Global Exponential Stabilization and Global $\mathcal{L}_p$ Performance of a Saturated Double Integrator

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***Abstract*—This letter presents novel results on the simultaneous global internal and external stabilization problem for the double integrator controlled by a saturated static state linear feedback. The methodology capitalizes on the distinctive characteristics of the smooth strictly increasing saturation function considered to strengthen and extend the stability properties reported in the existing literature for this canonical system. Concretely, as the main contributions, this letter demonstrates that the resulting closed-loop system, in the absence of disturbances, is globally exponentially stable and, in the face of non-input additive disturbances, is globally $\mathcal{L}_p$ stable for any given $p \in [1, \infty)$ and yields a bounded state trajectory. The simulation experiments showcase these new findings.**

***Index Terms*—Control nonlinearities, input saturation, robust stability, stability analysis.**

I. Introduction

A. Motivation and Background

THE DOUBLE integrator plant plays a prominent role in control applications, being the fundamental model for multiple electrical and mechanical systems (see [1] and references therein), describing, for instance, translational and rotational dynamics in one-dimensional space [2]. Within this widespread applicability, the limited control authority is a cross-cutting constraint in practical implementations. Hence, the saturated double integrator is a highly relevant system extensively explored in the literature over the past decades.

This letter focuses on the case of a double integrator controlled by a saturated static state linear feedback in the presence of a non-input additive external disturbance. Due to its simplicity, the linear controller tends to be more prevalent in practical applications [3]. Nonetheless, given the limited control authority, this subclass of feedback control is deceptively simple, posing an intricate problem involving simultaneous internal and external stability that still lacks complete understanding [4], [5]. Furthermore, the external stability results are instrumental for the robust stabilization of cascade and delay systems (see [6] and references therein).

B. Literature Review

In [7], Sussman and Yang demonstrated that a saturated static state linear feedback law renders the origin of the double integrator globally asymptotically stable. In terms of exponential convergence, Lin and Saberi proposed a saturated linear feedback that yields a semi-global exponential stability result [8]. The idea followed was also applied in [9, Proposition 1] and requires an *a priori* knowledge of a given compact set of initial conditions to find sufficiently low controller gains such that saturation does not occur. However, this approach has the drawback of the exponential result being obtained at the expense of the convergence rate [5, Sec. IV.5]. Furthermore, this logic relies on one assumption that may not hold in the presence of external disturbances. In [10, Th. 1], the authors globally exponentially stabilized the saturated double integrator in position tracking using linear feedback with any positive gains. Unlike [8], [9], since the initial conditions do not limit these parameters, the solution fully explores the available control capacity.

Most of the literature regards the $\mathcal{L}_p$-norm as an effective tool for evaluating external stability. In [11], Stoorvogel et al. established that the problem of simultaneous global internal stabilization with global finite-gain $\mathcal{L}_p$ performance for any $p \in [1, \infty]$ is intrinsically unsolvable for critically unstable systems in the presence of non-input additive disturbances. In this direction, the problem formulation must exclude the finite-gain condition or restrict the external disturbance to a given compact set in the $\mathcal{L}_p$ space. In [3], [12], the authors tackled this problem without finite gain for the double integrator controlled via saturated linear feedback, yielding a result that combines global asymptotic internal stability with global $\mathcal{L}_p$

Manuscript received 1 March 2024; revised 24 April 2024; accepted 13 May 2024. Date of publication 20 May 2024; date of current version 10 June 2024. This work was supported in part by the European Union under the Next Generation EU, through a Grant of the Portuguese Republic's Recovery and Resilience Plan (PRR) Partnership Agreement, within the scope of the Project PRODUTECH R3 – "Agenda Mobilizadora da Fileira das Tecnologias de Produção para a Reindustrialização," (Total Project Investment: 166.988.013,71 Euros; Total Grant: 97.111.730,27 Euros), and in part by Fundação para a Ciência e a Tecnologia (FCT) through the Project LAETA Base Funding (DOI: 10.54499/UIDB/50022/2020) and through the Project LAETA Programatic Funding (DOI: 10.54499/UIDP/50022/2020). The work of Luís Martins was supported by FCT for his Ph.D. Scholarship under Grant 2022.14126.BD. Recommended by Senior Editor K. Savla.
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Digital Object Identifier 10.1109/LCSYS.2024.3403471

stability and boundedness of states for any $p \in [1, 2]$. In addition, both works proved that global $\mathcal{L}_p$ performance for $p \in (2, \infty]$ is impossible since there are some external disturbances for which certain initial conditions lead to unbounded state trajectories. For the same closed-loop system, Shi and Saberi showcased in [13] the existence of disturbances with arbitrarily small $\mathcal{L}_\infty$-norm that lead to the unboundedness of the states, thereby precluding the attainment of the input-to-state stability (ISS) property [14, Definition 4.7]. Wen et al., in [15], broadened this result by adding another constrained set of disturbances. Notwithstanding, within the same context, the papers [4], [15] documented a class of uniformly integral bounded disturbances, encompassing, for instance, periodic and $\mathcal{L}_1$ signals, whose presence results in bounded state trajectories for any given initial condition.

