A Note on Impulsive Solutions to Nonlinear Control Systems

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***Abstract*—In the last decades, many authors provided different notions of *impulsive process*, seen as a suitably defined limit of a sequence of ordinary processes for a nonlinear control-affine system with unbounded, vector-valued controls. In particular, we refer to the impulsive processes introduced by Karamzin et al. –in which the control is given by a vector measure, a non-negative scalar measure, and a family of so-called attached controls that univocally determine the jumps of the corresponding trajectory– and to the graph completion processes developed by Bressan and Rampazzo et al. –in which an impulsive trajectory is seen as a spatial projection of a Lipschitzian trajectory in space-time. The equivalence between these notions is the crucial assumption of most results on optimal impulsive control problems, such as existence of an optimal process and necessary/sufficient optimality conditions. In this note we exhibit a counterexample which shows that, in presence of state constraints and endpoint constraints involving the total variation of the impulsive control, this equivalence may fail. Thus, we propose to replace the set of impulsive processes with a smaller class of impulsive processes, that we call *admissible*, which turns out to be actually in one-to-one correspondence with the set of graph completion processes.**

***Index Terms*—Impulsive control, optimal control, well-posedness of solutions.**

I. Introduction

In THIS letter we compare well-known concepts of generalized controls and corresponding generalized solutions for the following control system

$$\begin{cases} \frac{dx}{dt}(t) \ = f(t,x(t)) + \sum_{j=1}^m g_j(t,x(t))\, u_j(t), \\ \frac{dV}{dt}(t) = \|u(t)\|, \\ u(t) \in \mathcal{K} \quad \text{a.e. } t \in [0,T], \\ (x,v)(0) = \big(\check{x}_0, 0\big), \end{cases} \tag{S}$$

when the L^1 control $u = (u_1, \dots, u_m)$ –i.e., the measure $u(t)\,dt$ which is absolutely continuous with respect to the Lebesgue measure– is replaced with a finite and regular vector-valued measure μ, with range contained in $\mathcal{K}$, and the corresponding solution $(x,V) : [0,T] \to \mathbb{R}^{n+1}$ is a function of bounded variation. Notice that, for any $u \in L^1$, v is nothing but the total variation function of $u(t)\,dt$, as we set $\|w\| := \sum_{j=1}^m |w_j|$ for any $w \in \mathbb{R}^m$. The data comprise a fixed final time $T > 0$, an initial state $\check{x}_0 \in \mathbb{R}^n$, a closed convex cone $\mathcal{K} \subseteq \mathbb{R}^m$, and C^1, bounded functions[1] $f, g_j : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, $j = 1, \dots, m$. Under these hypotheses, for any control function $u \in L^1([0,T],\mathcal{K})$ there exists one and only one corresponding solution (x,v) to (S). We will refer to the triple (u,x,V) as a *strict sense process* for (S).

In several applications, e.g., to aerospace [6] or mechanics [7], [11], implementing an 'impulsive control' associated with (S) involves idealizing a highly intense control action within a short time interval. The appropriate definition of impulsive process is then that it should be the limit (in a suitable sense) of some sequence of strict sense processes. However, as it is well-known, the impulsive control cannot be simply identified with a limit measure μ, as different absolutely continuous approximations of μ can give rise to different state trajectories in the limit, unless we impose strict 'commutativity' conditions on the g_j's (see [9], [10], [12]).

In [19], [20] it has been proposed, for the *impulsive system*

$$\begin{cases} dx(t) \ = f(t,x(t))\,dt + \sum_{j=1}^m g_j(t,x(t))\,\mu_j(dt), \\ dV(t) = \nu(dt), \quad t \in [0,T], \\ \text{range}(\mu) \subseteq \mathcal{K}, \\ (x,v)(0) = \big(\check{x}_0, 0\big), \end{cases} \tag{IS}$$

a notion of *impulsive control* which includes, in addition to the vector-valued measure μ, a scalar, non negative measure ν, limit total variation of a specific approximating sequence to μ, and 'attached' controls describing instantaneous state evolution at each atom of ν. In this way, to each impulsive control it corresponds a unique solution (x,V) of (IS) (see Section III for the precise definitions).

Adopting instead the so-called graph completion approach in [8], [9], [21], (S) is embedded into the *space-time system*

$$\begin{cases} \frac{dy_0}{ds}(s) = \omega_0(s), \\ \frac{dy}{ds}(s) \ = f(y_0(s),y(s))\,\omega_0(s) + \sum_{j=1}^m g_j(y_0(s),y(s))\ \omega_j(s), \\ \frac{d\beta}{ds}(s) = \|\omega(s)\|, \\ (\omega_0,\omega)(s) \in W(\mathcal{K}) \text{ a.e. } s \in [0,S], \\ (y_0,y,\beta)(0) = \big(0,\check{x}_0,0\big), \quad y_0(S) = T, \end{cases} \tag{STS}$$

[1] These hypotheses, assumed for simplicity's sake, could be weakened.

