [2], virtual reference feedback tuning [3], fictitious reference iterative tuning [4], data-driven stabilization by state feedback and linear quadratic regulation [5], data-driven model predictive control [6], etc., have been studied.

In data-driven analysis and design of systems, data informativity has been known. This is a notion which represents the sufficiency of data to solve some analysis or design problem. Based on data informativity, it has been shown that several analysis and designs may be possible even if the data are not sufficient to uniquely determine the system. In [7], an example of stabilization by state feedback from the data that are not sufficient to identify the system has been illustrated.

So far, [7] has discussed on data informativity for stability analysis, identification, controllability analysis, stabilizability analysis, stabilization by state feedback, deadbeat control, linear quadratic regulation, and stabilization by dynamic measurement feedback. In addition, reachability analysis [8], output synchronization [9], H_2 and H_∞ control [10], model reduction [11], and model predictive control [12] have also been studied.

On the other hand, research on positive systems, while classical, has been actively conducted even in recent years. Real-world applications related to positive systems include analysis of the population dynamics of a pest described by a Leslie model [13], structural stability analysis of biochemical reaction networks [14], fluid network and congestion control [15], and stability analysis for SIS epidemiological models [16]. In latest works on positive systems, for example, identification of externally positive systems [17], distributed model predictive control of positive systems [18], and stabilization for a class of positive bilinear systems [19] have been considered. One of the studies on positive systems is distributed positive stabilization [20], [21], which considers stabilizing the system state to always be nonnegative using distributed controllers. To the best of our knowledge, data informativity for distributed positive stabilization has not been investigated in works on data informativity, including [7], [8], [9], [10], [11], [12].

In this letter, we investigate data informativity for distributed positive stabilization. Under the situation that the system model is unavailable but the measurement data are available, we address the problem of finding a distributed controller such that the closed-loop system by state feedback is positive and stable. In the problem, a distributed controller is designed based only on the available data. In this sense, it is more difficult than the model-based design problem [20], [21]. Our results are described as follows. First, we clarify a necessary and sufficient condition for the problem to be solvable. Second, a solution to the problem is derived. To demonstrate our results, we provide a numerical example. Furthermore, an application to the vehicle formation is given.

This letter is related to [7], [22]. However, it differs from their results in the following aspects. In [7] extending the result of [5], it has shown that the necessary and sufficient condition on data informativity for stabilization can be represented by

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the linear matrix inequalities (LMIs). Even by straightforwardly applying the techniques from [7], the necessary and sufficient condition for our problem to be solvable cannot be expressed as LMIs due to the structural constraint of feedback gain derived from distributed control. To address this challenge, we utilize the fact that there exists a positive definite "diagonal" solution to the Lyapunov inequality for stable positive systems [21]. In [20], this fact has been used to propose the LMI-based design method of distributed H_∞ controller based on the model-based approach. Based on the fact, in this letter, we derive a necessary and sufficient condition on data informativity for distributed positive stabilization using LMIs, and provide a data-driven distributed controller design method using the LMI technique. On the other hand, in [22], data-driven positive stabilization has been considered under the assumption of the persistent excitation (PE) condition [23]. Data satisfying the PE condition are informative for identification. Thus, our results have the following advantages over [22]. First, we propose a more generalized version of data-driven positive stabilization termed data-driven "distributed" positive stabilization. Second, we show that data satisfying the PE condition are not essential for data-driven distributed positive stabilization. This suggests that distributed positive stabilization may be achieved even without informative data for identification, and that positive distributed stabilization may be achieved with less data than is required to satisfy the PE condition.

The remainder of this letter is organized as follows: Problem formulation is presented in Section II. Our main results are given in Section III. In Section IV, a numerical example is provided. Moreover, an application to the vehicle formation is also given. Section V concludes this letter.

Notation: Let $\mathbf{R}$ and $\mathbf{R}_{\geq 0}$ be the real number field and the set of nonnegative real numbers, respectively. Let $\mathbf{PD}_n$ be the set of $n \times n$ real diagonal matrices whose all diagonal components are positive. The symbol I represents the identity matrix of appropriate size. For a matrix A, we denote the (i, j)-component of A by $[A]_{i,j}$. A right inverse of matrix A is denoted by $A^\dagger$. We describe the rank of matrix A as $\mathrm{rank}(A)$. If all components of matrix A are nonnegative, we say that A is nonnegative and it is denoted by $A \geq 0$. For a symmetric matrix S, we denote by $S \succ 0$ ($S \prec 0$) that S is a positive definite matrix (a negative definite matrix).