C. Contributions

This letter analyzes the internal and external stability of the double integrator controlled via saturated linear static state feedback. The majority of the existing literature on this subject relies on the standard saturation function $\sigma(s) = \operatorname{sgn}(s)\min\{M, |s|\}$ [3], [4], [12], [13], [15]. In contrast, this letter uses a smooth strictly increasing saturation function, commonly adopted in applications like artificial neural networks and AD/DA conversion systems, to restrict the linear feedback. Furthermore, in some cascade control applications (for instance, trajectory tracking with unmanned aerial vehicles [10], [16]), the outer loop consists of a saturated double integrator system that generates references for the inner loop. Since these references result from the derivatives of the outer-loop actuation, the saturation function resorted is smooth. This choice is pivotal since exploiting the unique properties of this latter function is what enables the attainment of the following novel internal and external stability results in this letter:

- Global exponential internal stabilization: For any given initial condition, the saturated static state linear feedback renders the origin of the double integrator system exponentially stable in the absence of disturbances. The initial condition does not restrict the feedback gains selection and the exponential convergence bound holds globally for any given positive gains even if it leads to saturation.
- Global $\mathcal{L}_p$ performance: The formulation considers a generic output map and the approach demonstrates that the closed-loop system, for any given initial condition and $d \in \mathcal{L}_p$, is $\mathcal{L}_p$ stable with $p \in [1, \infty)$ and is small signal finite gain $\mathcal{L}_p$ with $p \in [1, \infty]$. Furthermore, the state trajectory remains bounded for any given initial condition and any $d \in \mathcal{L}_p$ with $p \in [1, \infty)$.

In addition, this letter also presents some input-to-state implications resulting from the disturbance response characterization. Specifically, the system is integral input-to-state stable (iISS) [17, Definition II.1] and, for disturbances with sufficiently low $\mathcal{L}_\infty$-norm, is input-to-state stable. Simulation results illustrate the main contributions of this letter.

D. Organization

This letter unfolds as follows: Section II presents the notation, some definitions, and external stability concepts; Section III details the dynamic model and formalizes the control problem; Section IV focuses on the global exponential internal stabilization; Section V assesses the $\mathcal{L}_p$ performance by evaluating the disturbance response the system; Section VI displays and discusses the simulation results; lastly, Section VII draws some concluding remarks.

II. Notation and Preliminaries

A. Notation

In this letter, $\mathbb{R}^n$ represents the n-dimensional Euclidean space; $\mathbb{R}_{\geq 0}(\mathbb{R}_{>0})$ expresses the set of non-negative (positive) real numbers; $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices; $\mathbb{R}^{n\times n}_{\succ 0}$ represents the set of $n \times n$ positive definite matrices; dom V symbolizes the domain of the function V; for $\mathbf{S} \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(\mathbf{S})$ and $\lambda_{\min}(\mathbf{S})$ denote, respectively, the largest and smallest eigenvalues of $\mathbf{S}$; for $s \in \mathbb{R}$, $\operatorname{sgn}(s)$ represents the sign function, which satisfies $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(s) = s|s|^{-1}\ \forall\, s \neq 0$; $\|\cdot\|$ represents the Euclidean norm; $\mathcal{L}^n_\infty$ denotes the space of piecewise continuous bounded functions $\mathbf{s} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ and is equipped with the $\mathcal{L}_\infty$-norm given by $\|\mathbf{s}\|_{\mathcal{L}_\infty} := \sup_{t\geq 0}\|\mathbf{s}(t)\| < \infty$; for $p \in [1, \infty[$, $\mathcal{L}^n_p$ denotes the space of piecewise continuous p-integrable functions $\mathbf{s} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ and is endowed with the $\mathcal{L}_p$-norm given by $\|\mathbf{s}\|_{\mathcal{L}_p} := (\int_0^\infty \|\mathbf{s}(t)\|^p \mathrm{d}t)^{1/p} < \infty$; for any $\nu \in \mathbb{R} > 0$ and $p \in [1, \infty]$, $\mathcal{L}^n_p(\nu)$ denotes the set $\{\mathbf{s} \in \mathcal{L}^n_p : \|s\|_{\mathcal{L}_\infty} \leq \nu\}$; $\mathcal{C}_0$ is the set of all vanishing functions $\mathbf{s} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$, i.e., with the property $\lim_{t\to\infty}\mathbf{s}(t) = \mathbf{0}$. The class $\mathcal{K}$ and $\mathcal{K}_\infty$ comparison functions used are in accordance with [14, Definition 4.2]. The saturation function here considered is aligned with the following definition:

Definition 1: The mapping $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is a smooth odd strictly increasing function satisfying the following properties: (1) $\sigma(0) = 0$; (2) $s\sigma(s) > 0\ \ \forall\ \ s \neq 0$; (3) $\lim_{s\to\pm\infty}\sigma(s) = \pm M$, with $M > 0$; (4) $0 < \dot{\sigma}(s) \leq 1$; (5) $\ddot{\sigma}(s) < 0\ \ \forall\ \ s > 0$.