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where the new state variable is $(y_0, y, \beta) := (t, x, V)$ and

$$W(\mathcal{K}) := \{(w_0, w) \in]0, +\infty[\times \mathcal{K}\colon w_0 + \|w\| = 1\}. \quad (1)$$

If $t = y_0(s)$ is strictly increasing, i.e., $\omega_0 > 0$ a.e., a *space-time process* $(S, \omega_0, \omega, y_0, y, \beta)$ is simply a graph reparameterization of a strict sense process. However, (STS) allows to define discontinuous solutions, so-called *graph completion solutions*, to (S), as soon as $\omega_0 = 0$ on nondegenerate intervals. In this case, instead of attaching to a given measure μ a family of additional controls, one 'completes' the graph of $U(t) = \int_{[0,t]} \mu(dt')$ at the discontinuity points, and considers an arclength-type, 1-Lipschitz continuous parameterization (φ_0, φ) of this graph completion on some interval $[0, S]$. Then, the space-time control $(S, \omega_0, \omega) := (S, \frac{d\varphi_0}{ds}, \frac{d\varphi}{ds})$ and the corresponding solution (y_0, y, β) to (STS) identify the graph completion solution $(x, v) := (y, \beta) \circ \sigma$ to (S), in which σ is the right inverse of $t = y_0(s)$ (see Section III).

Our main goal is to compare the impulsive extension, say (P_{imp}), and the graph completion extension, say (P_{gc}), of the following optimization problem

$$\begin{cases} \text{minimize } \Psi(x(T), V(T)), \\ \text{over strict sense processes } (u, x, V) \text{ such that} \\ h(t, x(t)) \le 0, \quad t \in [0, T], \qquad (\textit{state constraint}) \\ (x(T), V(T)) \in \mathcal{S}, \qquad (\textit{terminal constraint}) \end{cases} \quad (P)$$

in which strict sense solutions are replaced by impulsive and graph completion solutions, respectively. (Indeed, according, e.g., to [4], [20], [22], [25], *feasible* solutions satisfy the state constraint also in their 'instantaneous evolution', during jumps –see Section III.) Let $\Psi : \mathbb{R}^{n+1} \to \mathbb{R}$, $h : \mathbb{R}^{1+n} \to \mathbb{R}^k$ be continuous,[2] and $\mathcal{S} \subseteq \mathbb{R}^{n+1}$ closed.

In impulsive optimal control, it is a common procedure to prove a one-to-one correspondence between impulsive processes and space-time processes, and then use it to derive existence of optimal controls and necessary/sufficient optimality conditions for the impulsive problem from the following space-time optimal control problem

$$\begin{cases} \text{minimize } \Psi(y(S), \beta(S)), \\ \text{over space-time processes } (S, \omega_0, \omega, y_0, y, \beta) \text{ such that} \\ h(y_0(s), y(s)) \le 0, \quad s \in [0, S], \\ (y, \beta)(S) \in \mathcal{S} \end{cases} \quad (P_{st})$$

equivalent to (P_{gc}) by definition. Note that (P_{st}) is a conventional optimization problem with bounded, measurable controls (ω_0, ω), to which classical results apply.

In this letter we show, by means of a counterexample, that the set of impulsive processes is not in one-to-one correspondence with the set of space-time processes and the minimum of (P_{imp}) may be smaller than the minimum of (P_{st}) (see Section V). We then introduce, in Section VI, the subset of *admissible* impulsive processes, for which we prove that the above mentioned correspondence is instead valid.

II. Notations and Some Preliminaries

Given $T > 0$ and a set $X \subseteq \mathbb{R}^k$, we write $L^1([0, T], X)$ and $BV([0, T], X)$ for the set of Lebesgue integrable and of bounded variation functions defined on $[0, T]$ and with values in X, respectively. We denote by $C^*([0, T], \mathbb{R}^k)$ the set of signed, finite and regular vector-valued measures from the Borel subsets of $[0, T]$ to $\mathbb{R}^k$. Moreover, we set $C^\oplus([0, T])$ for the elements of $C^*([0, T], \mathbb{R})$ taking nonnegative values and we define

$$C^*_X([0, T]) := \left\{\mu \in C^*\left([0, T], \mathbb{R}^k\right) : \ \text{range}(\mu) \subseteq X\right\}.$$

Given $\mu \in C^*([0, T], \mathbb{R}^k)$, $|\mu| \in C^\oplus([0, T])$ denotes the *total variation measure*, i.e., $|\mu| := \sum_{j=1}^k |\mu_j|$, while μ^c denotes the continuous component of μ with respect to the Lebesgue measure ℓ. Given a sequence $(\mu_i) \subset C^*([0, T], \mathbb{R}^k)$ and $\mu \in C^*([0, T], \mathbb{R}^k)$, we write $\mu_i \rightharpoonup^* \mu$ if $\lim_i \int_{[0,T]} \psi(t)\mu_{j_i}(dt) = \int_{[0,T]} \psi(t)\mu_j(dt)$, for all continuous maps $\psi : [0, T] \to \mathbb{R}$ and $j = 1, \dots, k$.

For any function $\varphi : [0, T] \to X$, for any $t \in]0, T[$ we write $\varphi(t^-)$ and $\varphi(t^+)$ to denote the left and the right limit of φ at t (if it exists). In particular, we set $\varphi(0^-) = \varphi(0)$ and $\varphi(T^+) = \varphi(T)$. Given $a, b \in]0, +\infty[$ and a (possibly not strictly) increasing function $\Gamma:[0, a] \to [0, b]$ such that $\Gamma(0) = 0$, $\Gamma(a) = b$, and Γ is right continuous on $]0, a[$, we define its *right inverse* as the function $\Lambda : [0, b] \to [0, a]$ such that $\Lambda(0) = 0$, $\Lambda(b) := a$, and $\Lambda(r) := \inf\{s \in [0, a]\colon \ \Gamma(s) > r\}$ for $r \in]0, b[$.