II. Problem Formulation

In this section, we formulate our problems. To do this, several definitions are provided.

A. Distributed Positive Feedback Systems

In this subsection, we explain distributed positive feedback systems.

Consider the system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$ are the state and input, respectively, and $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ are constant matrices. Suppose that the input $u(t)$ is given by

$$u(t) = Fx(t), \tag{2}$$

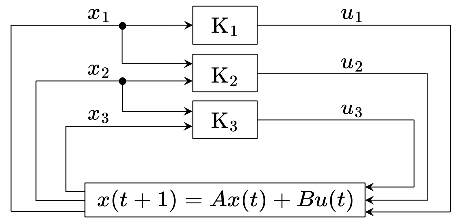

Fig. 1. The control of the system (1) by the distributed controllers K_1, K_2, and K_3. In this case, the distributed indices set is given by (3).

where $F \in \mathbf{R}^{m\times n}$ is a feedback gain.

Now, assume that the system (1) is controlled by m distributed controllers. Let x_i and $u_i(t)$ be the i-th component of $x(t)$ and $u(t)$, respectively. We represent the controller determining $u_i(t)$ as K_i, and suppose that each controller K_i can access to partial information of the state $x(t)$. This can be represented by F where $\mathbf{I} \subseteq \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ is the distributed indices set, and

$$\mathbf{S}(\mathbf{I}) := \{M \in \mathbf{R}^{m\times n} \mid [M]_{i,j} = 0 \quad (\forall (i,j) \notin \mathbf{I})\}$$

is the distributed gain set. Under the structural constraint $F \in \mathbf{S}(\mathbf{I})$, $[F]_{i,j}$ is constrained to be zero for all $(i,j) \notin \mathbf{I}$.

Therefore, if $(i,j) \notin \mathbf{I}$, the controller K_i cannot have access the information on $x_j(t)$ to determine $u_i(t)$.

Let us provide an example of the state feedback under the structural constraint $F \in \mathbf{S}(\mathbf{I})$. Consider the system (1) with 3-dimensional state and 3-dimensional input. Suppose that the distributed indices set

$$\mathbf{I} = \{(1,1), (2,1), (2,2), (3,2), (3,3)\} \tag{3}$$

is given. In this case, the input $u(t)$ is determined by

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} [F]_{1,1} & 0 & 0 \\ [F]_{2,1} & [F]_{2,2} & 0 \\ 0 & [F]_{3,2} & [F]_{3,3} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}. \tag{4}$$

On the state feedback (4), the available state information of K_i $(i = 1, 2, 3)$ can be seen in Fig. 1.

We here provide the notion of $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gains.

Definition 1: Consider the system (1). Suppose that a feedback gain $F \in \mathbf{S}(\mathbf{I})$ is given. If the closed-loop system by the state feedback (2)

$$x(t+1) = (A + BF)x(t) \tag{5}$$

is stable and $x(t) \in \mathbf{R}^n_{\geq 0}$ $(t = 1, 2, \ldots)$ for any nonnegative initial state $x(0) \in \mathbf{R}^n_{\geq 0}$, F is said to be an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain for (A, B).

The closed-loop system (5) by an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain is here called the $\mathbf{S}(\mathbf{I})$-distributed positive feedback system.

B. Data Informativity for Distributed Positive Stabilization

In this subsection, consider the situation that the system model is unknown but the measurement data are given. Under this situation, we address the problem of finding an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain for the system using the given data.

Consider the system

$$x(t+1) = A_s x(t) + B_s u(t), \tag{6}$$

where $A_s \in \mathbf{R}^{n\times n}$ and $B_s \in \mathbf{R}^{n\times m}$ are unknown constant matrices. For the system (6), suppose that the following input and state data are given:

$$U_- := \left[u(0)\ u(1)\ \cdots\ u(T-1)\right],$$
$$X := \left[x(0)\ x(1)\ \cdots\ x(T-1)\ x(T)\right].$$

The above represents the time series of input and state for the system (6) over the time interval $\{0, 1, \dots, T-1\}$.