B. External Stability

Consider the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)),\ \ \mathbf{x}(0) = \mathbf{x_0}\\ \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)),\end{aligned} \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$ represents time, $\mathbf{x}(t) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ denotes the state, $\mathbf{u}(t) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^m$ symbolizes the input, $\mathbf{y} \in \mathbb{R}^q$ is the output, $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $\mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^q$. The following definitions formalize the concepts of $\mathcal{L}_p$ stability and small-signal finite-gain $\mathcal{L}_p$ stability based on [14, Definitions 5.1 and 5.2].

Definition 2: For a given $p \in [1, \infty]$, the system (1) is $\mathcal{L}_p$ stable if for all $\mathbf{u} \in \mathcal{L}^m_p$ there exist a class $\mathcal{K}$ function $\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ and a constant $\beta \in \mathbb{R}_{\geq 0}$ such that

$$\|\mathbf{y}(t)\|_{\mathcal{L}_p} \leq \gamma\big(\|\mathbf{u}(t)\|_{\mathcal{L}_p}\big) + \beta \quad \forall \quad t \in \mathbf{R}_{\geq 0}.$$

If this inequality holds for any given initial condition $\mathbf{x_0} \in \mathbb{R}^n$, then the system is globally $\mathcal{L}_p$ stable.

Definition 3: For a given $p \in [1, \infty]$, the system (1) is small-signal finite-gain $\mathcal{L}_p$ stable if for all $\mathbf{u} \in \mathcal{L}^m_p(\nu)$ there exist constants $\eta, \beta \in \mathbb{R}_{\geq 0}$ such that

$$\|\mathbf{y}(t)\|_{\mathcal{L}_p} \leq \eta\|\mathbf{u}(t)\|_{\mathcal{L}_p} + \beta \quad \forall \quad t \in \mathbf{R}_{\geq 0}$$

holds for any given initial condition $\mathbf{x_0} \in \mathbb{R}^n$.

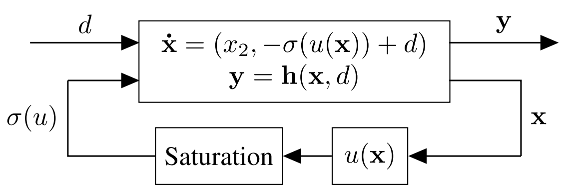


Fig. 1. Schematic representation of system (2).

III. Problem Formulation

This letter focuses on the internal exponential stabilization and dynamic response to external disturbances of a double integrator controlled by a smooth saturated linear static state feedback. The resulting closed-loop system has the form

$$\Sigma \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, d) \;\; = (x_2, -\sigma(u(\mathbf{x})) + d) \\ \mathbf{y} = \mathbf{h}(\mathbf{x}, d), \quad \mathbf{x}(0) = \mathbf{x_0} \end{cases}, \tag{2}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ denote the state, $d(t) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ models a non-input-additive disturbance, $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is a strictly increasing saturation function with $M > 0$ as saturation level, verifying the properties outlined in Definition 1, and $u(\mathbf{x}) : \mathbb{R}^2 \mapsto \mathbb{R}$ is the linear static state feedback given by $u(\mathbf{x}) = k_1x_1 + k_2x_2$, with $k_1, k_2 \in \mathbb{R}$ as constant control gains. Figure 1 provides a schematic representation of system (2). In addition, the output map satisfies the following assumption, which is relatively mild and ordinary.

Assumption 1: For some constants $\eta_1, \eta_2 \in \mathbb{R}_{\geq 0}$, the output map $\mathbf{h}(\mathbf{x}, d)$ verifies the inequality

$$\|\mathbf{h}(\mathbf{x}, d)\| \leq \eta_1\|\mathbf{x}\| + \eta_2|d| \quad \forall \quad (\mathbf{x}, d) \in \mathbb{R}^2 \times \mathbb{R}.$$

■

The following problem statement captures the dual goal driving this letter.

Problem 1: For the double integrator controlled by a smooth strictly increasing saturated linear static feedback, described by the closed-loop system (2)

1) In the absence of the non-input-additive disturbance $d(t)$, the origin $\mathbf{x} = \mathbf{0}$ is globally exponentially stable;
2) In the presence of any non-input-additive disturbance $d(t) \in \mathcal{L}_p$, with $p \in [1, \infty)$, the closed-loop system is globally $\mathcal{L}_p$ stable. ■

It is worth emphasizing that global $\mathcal{L}_\infty$ stability, corresponding to the behavior in the face of bounded external disturbances, is excluded from the problem formulation since any disturbance bound exceeding the saturation limit results invariably in unbounded trajectories, which renders this form of external stability unattainable.