III. Impulsive Processes and Graph Completions

A. Impulsive Controls and Trajectories

In this subsection we recall the concepts of *impulsive control* and corresponding *impulsive solution* to (IS) as introduced in [19] (and adopted, for instance, in [4], [20]).

Let $T > 0$ and $\mathcal{K} \subset \mathbb{R}^m$ be the final time and the closed convex cone considered in the Introduction, respectively. Given a measure $\mu \in C^*_{\mathcal{K}}([0, T])$, define the set

$$\mathcal{V}(\mu) := \{\nu \in C^\oplus([0, T]){:}\exists(\mu_i) \subset C^*_{\mathcal{K}}([0, T]) \text{ such that } \ (\mu_i, |\mu_i|) \rightharpoonup^* (\mu, \nu)\}. \quad (2)$$

In general, if $\nu \in \mathcal{V}(\mu)$, then one has $\nu \ge |\mu|$ and $|\mu|$ always belongs to $\mathcal{V}(\mu)$. Actually, if the range of μ is contained in a closed convex cone which belongs to one of the orthants of $\mathbb{R}^m$, then $\mathcal{V}(\mu) = \{|\mu|\}$.

Definition 1: An *impulsive control* for control system (IS) is an element $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]})$ comprising two Borel measures $\mu \in C^*_{\mathcal{K}}([0, T])$, $\nu \in \mathcal{V}(\mu)$, and a family of essentially bounded, measurable functions $\alpha^r : [0, 1] \to \mathcal{K}$ parameterized by $r \in [0, T]$, with the following properties:

(i) $\|\alpha^r(s)\| = \nu(\{r\})$ for a.e. $s \in [0, 1]$,

(ii) $\int_0^1 \alpha^r_j(s)\, ds = \mu_j(\{r\})$ for all $j = 1, \dots, m$.

A family of functions $\{\alpha^r\}_{r\in[0,T]}$ that satisfies conditions (i)–(ii) above is said to be *attached* to (μ, ν).[3] We refer to ν as the *total variation* of the impulsive control.

Definition 2: Let $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]})$ be an impulsive control. Then, $(x, v) \in BV([0, T], \mathbb{R}^{n+1})$ is an *impulsive solution* to (IS) corresponding to $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]})$ and $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]}, x, v)$

[2] As customary, $h \le 0$ means $h_j \le 0$ for any $j = 1, \dots, k$.

[3] By Def. 1, the set of atoms of μ is always a subset of the (countable) set the atoms of ν. Accordingly, we might have that ν has atoms, while μ does not. Moreover, (i) implies that $\alpha^r \ne 0$ for a countable set of r's only.

is an *impulsive process* if $(x, V)(0) = (\check{x}_0, 0)$ and, for $t \in]0, T]$,

$$\begin{aligned} x(t) = \check{x}_0 &+ \int_0^t f\big(t', x(t')\big)dt' \\ &+ \int_{[0,t]} \sum_{j=1}^m g_j\big(t', x(t')\big)\mu_j^c\big(dt'\big) \\ &+ \sum_{r\in[0,t]} \big(\zeta^r(1) \ - x(r^-)\big), \\ V(t) = \nu&([0, t]), \end{aligned}$$

where, for $r \in [0, T]$, the function $\zeta^r : [0, 1] \to \mathbb{R}^n$ satisfies

$$\begin{cases} \frac{d\zeta^r}{ds}(s) \ = \sum_{j=1}^m g_j(r, \zeta^r(s))\alpha_j^r(s), \text{ a.e. } s \in [0, 1], \\ \zeta^r(0) = x\big(r^-\big). \end{cases} \tag{3}$$

An impulsive process $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]}, x, v)$ is *feasible* for (P_{imp}) if $h(t, x(t)) \le 0$ for any $t \in [0, T]$, $h(r, \zeta^r(s)) \le 0$ for any $s \in [0, 1]$, $r \in [0, T]$, and $(x(T), V(T)) \in \mathcal{S}$.

We call *strict sense control* an impulsive control $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]})$ such that ν is absolutely continuous with respect to the Lebesgue measure and $\nu = |\mu|$. In this case, with a slight abuse of notation, we also call strict sense control the function $u \in L^1([0, T], \mathcal{K})$ such that $\mu(dt) = u(t)\,dt$ and $\nu(dt) = |u(t)|\,dt$.[4] This is justified, because the impulsive trajectory (x, v) corresponding to u is nothing but the classical, strict sense solution to (S) associated with u.

Hence, system (IS) can be interpreted as an extension of system (S), so we will sometimes call an impulsive process for (IS) also an *impulsive process for (S)*.

B. Graph Completion Controls and Trajectories

We now summarize the so-called graph completion approach, as introduced in [9] (see also [1], [8], [21], [24]).

For $W(\mathcal{K})$ as in (1), define the *set of space-time controls:*

$$\mathcal{W} := \bigcup_{S>0} \{S\} \times L^1([0, S]; W(\mathcal{K})).$$

Definition 3: For any $(S, \omega_0, \omega) \in \mathcal{W}$, we say that the absolutely continuous path $(y_0, y, \beta) : [0, S] \to \mathbb{R}^{1+n+1}$ is the corresponding *space-time trajectory* if it satisfies (STS). We call $(S, \omega_0, \omega, y_0, y, \beta)$ a *space-time process* for (STS). It is *feasible* for (P_{st}) if it satisfies the constraints in (P_{st}).