The pair (U_-, X) is here called the dataset, and denoted by $\boldsymbol{D}$. For the dataset $\boldsymbol{D}$, we define the set

$$\Sigma_{\boldsymbol{D}} := \{(A, B) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \mid X_+ = AX_- + BU_-\},$$

where $X_- := \left[x(0)\ x(1)\ \cdots\ x(T-1)\right]$ and $X_+ := \left[x(1)\ x(2)\ \cdots\ x(T)\right]$ are submatrices of X. Now, $\Sigma_{\boldsymbol{D}}$ is the set of all (A, B) that are consistent with $\boldsymbol{D}$. We see that $(A_s, B_s) \in \Sigma_{\boldsymbol{D}}$ since the dataset $\boldsymbol{D}$ is generated from (6).

On the dataset $\boldsymbol{D}$, we introduce the notion called data informativity for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization. This notion represents the sufficiency of the given dataset $\boldsymbol{D}$ to design an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain.

Definition 2: Consider the system (6). Suppose that the dataset $\boldsymbol{D}$ and the distributed indices set $\mathbf{I} \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ are given. If there exists an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain F for any $(A, B) \in \Sigma_{\boldsymbol{D}}$, the dataset $\boldsymbol{D}$ is said to be informative for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization.

If the condition of Definition 2 holds, it is possible to design an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain for (A_s, B_s) even if (A_s, B_s) cannot be uniquely determined from the dataset $\boldsymbol{D}$. Otherwise, additional data are required to design.

Now, our problems are formulated as follows.

Problem 1: Consider the system (6). Suppose that the dataset $\boldsymbol{D}$ and the distributed indices set $\mathbf{I} \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ are given. Then, determine whether the dataset $\boldsymbol{D}$ is informative for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization or not.

Problem 2: Consider the system (6). Suppose that the dataset $\boldsymbol{D}$ and the distributed indices set $\mathbf{I} \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ are given. Then, find an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain F for any $(A, B) \in \Sigma_{\boldsymbol{D}}$.

As seen from Definition 2, Problem 1 is the problem to verify the feasibility of Problem 2 on the given dataset $\boldsymbol{D}$.

Remark 1: For example, in [24], the problem imposing the constraint on the available information for each controller, such as $F \in \mathbf{S}(\mathbf{I})$, has been considered. This literature has noted that in some complex systems, the use of distributed controllers is important and they may not have access to all information on the system. As examples of such systems, power grids, automobiles on highways, and paper machining have been provided. The application to Eulerian models of air traffic flows in [25] indicates the applicability of distributed controllers. In particular, distributed positive stabilization has been considered in [20], [21], all of which are model-based approaches. However, data-driven distributed positive stabilization and data informativity for it have not been studied. These are expected to be useful when models are difficult to obtain, such as in complex systems.

III. Main Results

In this section, we present our main results for Problems 1 and 2.

A. Solutions to Problems 1 and 2

This subsection provides the theorems that correspond to solutions to Problems 1 and 2.

First, we provide a solution to Problem 1.

Theorem 1: Consider the system (6). Suppose that the dataset $\boldsymbol{D}$ and the distributed indices set $\mathbf{I} \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ are given. The dataset $\boldsymbol{D}$ is informative for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization if and only if there exists a matrix $\Theta \in \mathbf{R}^{T\times n}$ satisfying

$$\begin{bmatrix} X_-\Theta & X_+\Theta \\ \Theta^{\mathrm{T}} X_+^{\mathrm{T}} & X_-\Theta \end{bmatrix} \succ 0, \tag{7}$$
$$X_-\Theta \in \mathbf{PD}_n, \tag{8}$$
$$X_+\Theta \geq 0, \tag{9}$$
$$U_-\Theta \in \mathbf{S}(\mathbf{I}). \tag{10}$$

Next, we also provide a solution to Problem 2.

Theorem 2: Consider Problem 2. If there exists a matrix $\Theta \in \mathbf{R}^{T\times n}$ satisfying (7)–(10), then the matrix $F = U_-\Theta(X_-\Theta)^{-1}$ is an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain for any $(A, B) \in \Sigma_{\boldsymbol{D}}$.

Here, the conditions (7)–(10) are LMIs. Therefore, both Problems 1 and 2 can be solved as convex problems.