IV. Global Exponential Internal Stability

With the focus on the internal stabilization of the closed-loop system Σ, this section builds upon the foundation provided by a well-established conclusion from the existing literature: the global asymptotic stability of the origin of (2) in the absence of the external disturbance [7]. In this direction, the present work goes a step further by demonstrating the exponential nature of the convergence of the dynamical responses of (2) to its equilibrium point. To this end, the demonstration hinges on Lyapunov theory and explores the properties of the smooth saturation function considered to derive the required exponential bounds and, consequently, attain an improved stability result. Theorem 1 formalizes this novel result, tackling, in this way, the internal stabilization challenge posed in Problem 1.

Theorem 1: In the absence of the non-input additive disturbance, i.e., $d(t) = 0 \;\forall\; t \in \mathbb{R}_{\geq 0}$, the equilibrium point $\mathbf{x} = \mathbf{0}$ of the system (2) is globally exponentially stable.

Proof: Consider the Lyapunov function candidate $V : \mathbb{R}^2 \mapsto \mathbb{R}_{\geq 0}$, first proposed in [7], given by

$$V(\mathbf{x}) := k_1x_2^2 + \int_0^{u(\mathbf{x})} \sigma(\mu)\, \mathrm{d}\mu + \int_0^{k_1x_1} \sigma(\mu)\, \mathrm{d}\mu. \tag{3}$$

This function is continuously differentiable, radially unbounded, positive-definite, and satisfies the upper bound

$$V(\mathbf{x}) \leq \frac{1}{2}u(\mathbf{x})^2 + \frac{1}{2}(k_1x_1)^2 + k_1x_2^2, \tag{4}$$

which, in turn, leads to $V(\mathbf{x}) \leq \lambda_{\max}(\mathbf{A})\|\mathbf{x}\|^2$ with $\mathbf{A} \in \mathbb{R}^{2\times 2}_{>0}$ given by

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 2k_1^2 & k_1k_2 \\ k_1k_2 & k_2^2 + 2k_1 \end{bmatrix}.$$

The derivative of V along the trajectories of (2) verifies

$$\dot{V}(\mathbf{x}) = -k_2\sigma(u(\mathbf{x}))^2 - k_1x_2(\sigma(u(\mathbf{x})) - \sigma(k_1x_1)) = -W(\mathbf{x}),$$

where $W(\mathbf{x}) : \mathbb{R}^2 \mapsto \mathbb{R}_{\geq 0}$ is a continuous function. Given that the saturation function is strictly increasing,

$$\frac{\sigma(u(\mathbf{x})) - \sigma(k_1x_1)}{k_2x_2} > 0 \quad \text{for} \quad x_2 \neq 0,$$

which leads to $-k_1x_2(\sigma(u(\mathbf{x})) - \sigma(k_1x_1)) \leq 0$. Therefore, since the first and second terms of $\dot{V}$ are negative definite with respect to the sets $\{\mathbf{x} \in \mathbb{R}^2 : k_1x_1 = -k_2x_2\}$ and $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = 0\}$, respectively, the time derivative of $V(\mathbf{x})$ is negative definite and the function $W(\mathbf{x})$ is positive definite. Hence, based on [14, Th. 4.9], the equilibrium point $\mathbf{x} = \mathbf{0}$ is globally uniformly asymptotically stable for the system (2). In light of this result, it follows that $V(\mathbf{x}(t)) \leq V(\mathbf{x_0}) \;\forall\; t \in \mathbb{R}_{>0}$. Since $|u(\mathbf{x})|$ and $|k_1x_1|$ are continuous, and the set $\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^2 : V(\mathbf{x}) \leq V(\mathbf{x_0})\}$ is compact, according to the Weierstrass's extreme value theorem, these functions have a maximum on $\mathcal{V}$. In this way, for all $t \in \mathbb{R}_{\geq 0}$, one can write $|u(\mathbf{x}(t))| \leq \alpha$ and $|k_1x_1(t)| \leq \alpha$, where $\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ results from $\alpha = \max\{\alpha_u, \alpha_{x_1}\}$ with

$$\alpha_u = \max_{\text{s.t. } \mathbf{x} \in \mathcal{V}} |u(\mathbf{x})|, \quad \alpha_{x_1} = \max_{\text{s.t. } \mathbf{x} \in \mathcal{V}} |k_1x_1|.$$