Note that the set of strict sense processes (u, x, V) for (S) is in one-to-one correspondence with the subset of space-time processes $(S, \omega_0, \omega, y_0, y, \beta)$ for (STS) with $\omega_0 > 0$ a.e.. Indeed, with each (u, x, V), by means of the inverse y_0 of the following arc length-type reparameterization

$$\sigma(t) := t + V(t) = t + \int_0^t \big|u\big(t'\big)\big|\,dt', \quad t \in [0, T],$$

we can associate the space-time process $(S, \omega_0, \omega, y_0, y, \beta)$, where $S := \sigma(T)$, $(\omega_0, \omega) := \frac{dy_0}{ds} \cdot (1, u \circ y_0)$, $(y_0, y, \beta) := (id, x, V) \circ y_0$ (id is the identity function). Clearly, $\omega_0 > 0$ a.e.. Conversely, if $(S, \omega_0, \omega, y_0, y, \beta)$ is a space-time process with $\omega_0 > 0$ a.e., the absolutely continuous inverse $\sigma : [0, T] \to [0, S]$ of y_0, allows us to define the strict sense process (u, x, v), given by $u := \frac{\omega}{\omega_0} \circ \sigma$, $(x, v) := (y, \beta) \circ \sigma$.

The extension is to consider space-time processes with ω_0 possibly zero on some non-degenerate intervals.

Definition 4: Let $(S, \omega_0, \omega, y^0, y, \beta)$ be a space-time process for (STS). We call *graph completion*, in short *g.c., solution* of (S) associated with the space-time control (S, ω_0, ω), the pair $(x, v) \in BV([0, T], \mathbb{R}^{n+1})$ defined as

$$(x, v)(t) := (y, \beta)(\sigma(t)) \qquad \text{for all } t \in [0, T], \tag{4}$$

in which $\sigma : [0, T] \to [0, S]$ is the right inverse of y_0. Then, $(S, \omega_0, \omega, x, v)$ is a *graph completion process*, which is *feasible* for (P_{gc}) if $(S, \omega_0, \omega, y^0, y, \beta)$ is feasible for (P_{st}).

Remark 1: As mentioned in the Introduction, the name 'graph completion' comes from the fact that assigning a measure $\mu \in C^*_{\mathcal{K}}([0, T])$ is equivalent to assigning a BV function U, such that $U(0) = 0$ and $U(t) = \int_{[0,t]} \mu(dt')$ for any $t \in]0, T]$. For this U, a *graph completion* is any pair of Lipschitz continuous functions $(\varphi_0, \varphi) : [0, S] \to \mathbb{R}^{1+m}$ for some $S > 0$, satisfying the following conditions: (i) $(\varphi_0, \varphi)(0) = (0, 0)$, $(\varphi_0, \varphi)(S) = (T, U(T))$; (ii) for all $t \in [0, T]$, there exists $s \in [0, S]$ such that $(t, u(t)) = (\varphi_0, \varphi)(s)$; (iii) $(\omega_0, \omega) := (\frac{d\varphi_0}{ds}, \frac{d\varphi}{ds}) \in L^1([0, S]; W(\mathcal{K}))$. Clearly, $(S, \omega_0, \omega) \in \mathcal{W}$. Hence, a graph completion associates with a measure μ a space-time control. Conversely, any $(S, \omega_0, \omega) \in \mathcal{W}$ identifies a graph completion of the BV function U such that $U(0) := 0$ and $U(t) := \int_0^{\sigma(t)} \omega(s)\,ds$ for any $t \in]0, T]$, where $\sigma : [0, T] \to [0, S]$ is the right inverse of y_0.

In conclusion, any g.c. solution (x, v) of (S) corresponds to a graph completion of a measure $\mu \in C^*_{\mathcal{K}}([0, T])$.

IV. Some Preliminary Lemmas

Lemma 1: Let $F : [0, +\infty[\to [0, +\infty[$ be given by

$$F(x) = \frac{1}{x} \int_0^x |\sin(t)|dt.$$

Then $\lim_{x\to+\infty} F(x) = \frac{2}{\pi}$.

Proof: For any $k \in \mathbb{N}$ we have

$$F(2k\pi) = \frac{1}{2k\pi} \int_0^{2k\pi} |\sin(t)|dt = \frac{1}{\pi} \int_0^{\pi} \sin(t)dt = \frac{2}{\pi}.$$

In particular, from the above calculations it follows that

$$\int_0^{2k\pi} |\sin(t)|dt = 4k.$$

Using the above equality, for $x \in [2k\pi, 2(k + 1)\pi]$, we get

$$F(x) \le \frac{1}{2k\pi} \int_0^{2(k+1)\pi} |\sin(t)|dt = \frac{2}{\pi} + \frac{2}{k\pi}$$

and, similarly,

$$F(x) \ge \frac{1}{2(k + 1)\pi} \int_0^{2k\pi} |\sin(t)|dt = \frac{2}{\pi} - \frac{2}{(k + 1)\pi}.$$

As a consequence, for all $x \in [2k\pi, 2(k + 1)\pi]$, we obtain

$$\left|F(x) - \frac{2}{\pi}\right| \le \max\left\{\frac{2}{k\pi},\ \frac{2}{(k + 1)\pi}\right\} = \frac{2}{k\pi},$$