If there exists no matrix Θ satisfying (7)–(10) for the given $\boldsymbol{D}$ and $\mathbf{I}$, $\boldsymbol{D}$ is not informative for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization. Then, one strategy to achieve $\mathbf{S}(\mathbf{I})$-distributed positive stabilization in a data-driven manner is to repeatedly collect additional data until the LMI condition is satisfied.

B. Proofs of Theorems 1 and 2

This subsection provides the proofs of Theorems 1 and 2. In order to do this, we define the set

$$\Sigma(F) := \{(A, B) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \mid A + BF \text{ is Schur}\}.$$

For the given matrix $F \in \mathbf{R}^{m\times n}$, this represents the set of (A, B) for which the closed-loop system (5) is stable.

Moreover, we also define the set

$$\Sigma_{\geq 0}(F) := \big\{(A, B) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \mid A + BF \text{ is nonnegative and Schur}\}.$$

This is a subset of $\Sigma(F)$ for which the state of closed-loop system (5) satisfies that $x(t) \in \mathbf{R}^n_{\geq 0}$ $(t = 1, 2, \dots)$ for any nonnegative initial state $x(0) \in \mathbf{R}^n_{\geq 0}$ (see [26]).

If there exists an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain F for any $(A, B) \in \Sigma_{\boldsymbol{D}}$, the following relation holds:

$$\Sigma_{\boldsymbol{D}} \subseteq \Sigma_{\geq 0}(F).$$

Moreover, we introduce the following result, which is well-known in stability analysis of positive systems.

Lemma 1 (See [21]): For the given matrix $A \in \mathbf{R}^{n\times n}_{\geq 0}$, the following statements are equivalent.

1) The matrix A is Schur.
2) There exists a diagonal matrix $P \in \mathbf{PD}_n$ such that $APA^{\mathrm{T}} - P \prec 0$.

Now, the proofs of Theorem 1 and 2 are provided as follows.

[Proof of Theorem 1] To prove the 'if' part of the statement, suppose that there exists a matrix $\Theta \in \mathbf{R}^{T\times n}$ satisfying (7)–(10). Then the proof can be shown from the facts "(7)–(9) $\Rightarrow$ (a)," "(a) $\Rightarrow$ (b)," and "(a), (b), and (10) $\Rightarrow$ (c)" for the following statements.

1) There exists the inverse $(X_-\Theta)^{-1} \in \mathbf{PD}_n$ and $X_+X_-^\dagger$ is nonnegative and Schur, where $X_-^\dagger := \Theta(X_-\Theta)^{-1}$.
2) $X_+X_-^\dagger = A + BU_-X_-^\dagger$ holds for any $(A, B) \in \Sigma_{\boldsymbol{D}}$.
3) $F := U_-X_-^\dagger$ satisfies $F \in \mathbf{S}(\mathbf{I})$ and $\Sigma_{\boldsymbol{D}} \subseteq \Sigma_{\geq 0}(F)$.

First, we prove "(7)–(9) $\Rightarrow$ (a)." From (8), there exists the inverse $(X_-\Theta)^{-1}$ of $X_-\Theta$ and $(X_-\Theta)^{-1} \in \mathbf{PD}_n$. Applying Schur complement to (7), we have

$$X_-\Theta - X_+\Theta(X_-\Theta)^{-1}\Theta^{\mathrm{T}}X_+^{\mathrm{T}} \succ 0. \tag{11}$$

Because $X_-\Theta(X_-\Theta)^{-1} = I$, (11) can also be represented as

$$X_+\Theta(X_-\Theta)^{-1}X_-\Theta(X_-\Theta)^{-1}\Theta^{\mathrm{T}}X_+^{\mathrm{T}} - X_-\Theta \prec 0. \tag{12}$$

We here define the matrix $X_-^\dagger := \Theta(X_-\Theta)^{-1}$. It follows from $(X_-\Theta)^{-1} \in \mathbf{PD}_n$ and (9) that $X_+X_-^\dagger = X_+\Theta(X_-\Theta)^{-1}$ is a nonnegative matrix. Moreover, from this and (8), we can apply Lemma 1 to (12). Therefore, we see that $X_+X_-^\dagger$ is Schur.

Second, we prove "(a) $\Rightarrow$ (b)." Because $X_-^\dagger := \Theta(X_-\Theta)^{-1}$ is a right inverse of X_-, we see that

$$X_+X_-^\dagger = \begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} X_- \\ U_- \end{bmatrix}X_-^\dagger = A + BU_-X_-^\dagger \tag{13}$$

holds for any $(A, B) \in \Sigma_{\boldsymbol{D}}$.