It is noteworthy that the positive definiteness of the function $V(\mathbf{x})$ implies $\alpha = 0$ exclusively when $\mathbf{x_0} = \mathbf{0}$. Furthermore, stemming from the global uniform asymptotic stability result, this initial condition leads to the trivial solution $\mathbf{x}(t) = \mathbf{0}$. In the sequel, to evaluate the exponential convergence of nontrivial solutions, the derivation of the required bounds considers the case $\alpha \in \mathbb{R}_{>0}$. First, note that the last property outlined in Definition 1 implies

$$|\sigma(u(\mathbf{x}))| \geq \sigma(\alpha)\alpha^{-1}|u(\mathbf{x})|, \tag{5a}$$

$$|\sigma(u(\mathbf{x})) - \sigma(k_1x_1)| \geq |u(\mathbf{x}) - k_1x_1|\dot{\sigma}(\alpha) = |k_2x_2|\dot{\sigma}(\alpha). \tag{5b}$$

Based on the former inequality, one arrives to $V(\mathbf{x}) \geq \lambda_{\min}(\mathbf{B}(\alpha))$, where $\mathbf{B}(\alpha) : \mathbb{R}_{>0} \mapsto \mathbb{R}^{2\times 2}_{>0}$ verifies

$$\mathbf{B}(\alpha) = \frac{1}{2}\begin{bmatrix} 2k_1^2\sigma(\alpha)\alpha^{-1} & k_1k_2\sigma(\alpha)\alpha^{-1} \\ k_1k_2\sigma(\alpha)\alpha^{-1} & k_2^2\sigma(\alpha)\alpha^{-1} + 2k_1 \end{bmatrix}.$$

Note that $\mathbf{B}(\alpha)$ is Hermitian and bear in mind that k_1 and k_2 are positive, and σ is strictly increasing. Then, based on Sylvester's criterion, it follows that $\mathbf{B}(\alpha)$ is positive definite for any given $\alpha \in \mathbb{R}_{>0}$. Furthermore, combining the inequalities (5a) and (5b) leads to

$$\dot{V}(\mathbf{x}) \leq -k_1k_2\dot{\sigma}(\alpha)x_2^2 - k_2\sigma(\alpha)^2\alpha^{-2}u^2.$$

Then, it follows that $\dot{V}(\mathbf{x}) \leq -\lambda_{\min}(\mathbf{C}(\alpha))\|\mathbf{x}\|^2$, with $\mathbf{C}(\alpha):\mathbb{R}_{>0} \mapsto \mathbb{R}^{2\times 2}_{>0}$ defined by

$$\mathbf{C}(\alpha) = \begin{bmatrix} k_1^2k_2\sigma(\alpha)^2\alpha^{-2} & k_1k_2^2\sigma(\alpha)^2\alpha^{-2} \\ k_1k_2^2\sigma(\alpha)^2\alpha^{-2} & k_2^3\sigma(\alpha)^2\alpha^{-2} + k_1k_2\dot{\sigma}(\alpha) \end{bmatrix},$$

Analogous to $\mathbf{B}(\alpha)$, $\mathbf{C}(\alpha)$ is Hermitian and positive definite for any given $\alpha \in \mathbb{R}_{>0}$. In this direction, one has

$$\dot{V}(\mathbf{x}) \leq -\frac{\lambda_{\min}(\mathbf{C}(\alpha))}{\lambda_{\max}(\mathbf{A})}V(\mathbf{x}). \tag{6}$$

Hence, based on [14, Th. 4.10], $\mathbf{x} = \mathbf{0}$ is globally exponentially stable for (2). In detail, the state verifies

$$\|\mathbf{x}(t)\| \leq \sqrt{c_2c_1^{-1}}\|\mathbf{x_0}\|e^{-\frac{c_3}{2c_2}t} \quad \forall \quad t \in \mathbb{R}_{\geq 0}, \tag{7}$$

where $c_1 = \lambda_{\min}(\mathbf{B})$, $c_2 = \lambda_{\max}(\mathbf{A})$, and $c_3 = \lambda_{\min}(\mathbf{C})$. ■

Remark 1: Under the same arguments of the previous proof, the widely-used standard saturation function $\sigma(s) = \text{sgn}(s)\min\{M, |s|\}$, which is not continuously differentiable nor strictly increasing, does not yield global exponential stability since $\dot{\sigma}(s) = 0\ \forall\ |s| > M$ and, consequently, the matrix $\mathbf{C}(\alpha)$ would be positive semi-definite, leading to $\lambda_{\min}(\mathbf{C}(\alpha)) = 0$, which prevents from concluding the required condition for exponential stability (6). ■

Remark 2: The authors applied this result to a position tracking problem in [10]. Compared to [10, Th. 1], this proof resorts to an optimization problem to obtain the minimum value for α, ultimately leading to a less conservative exponential bound for the state trajectory. Furthermore, it elaborates further on the derivation of the bounds for $V(\mathbf{x})$ and $\dot{V}(\mathbf{x})$ and specifies the resulting exponential bound. ■

V. Global External Stability

Having assessed the internal stability of (2), the focus now shifts to analyzing its behavior in the presence of the disturbance $d(t)$ within the framework of $\mathcal{L}_p$ stability, whose concepts are frequently resorted to evaluate cascade control architectures (see [10, Th. 3] for an example of application). This external stability analysis builds upon the previously established exponential result to tackle the second part of Problem 1. Theorem 2 leverages Lyapunov tools to demonstrate the small-signal finite-gain $\mathcal{L}_p$ stability of (2).