[4] Since ν has no atoms, by Def. 1 (i), $\alpha_i^r = 0$ a.e. for any $r \in [0, T]$.

where the sequence $\frac{2}{k\pi}$ is decreasing to 0, so that

$$\left|F(x) - \frac{2}{\pi}\right| \le \frac{2}{k\pi} \qquad \text{for all } x \ge 2k\pi.$$

This concludes the proof. ∎

For each $n \in \mathbb{N}$, consider in $C^*_{\mathbb{R}}([0,1])$ the measure

$$\mu_n(dt) := \frac{\pi}{2}\sin(nt)dt, \tag{5}$$

so that its bounded variation measure $|\mu_n| \in C^{\oplus}([0,1])$ is [5]

$$|\mu_n|(dt) := \frac{\pi}{2}|\sin(nt)|dt. \tag{6}$$

Lemma 2: Let $|\mu_n| \subset C^{\oplus}([0,1])$ be as in (6) for any $n \in \mathbb{N}$. Then $\lim_n |\mu_n|([0,t]) = t$ for all $t \in [0,1]$. As a consequence, it holds

$$|\mu_n| \rightharpoonup^* \ell, \tag{7}$$

where ℓ is the Lebesgue measure.

Proof: By Lemma 1, for any $t \in]0,T]$, we get

$$|\mu_n|([0,t]) = \frac{\pi}{2}\int_0^t |\sin(nt')|dt' = \frac{\pi}{2}\frac{1}{n}\int_0^{nt} |\sin(s)|ds$$
$$= \frac{\pi}{2}tF(nt) \to t \quad \text{as } n \to +\infty.$$

The conclusion then follows by [17, Lemma 2.9, (i)]. ∎

The proof of the next lemma is very similar to the previous one, hence we omit it.[6]

Lemma 3: Let $\mu_n \in C^*_{\mathbb{R}}([0,1])$ be as in (5) for any $n \in \mathbb{N}$. Then $\mu_n([0,t]) \to 0$ for all $t \in [0,1]$, so that

$$\mu_n \rightharpoonup^* \mu \equiv 0. \tag{8}$$

As a consequence of Lemmas 2 and 3, the sequence (μ_n) as in (5) satisfies $(\mu_n, |\mu_n|) \rightharpoonup^* (\mu, \nu) := (0, \ell)$, so that, following Def. 2, $\nu \in \mathcal{V}(\mu)$ and

$$\big(\mu, \nu, \{\alpha^r\}_{r\in[0,1]}\big) = \big(0, \ell, \{\alpha^r = 0\}_{r\in[0,1]}\big)$$

is an impulsive control (for $\mathcal{K} = \mathbb{R}$).

V. A Counter Example

Consider the following impulsive optimization problem

$$\begin{cases} \text{minimize } \Psi(x(1)), \\ \text{over impulsive processes } \big(\mu, \nu, \{\alpha^r\}_{r\in[0,1]}, x, v\big) \text{ s.t.} \\ dx(t) = f(x(t))\,dt + g_1(x(t))\mu_1(dt) + g_2(x(t))\mu_2(dt), \\ dV(t) = \nu(dt), \quad t \in [0,1], \\ \text{range}(\mu) = \text{range}(\mu_1,\mu_2) \subseteq \mathcal{K} := \mathbb{R} \times [0,+\infty[, \\ x(0) = (x_1,x_2,x_3)(0) = (0,0,0), \quad V(0) = 0, \\ x_2(t) \le 0, \quad t \in [0,1], \\ v(1) \le 2, \quad x_1(1) + v(1) \ge 1/2 \end{cases} \tag{P$_{\text{imp}}$}$$

in which $\Psi(x) = \Psi(x_1,x_2,x_3) := x_1^2 + x_3^2$ and[7]

$$f(x) = \begin{pmatrix} 0 \\ x_1^2 \\ 0 \end{pmatrix}, \quad g_1(x) = \begin{pmatrix} 1 \\ x_1 \\ x_3 \end{pmatrix}, \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

From the results in the previous section it follows that

$$\big(\bar\mu, \bar\nu, \{\bar\alpha^r\}_{r\in[0,1]}\big) = \big((0,0), \ell, \{\bar\alpha^r = 0\}_{r\in[0,1]}\big),$$

is an impulsive control with corresponding trajectory

$$\big(\bar x, \bar V\big)(t) = \big((\bar x_1, \bar x_2, \bar x_3), \bar V\big)(t) = ((0,0,0), t), \quad t \in [0,1].$$

The impulsive process $(\bar\mu, \bar\nu, \{\bar\alpha^r\}_{r\in[0,1]}, \bar x, \bar V)$ is actually a minimizer for (P$_{\text{imp}}$), as it is feasible and $\Psi(\bar x(1)) = 0$.