Finally, we prove "(a), (b), and (10) $\Rightarrow$ (c)." We define the matrix $F := U_-X_-^\dagger$. It follows from $(X_-\Theta)^{-1} \in \mathbf{PD}_n$ and (10) that $F = U_-X_-^\dagger = U_-\Theta(X_-\Theta)^{-1} \in \mathbf{S}(\mathbf{I})$. Since (a) and (b) hold, $X_+X_-^\dagger = A + BF$ is nonnegative and Schur for any $(A, B) \in \Sigma_{\boldsymbol{D}}$. From the above, F satisfies $F \in \mathbf{S}(\mathbf{I})$ and $\Sigma_{\boldsymbol{D}} \subseteq \Sigma_{\geq 0}(F)$.

Next, to prove the 'only if' part of the statement, suppose that F is an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain for any $(A, B) \in \Sigma_{\boldsymbol{D}}$. Then we prove that there exists a matrix $\Theta \in \mathbf{R}^{T\times n}$ satisfying (7)–(10).

Since F is an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain for any $(A, B) \in \Sigma_{\boldsymbol{D}}$, the relation $\Sigma_{\boldsymbol{D}} \subseteq \Sigma_{\geq 0}(F)$ holds. From $\Sigma_{\geq 0}(F) \subseteq \Sigma(F)$, we can apply Lemma 2 in Appendix. Therefore, we see that X_- has full row rank and F is of the form $F = U_-X_-^\dagger$ for $X_-^\dagger$ satisfying that $X_+X_-^\dagger$ is Schur. Because $X_-^\dagger$ is a right inverse of X_- and $\Sigma_{\boldsymbol{D}} \subseteq \Sigma_{\geq 0}(F)$, (13) holds and this is nonnegative and Schur for any $(A, B) \in \Sigma_{\boldsymbol{D}}$. Applying Lemma 1 in Appendix to $X_+X_-^\dagger$, there exists a diagonal matrix $P \in \mathbf{PD}_n$ such that

$$X_+X_-^\dagger P\left(X_+X_-^\dagger\right)^{\mathrm{T}} - P \prec 0. \tag{14}$$

We here define the matrix

$$\Theta := X_-^\dagger P \tag{15}$$

for some diagonal matrix $P \in \mathbf{PD}_n$ satisfying (14). Now we prove that (7)–(10) hold for the matrix Θ.

First, we prove (7) and (8). Using the matrix Θ, (14) can also be expressed as

$$X_+\Theta P^{-1}\Theta^{\mathrm{T}}X_+^{\mathrm{T}} - P \prec 0. \tag{16}$$

Applying Schur complement to (16), we obtain

$$\begin{bmatrix} P & X_+\Theta \\ \Theta^{\mathrm{T}}X_+^{\mathrm{T}} & P \end{bmatrix} \succ 0. \tag{17}$$

Multiplying X_- from the left-hand side of (15), we see that

$$P = X_-\Theta. \tag{18}$$

Substituting (18) into (17), we have (7). Because $P = X_-\Theta \in \mathbf{PD}_n$, we also have (8).

Next, we prove (9). From $X_+X_-^\dagger \geq 0$, $P = X_-\Theta \in \mathbf{PD}_n$, and (15), we see that $X_+X_-^\dagger P = X_+\Theta \geq 0$.

Finally, we prove (10). Because $P \in \mathbf{PD}_n$, there exists the inverse P^{-1} of P. From this and (15), we have

$$X_-^\dagger = \Theta P^{-1}. \tag{19}$$

Substituting (18) into (19), we obtain $X_-^\dagger = \Theta(X_-\Theta)^{-1}$. Since this holds and F is of form $F = U_-X_-^\dagger$, we see that

$$F = U_-\Theta(X_-\Theta)^{-1}. \tag{20}$$

It follows from $F \in \mathbf{S}(\mathbf{I})$, (8), and (20) that $F(X_-\Theta) = U_-\Theta \in \mathbf{S}(\mathbf{I})$. This completes the proof.

[Proof of Theorem 2] The proof can be provided by the proof of the 'if' part of the statement of Theorem 1.