Theorem 2: For any given initial condition $\mathbf{x_0} \in \mathbb{R}^2$, the system (2) is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$. In detail, for each $\mathbf{d} \in \mathcal{L}_p(\nu)$, with

$$\nu = \min\left\{\sigma(M), \sqrt{c_1V(\mathbf{x_0})}c_3(c_2c_4)^{-1}\right\}, \tag{8}$$

where $c_4 = \sqrt{(2k_1^2 + k_1k_2)^2 + (k_1k_2 + 2k_1 + k_2^2)^2}$, the output $\mathbf{y}(t)$ satisfies

$$\|\mathbf{y}(t)\|_{\mathcal{L}_p} \leq (\eta + \eta_2)\|d(t)\|_{\mathcal{L}_p} + \beta \quad \forall \quad t \in \mathbf{R}_{\geq 0}, \tag{9}$$

where

$$\eta = \frac{\eta_1c_2c_4}{c_1c_3},\ \beta = \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}}\rho\|\mathbf{x_0}\|,\ \text{with } \rho = \begin{cases} 1 & \text{, if } p = \infty \\ \left(\frac{2c_2}{pc_3}\right)^{\frac{1}{p}} & \text{, if } p \in [1, \infty) \end{cases}.$$

Proof: By definition, $d \in \mathcal{L}_p$ implies being a piecewise continuous function of t. In this way, in light of $\mathbf{f}(\mathbf{x}, 0)$ being continuously differentiable in $\mathbb{R}^2$, the function $\mathbf{f}(\mathbf{x}, d)$ is also piecewise continuous in t. Moreover, for all $\mathbf{w}, \mathbf{y} \in \mathbb{R}^2$,

$$\|\mathbf{f}(\mathbf{w}, d) - \mathbf{f}(\mathbf{y}, d)\|^2 \leq \lambda_{\max}\left(\begin{bmatrix} k_1^2 & k_1k_2 \\ k_1k_2 & k_2^2 + 1 \end{bmatrix}\right)\|\mathbf{w} - \mathbf{y}\|.$$

Therefore, the system (2) is globally Lipschitz. In addition, the condition $\|\mathbf{f}(\mathbf{x}, d) - \mathbf{f}(\mathbf{x}, 0)\| = |d|$ also holds. The Lyapunov function $V(\mathbf{x})$, defined in (3)

$$c_1\|\mathbf{x}\|^2 \leq V(\mathbf{x}) \leq c_2\|\mathbf{x}\|^2. \tag{10}$$

and, for $d = 0$, its derivative satisfies $\dot{V}(\mathbf{x}) \leq -c_3\|\mathbf{x}\|^2$. Note that the condition $V(\mathbf{x}(t)) \leq V(\mathbf{x_0})\ \forall\ t \in \mathbb{R}_{\geq 0}$, required for the exponential stability of (2) in the absence of the external disturbance d, implies $\|\mathbf{x}\| \leq \sqrt{c_1^{-1}V(\mathbf{x_0})}$. The gradient $\nabla V(\mathbf{x}) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ verifies

$$\nabla V(\mathbf{x}) = \left[\sigma(u(\mathbf{x}))k_1 + \sigma(k_1x_1)k_1 \;\; \sigma(u(\mathbf{x}))k_2 + 2k_1x_2\right],$$

yielding

$$\|\nabla V(\mathbf{x})\| \leq c_4\|\mathbf{x}\|. \tag{11}$$

Thus, in virtue of these results and Assumption 1, it follows from [14, Th. 5.1] that for each $p \in [1, \infty]$ and $d \in \mathcal{L}_p(\nu)$, the output $\mathbf{y}$ verifies (9)–(2) is small-signal finite gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$. ■

Remark 3: According to [13, Th. 1], resorting to the standard saturation function $\sigma(s) = \text{sgn}(s)\min\{M, |s|\}$ to constraint the linear feedback in (2) leads to a closed-loop system that is not ISS even for $d \in \mathcal{L}_\infty(\delta)$ with $\delta \in \mathbb{R}_{>0}$ arbitrarily small. In contrast, by using a smooth strictly increasing saturation function in line with Definition 1, for any $d \in \mathcal{L}_\infty(\nu)$ and any given initial condition $\mathbf{x_0} \in \mathbb{R}^2$, the system (2) is finite gain $\mathcal{L}_\infty$ stable and, based on the proof of [14, Th. 5.1], the following condition holds

$$\|\mathbf{x}(t)\| \leq \left(\frac{c_2}{c_1}e^{-\frac{c_3}{c_2}t}\right)^{\frac{1}{2}}\|(\mathbf{x_0})\| + \frac{c_4}{2c_1}\int_0^t e^{-\frac{c_3}{2c_2}(t-\tau)}|d(\tau)|\,\mathrm{d}\tau.$$