The corresponding space-time optimization problem is

$$\begin{cases} \text{minimize } \Psi(y(S)), \\ \text{over space-time processes } (S, \omega_0, \omega, y_0, y, \beta) \text{ s.t. } S > 0, \\ \frac{dy_0}{ds}(s) = \omega_0(s), \\ \frac{dy}{ds}(s) = f(y(s))\,\omega_0(s) + g_1(y(s))\omega_1(s) + g_2(y)\omega_2(s), \\ \frac{d\beta}{ds}(s) = \|\omega(s)\|, \quad s \in [0,S], \\ (\omega_0, \omega)(s) \in W(\mathcal{K}) \quad \text{a.e. } s \in [0,S], \\ y_0(0) = 0, \quad y(0) = (y_1,y_2,y_3)(0) = (0,0,0), \quad \beta(0) = 0, \\ y_2(s) \le 0, \quad s \in [0,S], \\ \beta(S) \le 2, \quad y_1(S) + \beta(S) \ge 1/2 \end{cases} \tag{P$_{\text{st}}$}$$

($W(\mathcal{K})$ as in (1)). Following the usual construction (see, e.g., the proof of [20, Th. 5.1]), we set $\bar\sigma(t) := t + \bar V(t) = 2t$, $t \in [0,1]$, and associate with $(\bar\mu, \bar\nu, \{\bar\alpha^r\}_{r\in[0,1]})$ the space-time control $(\bar S, \bar\omega_0, \bar\omega)$, given by $\bar S := \bar\sigma(1) = 2$ and

$$(\bar\omega_0, \bar\omega)(s) := (m_1, m_2)(\theta(s)), \ \text{ for any } s \in \big[0, \bar S\big],$$

where $\theta(s) = \frac{s}{2}$ is the inverse of $\bar\sigma$ and m_1, m_2 are the Radon-Nicodym derivatives of ℓ, $\bar\mu^c = \bar\mu = (0,0)$ w.r.t. $\nu + \ell = 2\ell$. Hence, $m_1 \equiv \frac{1}{2}$, $m_2 \equiv (0,0)$, so that $(\bar S, \bar\omega_0, \bar\omega) = (2, 1/2, 0, 0)$ and $\bar\omega_0(s) + \|\bar\omega(s)\| = \frac{1}{2}$ for $s \in [0,2]$. Thus, the control obtained does not take values in $W(\mathcal{K})$. However, if we consider the space-time control $(\hat S, \hat\omega_0, \hat\omega) := (1,1,0,0)$,[8] the corresponding space-time trajectory $(\hat y_0, \hat y, \hat\beta)(s) = (s, (0,0,0), 0)$ for $s \in [0,1]$, is simply a reparameterization of the solution to the control system in (P$_{\text{st}}$) corresponding to $(\bar S, \bar\omega_0, \bar\omega)$ by means of the time-change $s' = \theta(s)$. Hence, both controls $(\bar S, \bar\omega_0, \bar\omega)$, $(\hat S, \hat\omega_0, \hat\omega)$ identify the same g.c. solution $(\hat x, \hat V) \equiv (0,0)$, which does not coincide with $(\bar x, \bar V)$ and actually is not feasible for (P$_{\text{st}}$).

Furthermore, any space-time process $(S, \omega_0, \omega, y_0, y, \beta)$ satisfying the state constraint $y_2(s) \le 0$ for any $s \in [0,S]$ has $y_1 \equiv 0$,[9] which in turn implies that the control component ω_1 is constantly equal to 0. Therefore, the terminal constraint $y_1(S) + \beta(S) \ge 1/2$ implies that a minimizer for (P$_{\text{st}}$) corresponds, for instance, to the space-time control $(\tilde S = 3/2, \tilde\omega_0, \tilde\omega)$, in which

$$(\tilde\omega_0, \tilde\omega_1, \tilde\omega_2)(s) = \begin{cases} (1/2, 0, 1/2) & s \in [0,1], \\ (1,0,0) & s \in]1, 3/2], \end{cases}$$

with associated cost equal to 1/4.

Thus, problems (P$_{\text{imp}}$), (P$_{\text{st}}$) are *not* equivalent, since:

(i) the space-time process associated with the optimal impulsive process $(\bar\mu, \bar\nu, \{\bar\alpha^r\}_{r\in[0,1]}, \bar x, \bar V)$ according to the usual construction, is not feasible for (P$_{\text{st}}$);

(ii) the minimum of the two problems is not the same.

[5] Recall that the absolute value of the Radon-Nicodym derivative of a measure μ with respect to the Lebesgue measure ℓ coincides with the Radon-Nicodym derivative of the total variation measure $|\mu|$ with respect to ℓ.

[6] Note that $\lim_{x\to+\infty} \frac{1}{x}\int_0^x \sin(t)dt = 0$, as $\int_0^x \sin(t)dt \in [0,2]$ $\forall x \ge 0$.

[7] Incidentally, this control system is not commutative, as $[g_1, g_2](x) = -(0,0,1)^t \ne 0$ (the suffix ^{t} means transposition).

[8] I.e., the *canonical parameterization* of $(\bar S, \bar\omega_0, \bar\omega)$, according, e.g., to [9].

[9] Indeed, $y_2(s) = \int_0^s [(y_1(s'))^2\omega_0(s') + y_1(s')\frac{dy_1}{ds}(s')]ds' = \int_0^s (y_1(s'))^2\omega_0(s')ds' + \frac{y_1^2(s)}{2} \ge 0$ for any $s \in [0,S]$.