IV. Numerical Examples

In this section, we provide the following examples to demonstrate our results. Examples 1 and 2 show that identification is not required for distributed positive stabilization.

Example 1: Consider the system (6), where

$$A_{\mathrm{s}} = \begin{bmatrix} 1.1 & 0.1 & 0.3 \\ 0.1 & 0.3 & 0.1 \\ 0.2 & 0.2 & 1.0 \end{bmatrix}, \quad B_{\mathrm{s}} = \begin{bmatrix} 0.9 & 0.0 & 0.1 \\ 0.1 & 0.6 & 0.1 \\ 0.0 & 0.2 & 0.8 \end{bmatrix}. \tag{21}$$

We see that the matrix A_{s} in (21) is not Schur because its eigenvalues are 0.2708, 0.8, and 1.3292. Suppose that A_{s} and B_{s} are unknown but the following data are available:

$$U_- = \begin{bmatrix} 1.0 & 0.8 & 0.6 & 0.5 & 0.4 \\ 0.8 & 0.6 & 0.4 & 0.3 & 0.2 \\ 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \end{bmatrix},$$
$$X = \begin{bmatrix} 0 & 0.95 & 2.036 & 3.2864 & 4.8383 & 6.7794 \\ 0 & 0.63 & 0.820 & 0.9112 & 1.0727 & 1.3023 \\ 0 & 0.56 & 1.316 & 2.2072 & 3.2667 & 4.5689 \end{bmatrix}. \tag{22}$$

We here give the remark for the dataset $\boldsymbol{D} = (U_-, X)$. The dataset $\boldsymbol{D}$ does not satisfy the PE condition because $\mathrm{rank}(\begin{bmatrix} X_-^{\mathrm{T}} & U_-^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}) = 5 < n + m$. In other words, $\boldsymbol{D}$ is not informative for identification [7, Proposition 6]. Therefore, it is not possible to design $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gains based on $(A_{\mathrm{s}}, B_{\mathrm{s}})$ identified from $\boldsymbol{D}$.

Under the situation that the dataset $\boldsymbol{D}$ and the distributed indices set $\mathbf{I}$ in (3) are given, we first consider Problem 1. By solving LMIs in (7)–(10), we obtain

$$\Theta = \begin{bmatrix} 5.5097 & -4.6275 & -15.8172 \\ -11.9852 & 12.7322 & 53.7382 \\ 124.0195 & -21.9785 & -178.7098 \\ -229.3669 & 21.5912 & 237.3701 \\ 107.0679 & -7.9169 & -96.5813 \end{bmatrix}. \tag{23}$$

Therefore, from Theorem 1, we see that the dataset $\boldsymbol{D}$ is informative for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization.

Next, we consider Problem 2. From Theorem 2, we obtain

$$F = U_-\Theta(X_-\Theta)^{-1} = \begin{bmatrix} -0.2844 & 0 & 0 \\ -0.1068 & 0.0338 & 0 \\ 0 & -0.2439 & -0.6637 \end{bmatrix} \quad (24)$$

using Θ in (23). For the distributed indices set $\mathbf{I}$ in (3), the matrix in (24) satisfies $F \in \mathbf{S}(\mathbf{I})$.

In fact, applying the $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain F in (24) to the system (6), we obtain

$$A_{\mathrm{s}} + B_{\mathrm{s}}F = \begin{bmatrix} 0.8440 & 0.0756 & 0.2336 \\ 0.0075 & 0.2959 & 0.0336 \\ 0.1786 & 0.0116 & 0.4691 \end{bmatrix}. \quad (25)$$

This matrix is nonnegative and Schur from that its eigenvalues are 0.2409, 0.3961, and 0.9556. Therefore, we see that the close-loop system with the matrix in (25) is an $\mathbf{S}(\mathbf{I})$-distributed positive feedback system.

Moreover, we conduct an experiment to investigate the minimal informative dataset for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization. In this experiment, we check for the existence of a matrix Θ that satisfies the LMI condition of Theorem 1 for 1000 randomly generated datasets with $T = 4$. As a result, there is no matrix Θ satisfying the LMI condition. Thus, the experiment implies that the dataset in (22) with $T = 5$ is a minimal dataset in Example 1.