This inequality leads to

$$\|\mathbf{x}(t)\| \leq \left(\frac{c_2}{c_1}e^{-\frac{c_3}{c_2}t}\right)^{\frac{1}{2}}\|(\mathbf{x_0})\| + \frac{c_4c_2}{c_1c_3}\|d(\tau)\|_{\mathcal{L}_\infty}.$$

Hence, as a result of using the smooth strictly increasing saturation function, the system (2) is ISS for $d \in \mathcal{L}_\infty(\nu)$. ■

Before advancing to the main result of this letter, Theorem 3 builds on the bounds established to demonstrate the meaningful property of integral input-to-state stability, which provides a qualitative perception for the overshoot of the states under the influence of a finite energy disturbance [17].

Theorem 3: The system (2) is iISS.

Proof: Consider the function U : dom $V \mapsto \mathbb{R}_{\geq 0}$ given by $U(\mathbf{x}) := V^{1/2}(\mathbf{x})$. For $V \neq 0$, since $\dot{U} = 2^{-1}\dot{V}V^{-1/2}$ and in light of $V \geq k_1 x_2^2$ and $V \geq \frac{1}{2}\sigma(u(\mathbf{x}))^2$, one has

$$\dot{U}(\mathbf{x}, d) \leq \frac{-W(\mathbf{x})}{2\sqrt{V(\mathbf{x})}} + \frac{\sqrt{2}}{2}\max\{\sqrt{c_4}, \sqrt{2k_1} + k_2|d(t)|. \quad (12)$$

On the other hand, for $V(\mathbf{x}(t)) = 0$, first, note that

$$\limsup_{h\to 0^+} \frac{1}{h}(U(\mathbf{x}(t+h)) - U(\mathbf{x}(t))) = \limsup_{h\to 0^+} \frac{1}{h}\sqrt{V(\mathbf{x}(t+h))}.$$

Since V is continuously differentiable, positive definite, and maps a convex domain into $\mathbb{R}_{\geq 0}$, for $h^* \in [0, 1]$, applying the fundamental theorem of calculus followed by a change of variables yields

$$V(\mathbf{x}(t+h)) = \int_0^{\mathbf{x}(t+h)} \nabla V(\boldsymbol{\mu})\, \mathrm{d}\boldsymbol{\mu} = \int_0^1 \nabla V\big(h^*\mathbf{x}(t+h)\big)\, \mathrm{d}h^*\mathbf{x}(t+h).$$

Then, in light of (11), if follows that $V(\mathbf{x}(t+h)) \leq 2^{-1}c_4\|\mathbf{x}\|^2$. Consequently, given (10), $c_4 \geq 2c_1$. The vector $\mathbf{x}(t+h)$ can be expanded using a first-order Taylor polynomial and a sensitivity function $\mathbf{o}(h)$:$\mathbb{R} \mapsto \mathbb{R}^2$ satisfying $\lim_{h\mapsto 0} \mathbf{o}(h) = \mathbf{0}$, yielding $\mathbf{x}(t+h) = h\dot{\mathbf{x}}(t) + h\mathbf{o}(h)$. In this direction, one obtains $\|\mathbf{x}(t+h)\|^2 = h^2 d^2(t) + h^2\mathbf{o}(h)$, which, combined with the previous notions, leads to

$$\limsup_{h\to 0^+} \frac{1}{h}\sqrt{V(\mathbf{x}(t+h))} \leq \frac{\sqrt{2}}{2}\max\{\sqrt{c_4}, \sqrt{2k_1} + k_2|d|.$$

Therefore, U verifies (12) for any given value of V. Given that V is positive definite, it follows from [14, Lemma 4.3] that there exists class $\mathcal{K}_\infty$ functions γ_1 and γ_2 such that $\gamma_1(\mathbf{x}) \leq U(\mathbf{x}) \leq \gamma_2(\mathbf{x})$. Furthermore, note that $W(\mathbf{x})(2\sqrt{V(\mathbf{x})})^{-1}$ is positive definite. Then, based on [17, Defintion II.2], the continuously differentiable function U is an iISS Lyapunov function for system (2) for all $\mathbf{x} \in \mathbb{R}^2$ and all $d \in \mathbb{R}$. In this way, according to [17, Th. 1], the system (2) is iISS. ■

The sequel relies on a pivotal notion of equivalence of external stability that provides a framework in which it is sufficient to consider a bounded vanishing $\mathcal{L}_p$ disturbance to obtain the intended global $\mathcal{L}_p$ stabilization from the previous small-signal result. In this direction, by levering this equivalence, Theorem 4 tackles the second part of Problem 1 and states the main result of this letter.