Note that the optimal control problem over strict sense processes, say $(\mathcal{P})$, of which (P$_{imp}$) and (P$_{st}$) are extensions, has the same minimum as (P$_{st}$), with minimizing strict sense control (corresponding to $(\tilde{S}, \tilde{\omega}_0, \tilde{\omega})$), given by

$$(\tilde{u}_1, \tilde{u}_2)(t) = \begin{cases} (0,1) & t \in [0, 1/2], \\ (0,0) & t \in]1/2, 1]. \end{cases} \quad (9)$$

VI. Admissible Impulsive Processes

The example in the previous section suggests that the set of impulsive processes for (IS) as in Definition 2 is too large, at least when associated with the extension of optimization problems with constraints, such as (P) in the Introduction.

Given a measure $\mu \in C^*_{\mathcal{K}}([0, T])$, we introduce the following subset $\mathcal{V}_c(\mu) \subseteq \mathcal{V}(\mu)$, defined as

$$\mathcal{V}_c(\mu) := \{\nu \in \mathcal{V}(\mu): \quad \nu^c = |\mu^c|\}, \quad (10)$$

where we recall that for any measure $\tilde{\mu}$, $\tilde{\mu}^c$ is the continuous component of $\tilde{\mu}$ w.r.t. ℓ. We propose to modify the notion of impulsive control and impulsive process, limiting ourselves to considering those for which $\nu \in \mathcal{V}_c(\mu)$. Precisely:

Definition 5: We call *admissible impulsive control* any impulsive control $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]})$ such that $\nu \in \mathcal{V}_c(\mu)$. We refer to the corresponding impulsive solution (x, v) to (IS) and the impulsive process $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]}, x, v)$ as *admissible impulsive solution* and *admissible impulsive process* for (IS), respectively.

Hence, if $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]})$ is an admissible impulsive control, the measure ν may differ from the total variation measure $|\mu|$ over a countable set of jump instants only. In particular, the corresponding admissible impulsive process is strict sense as soon as ν is absolutely continuous with respect to the Lebesgue measure.

Notice that, given the scalar measure $\bar{\mu} = 0 \in C^*_{\mathbb{R}}([0, 1])$ as in Section V, the measure $\bar{\nu} = \ell$ $(\in \mathcal{V}(\bar{\mu}))$ does not belong to $\mathcal{V}_c(\bar{\mu})$, so that the minimizing impulsive control in the example is not admissible. It is easy to see that a minimizing control for (P$_{imp}$) over admissible impulsive controls is actually the strict sense control in (9) with cost 1/4, as for the space-time problem (P$_{st}$). This equivalence is indeed a general result.

Theorem 1: (i) Let $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]}, x, v)$ be an admissible impulsive process. Set $\sigma(0) = 0$, $\sigma(t) := t + \nu([0, t])$ for any $t \in]0, T]$ and let $\mathcal{J}$ be the countable set of discontinuity points of σ. Set $S := \sigma(T)$ and define y_0 as the right inverse of σ.[10] Let m_0, m, m^ν be the Radon-Nicodym derivatives w.r.t. $d\sigma$ of ℓ, μ^c, and ν^c, respectively, and, for every $r \in \mathcal{J}$ and any $s \in \Sigma^r := [\sigma(r^-), \sigma(r^+)]$, set

$$\gamma^r(s) := \frac{s - \sigma(r^-)}{\sigma(r^+) - \sigma(r^-)}.$$

Consider the control pair (ω_0, ω), in which $\omega_0 := \frac{dy_0}{ds}$ and

$$\omega(s) := \begin{cases} m(y_0(s)) & \text{if } s \in [0, S] \setminus \bigcup_{r\in\mathcal{J}} \Sigma^r \\ \frac{\alpha^r(\gamma^r(s))}{\sigma(r^+)-\sigma(r^-)} & \text{if } s \in \Sigma^r, \quad r \in \mathcal{J}. \end{cases}$$

Finally, let ζ^r be as in Def. 2, $\theta^r(s') := v(r^-) + [v(r^+) - v(r^-)]s'$ for $s' \in [0, 1]$, and set

$$(y, \beta)(s) = \begin{cases} (x, v)(y_0(s)), & s \in]0, S[\setminus \cup_{r\in\mathcal{J}} \Sigma^r, \\ (\zeta^r, \theta^r)(\gamma^r(s)) & s \in]0, S[\cap \Sigma^r, \quad r \in \mathcal{J}. \end{cases}$$

Then $(S, \omega_0, \omega, y_0, y, \beta)$ is a space-time process and (x, v) coincides with the g.c. solution associated with (S, ω_0, ω).

(ii) Conversely, let $(S, \omega_0, \omega, y_0, y, \beta)$ be a space-time process. Let σ be the right inverse of y_0 and define the measures $\mu \in C^*_{\mathcal{K}}([0, T])$ and $\nu \in C^{\oplus}([0, T])$ via their distribution functions, as follows:

$$\mu([0, t]) = \int_0^{\sigma(t^+)} \omega(s)ds, \quad \nu([0, t]) = \int_0^{\sigma(t^+)} \|\omega(s)\|ds.$$

For any $r \in [0, T]$ and $s \in [0, 1]$, set

$$\alpha^r(s) := \left(\sigma(r^+) - \sigma(r^-)\right)\omega\Big(\left(\sigma(r^+) - \sigma(r^-)\right)s + \sigma\left(r^-\right)\Big).$$

Finally, let (x, v) be the g.c. solution associated with (S, ω_0, ω). Then, $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]}, x, v)$ is an admissible impulsive process. Moreover, its corresponding space-time process according to statement (i), is precisely $(S, \omega_0, \omega, y_0, y, \beta)$.