Example 2: Consider the system composed of one leader vehicle and three follower vehicles on a straight line, as shown in Fig. 2. The leader runs autonomously. The follower $\#k$ $(k = 1, 2, 3)$ has the given reference value of distance to the one forward vehicle, and it is denoted by x_k^0. Suppose that the vehicle $\#k$ $(k = 1, 2, 3)$ can obtain the distance between itself and the one forward vehicle using the distance sensor. Then the distance can be expressed as $x_k^0 + x_k$, where x_k is the deviation between the position of vehicle $\#k$ and x_k^0. Based on the information $x_k^0 + x_k$ and x_k^0, the vehicle $\#k$ can obtain x_k. Additionally, the vehicle $\#k$ $(k = 2, 3)$ can obtain the information x_{k-1} through digital communication with the one forward follower vehicle $\#k - 1$. Therefore, the vehicle $\#1$ has only x_1, while the vehicle $\#k$ $(k = 2, 3)$ can obtain x_k and x_{k-1}. Moreover, the input u_k of follower $\#k$ $(k = 1, 2, 3)$ is the adjustment value determined by the available state information.

For the above, suppose that the dynamics of follower $\#k$ $(k = 1, 2, 3)$ is given as follows:

$$x_k(t+1) = x_k(t) + (u_k(t) - \alpha_k x_k(t)), \quad (26)$$

where $\alpha_k \in [0, 1)$ $(k = 1, 2, 3)$ is the parameter that represents wind resistivity, and the value of $\alpha_k x_k(t)$ becomes smaller as the distance between vehicles decreases. The dynamics in (26) can also be described as

$$x_k(t+1) = (1 - \alpha_k)x_k(t) + u_k(t). \quad (27)$$

Moreover, based on (27), the system model for the vehicle formation can be denoted by (6), where

$$A_{\mathrm{s}} = \begin{bmatrix} 1-\alpha_1 & 0 & 0 \\ 0 & 1-\alpha_2 & 0 \\ 0 & 0 & 1-\alpha_3 \end{bmatrix}, \; B_{\mathrm{s}} = I,$$

$x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^{\mathrm{T}}$, and $u(t) = [u_1(t)\ u_2(t)\ u_3(t)]^{\mathrm{T}}$.

For this system, the scenario is given as follows. Consider stabilizing the state of each follower to the reference value x_k^0 by state feedback. While, note that each distance between vehicles must not be closer than x_k^0 to avoid vehicle collisions. This corresponds to guaranteeing that $x(t) \in \mathbf{R}^n_{\geq 0}$ $(t = 1, 2, \ldots)$ for any nonnegative initial state $x(0) \in \mathbf{R}^n_{\geq 0}$. From the limitation that each follower can access the available state information, Since each follower can only access limited state information, it is necessary to design a stabilizing feedback gain F such that $F \in \mathbf{S}(\mathbf{I})$, where the distributed indices set $\mathbf{I}$ is given by (3). Considering the situation that A_{s} and B_{s} are unknown but the dataset $\boldsymbol{D}$ is given, we address the problem of finding an $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain F for $(A_{\mathrm{s}}, B_{\mathrm{s}})$. This is exactly a case of Problem 2.

In this example, let $\alpha_1 = 0$, $\alpha_2 = 0.1$, and $\alpha_3 = 0.2$, and suppose that the following data are given:

$$U_- = \begin{bmatrix} 0 & 0.9 & 1.2 & 1.5 & 1.6 \\ 0 & 0.7 & 1.1 & 1.3 & 1.4 \\ 0 & 0.5 & 1.0 & 1.2 & 1.3 \end{bmatrix},$$
$$X = \begin{bmatrix} 3 & 3.0 & 3.90 & 5.100 & 6.6000 & 8.2000 \\ 2 & 1.8 & 2.32 & 3.188 & 4.1692 & 5.1523 \\ 1 & 0.8 & 1.14 & 1.912 & 2.7296 & 3.4837 \end{bmatrix}. \quad (28)$$

As in Example 1, this dataset $\boldsymbol{D} = (U_-, X)$ also does not satisfy the PE condition.

Under this situation, we first consider Problem 1 to verify the feasibility of Problem 2. By solving LMIs in (7)–(10), we obtain the matrix

$$\Theta = \begin{bmatrix} -0.0053 & 0.0262 & -0.0197 \\ 0.0014 & -0.0169 & 0.0045 \\ 0.0270 & -0.0542 & -0.0051 \\ -0.0519 & 0.1357 & -0.0523 \\ 0.0266 & -0.0771 & 0.0503 \end{bmatrix}. \quad (29)$$

It follows from Theorem 1 that the dataset $\boldsymbol{D}$ is informative for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization.

Moreover, we consider Problem 2. From Theorem 2, we obtain a solution

$$F = U_-\Theta(X_-\Theta)^{-1} = \begin{bmatrix} -0.3682 & 0 & 0 \\ 0.0995 & -0.3871 & 0 \\ 0 & 0.0001 & -0.0128 \end{bmatrix} \quad (30)$$

using Θ in (29). For the distributed indices set $\mathbf{I}$ in (3), we see that (30) satisfies $F \in \mathbf{S}(\mathbf{I})$.

In fact, applying the $\mathbf{S}(\mathbf{I})$-distributed positive stabilizing gain F in (30) to the system (6), we obtain the matrix

$$A_{\mathrm{s}} + B_{\mathrm{s}}F = \begin{bmatrix} 0.6318 & 0 & 0 \\ 0.0995 & 0.5129 & 0 \\ 0 & 0.0001 & 0.7872 \end{bmatrix}. \quad (31)$$

We can verify this matrix is nonnegative and Schur from that its eigenvalues are 0.7010, 0.7863, and 0.7872. Therefore, we see that the close-loop system with the matrix in (31) is an $\mathbf{S}(\mathbf{I})$-distributed positive feedback system.

An example of vehicle formation represented by the distributed positive feedback system with the matrix in (31) is provided as follows. Here, let $x_1^0 = x_2^0 = x_3^0 = 5$, and $x(0) = [4\quad 8\quad 12]^{\mathrm{T}}$. Fig. 3 illustrates the time evolution of each

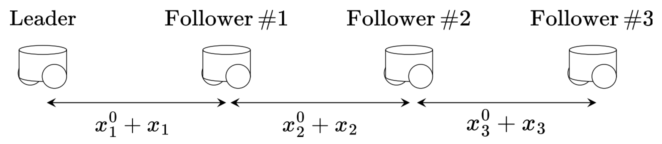

Fig. 2. The formation of one leader vehicle and three follower vehicles.

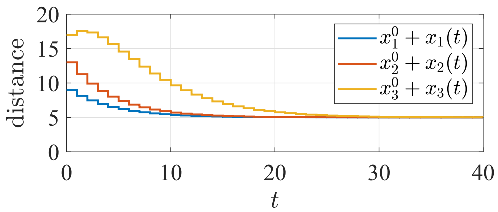

Fig. 3. The time evolution of each distance between vehicles $x_k^0 + x_k(t)$ ($k = 1, 2, 3$).

distance between vehicles $x_k^0 + x_k(t)$ ($k = 1, 2, 3$). Thus, we see that all distances converge to the reference values.

As a similar manner to Example 1, we conduct an experiment to investigate the minimal informative dataset for $\mathbf{S}(\mathbf{I})$-distributed positive stabilization. The result implies that the dataset in (28) with $T = 5$ is a minimal dataset.

V. Conclusion

In this letter, we discussed on data informativity for distributed positive stabilization. For the data-driven distributed positive stabilization, we clarified a necessary and sufficient condition for the problem to be solvable. It has been shown that this condition is characterized by LMIs. Moreover, an LMI-based solution to the problem has been provided. Our main results have been demonstrated by numerical examples.

Appendix

To prove Theorems 1 and 2, the following definition and lemma are given.

Definition 3 (See [7]): Consider the system (6). Suppose that the dataset $\boldsymbol{D}$ is given. If there exists an $F \in \mathbf{R}^{m\times n}$ such that (5) is stable for any $(A, B) \in \Sigma_{\boldsymbol{D}}$, the dataset $\boldsymbol{D}$ is said to be informative for stabilization.

Lemma 2 (See [7]): Consider the system (6). Suppose that the dataset $\boldsymbol{D}$ is given. Then, the following statements hold.

1) The dataset $\boldsymbol{D}$ is informative for stabilization if and only if X_- has full row rank and there exists a right inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is Schur.
2) The matrix F satisfies $\Sigma_{\boldsymbol{D}} \subseteq \Sigma(F)$ if and only if F is of the form $F = U_-X_-^\dagger$ for the matrix $X_-^\dagger$ satisfying that $X_+X_-^\dagger$ is Schur.

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