Proof: The proof follows exactly the same lines as, e.g., the proof of [20, Th. 5.1], where however there is a small error, due to the fact that non-admissible impulsive processes are not excluded (and this generates the problem of non-equivalence highlighted in the counterexample of Section V). Therefore, we limit ourselves to pointing out where the need to consider *admissible* impulsive controls comes into play.

In the proof of statement (i), we use the assumption that $\nu^c = |\mu^c|$ to be able to deduce from well-known properties of Radon-Nicodym derivatives that $m^\nu = \|m\|$. Hence, $0 \le m_0(r) \le 1$, $0 \le \|m(r)\| \le 1$ $d\sigma$-a.e., and

$$m_0(r) + \|m(r)\| = 1, \quad d\sigma\text{-a.e. } r \in [0, T]\setminus\mathcal{J}, \quad (11)$$

Since $\sigma(r^+)-\sigma(r^-) = \nu(r^+)-\nu(r^-)$ by definition, this allows to obtain that $\omega_0(s) + \|\omega(s)\| = 1$ for a.e. $s \in [0, S]$, so that (S, ω_0, ω) turns out to be a space-time control.[11]

On the other hand, starting from a space-time process $(S, \omega_0, \omega, y_0, y, \beta)$ as in (ii), the impulsive process $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]}, x, v)$ identified in statement (ii) is always admissible. Indeed, set $\mathcal{J}' := \{r \in [0, T]: \mu(\{r\}) \neq 0\}$, $\mathcal{J} := \{r \in [0, T] : \nu(\{r\}) \neq 0\}$, and define

$$D_\nu := \cup_{r\in\mathcal{J}} \Sigma^r, \quad D_\mu := \cup_{r\in\mathcal{J}'} \Sigma^r.$$

By construction, $D_\mu \subseteq D_\nu$ and $\int_{D_\nu\setminus D_\mu} \omega(s)ds = 0$. Accordingly, one has

$$\nu^c([0, t]) = \int_{[0,\sigma^+(t)]\setminus D_\nu} \|\omega(s)\| \, ds,$$

[10]From the very definition of σ it follows that y_0 is 1-Lipschitz continuous and increasing. Moreover, it is constant exactly on the intervals $[\sigma(r^-), \sigma(r^+)]$, $r \in \mathcal{J}$. Then, the Lebesgue measure ℓ and the continuous components μ^c, ν^c are absolutely continuous w.r.t. the measure $d\sigma$.

[11]In particular, we use Def. 1(i) and the fact that $\frac{dy_0}{ds} = 0$ on Σ^r.

$$\mu^c([0,t]) = \int_{[0,\sigma^+(t)]\setminus D_\mu} \omega(s)\,ds = \int_{[0,\sigma^+(t)]\setminus D_\nu} \omega(s)\,ds.$$

From these relations it immediately follows that $\nu^c = |\mu^c|$. ∎

Remark 2: Under the following additional hypotheses, fulfilled in many applications, that penalize the use of controls with large total variation: (i) $V \mapsto \Psi(x, V)$ *increasing for all* x, and (ii) $\mathcal{S} = \mathcal{C} \times [0, K]$, *where* $\mathcal{C} \subset \mathbb{R}^n$ *is a closed set and* K *is a positive constant*, the infima of the impulsive extension and the space-time extension of (P) do coincide. Indeed, in this case we can associate with each impulsive process $(\mu, \nu, \{\alpha^r\}_{r\in[0,T]}, x, V)$ an admissible impulsive process $(\tilde{\mu}, \tilde{\nu}, \{\tilde{\alpha}^r\}_{r\in[0,T]}, \tilde{x}, \tilde{V})$ such that $\tilde{\mu} = \mu$, $\tilde{\alpha}^r = \alpha^r$ for any r, and $\tilde{x} = x$, but $\tilde{V} \le V$ (and $\tilde{V} \le V$). However, even for these problems, the results in the literature on the existence of optimal controls, or the necessary conditions of optimality, are valid for admissible impulsive processes only.

VII. Conclusion

The purpose of this note is to address a problem related to the notion of impulsive process developed in [19]. This problem emerges in particular when we consider an associated optimization problem with constraints and costs involving the total variation of the process, as frequently happens in applications. In particular, we highlight by means of a counterexample that this notion is not equivalent to the impulsive process defined through graph completions. To resolve this, we introduce the subset of *admissible* impulsive processes which ensures equivalence and validates the results that have already been obtained. This new definition also serves as a starting point for a new line of research in collaboration with Vinter. This involves defining a well-posed solution for an impulsive system with time delays in the state, obtaining results related to the existence of an optimal process and necessary optimality conditions for an associated optimal control problem. So far, we have results for systems with vector-valued impulsive controls with delays in the drift term only [17], or for systems with non-negative scalar valued impulsive controls and delays both in the drift f and in the control coefficients $(g_j)_{j=1,\dots,m}$ [18], case in which any impulsive process is actually admissible. However, there is still much to be explored in this context, such as considering more general delayed impulsive optimization problems with vector-valued controls, analyzing the case with state constraints (to extend, for instance, the results in [3], [4], [5], [25] to the case with delays), and determining sufficient conditions to prevent a gap between the minimum of the impulsive problem with time delays and the infimum of the problem with unbounded controls of which the impulsive problem is an extension, as done, e.g., in [2], [13], [14], [15], [16], [23] for the case without delays.

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