Theorem 4: For any initial condition $\mathbf{x_0} \in \mathbb{R}^2$ and disturbance $d(t) \in \mathcal{L}_p$, with $p \in [1, \infty)$, the system (2) is $\mathcal{L}_p$ stable.

Proof: Consider the system

$$\Sigma_z \quad \begin{cases} \dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, d_z(t)) \\ \mathbf{y_z} = \mathbf{z}, \quad \mathbf{z}(0) = \mathbf{z_0}, \end{cases}$$

where $\mathbf{z} := (z_1, z_2) \in \mathbb{R}^2$ denotes the state, $\mathbf{y_z} \in \mathbb{R}^2$ is the output, and $d_z \in \mathcal{L}_p \cap \mathcal{L}_\infty \cap \mathcal{C}_0$, with $p \in [1, \infty)$, represents a vanishing bounded non-input additive disturbance. Since

$$\|u(\mathbf{w}) - u(\mathbf{s})\| \leq (k_1 + k_2)\|\mathbf{w} - \mathbf{s}\| \quad \forall \quad \mathbf{w}, \mathbf{s} \in \mathbb{R}^2,$$

the linear feedback $u(\mathbf{x})$ is globally Lipschitz continuous in $\mathbf{x}$. Given that the strictly increasing saturation function σ is also globally Lipschitz continuous, $|\sigma(w) - \sigma(s)| \leq |w - s|\ \forall\ w, s \in \mathbb{R}$, and the composition of such functions is itself Lipschitz continuous, according to [5, Lemma 13.21], for $p \in [1, \infty)$ and $\mathbf{y} = \mathbf{x}$, the system Σ is $\mathcal{L}_p$ stable for all $d \in \mathcal{L}_p$ if and only if the system Σ_z is $\mathcal{L}_p$ stable for all d_z. Furthermore, it also follows from [5, Lemma 13.21] that the small-signal finite-gain $\mathcal{L}_p$ stability result presented in Theorem 2 also holds for Σ_z. In light of $d_z(t) \in \mathcal{L}_p$ and the system Σ_z being piecewise continuous in t and globally Lipschitz, based on [14, Th. 3.2], the system Σ_z has a unique solution $\forall\ t > t_0$ and, thereby, cannot have a finite escape time. In this direction, since $d_z \in \mathcal{C}_0$, there exists a time instant $\bar{t}$ such that $d_z(t) \in \mathcal{L}_p(\nu) \cap \mathcal{L}_\infty(\nu)\ \forall\ t \geq \bar{t}$, satisfying, thus, (8) Therefore, bearing in mind Theorem 2, for any given initial condition z(0) and external disturbance $d_z \in \mathcal{L}_p$, the system $\sum_z$ is $\mathcal{L}_p$ stable for each $p \in [1, \infty)$. Then, using the equivalence notion presented in [5, Lemma 13.21], one concludes that, for any $d \in \mathcal{L}_p$ with $p \in [1, \infty)$, the state of the system (2) verifies $\mathbf{x} \in \mathcal{L}_p^2$. In this way, by virtue of Assumption 1, it directly follows that the system (2) is globally $\mathcal{L}_p$ stable for each $p \in [1, \infty)$. ■

Corollary 1: For any given $\mathbf{x_0} \in \mathbb{R}^2$ and $d(t) \in \mathcal{L}_p$, with $p \in [1, \infty)$, the state of system (2) is bounded, i.e., $\mathbf{x} \in \mathcal{L}_\infty^2$.

Proof: The system (2) being globally $\mathcal{L}_p$ stable for all $p \in [1, \infty)$ results in $\mathbf{x} \in \mathcal{L}_p^2$. Furthermore, given that $|s| \geq \sigma(|s|)$ and by applying the Minkowski inequality [18, Th. 2.11.9], one has

$$\|\sigma(u(\mathbf{x}))\|_{\mathcal{L}_p} \leq \|u(\mathbf{x})\|_{\mathcal{L}_p} \leq (k_1 + k_2)\|\mathbf{x}\|_{\mathcal{L}_p} < \infty.$$

Hence, since $d(t) \in \mathcal{L}_p$, $\dot{\mathbf{x}} \in \mathcal{L}_p^2$. Thereby, in light of [5, Lemma 2.5], it directly follows that, for any given $\mathbf{x_0}$ and $d(t) \in \mathcal{L}_p$, with $p \in [1, \infty)$, $\mathbf{x} \in \mathcal{L}_\infty^2$. ■

VI. Simulation Results

To demonstrate the novel results presented in Theorem 1 and Theorem 4, the authors conducted two simulation tests capturing scenarios excluded from the global results found in the literature. Specifically, to illustrate the exponential stability result, the test consists of considering an initial condition verifying $u(\mathbf{x_0}) > M$, and, to showcase the $\mathcal{L}_p$ performance, the experiment considers an external disturbance verifying $d \in \mathcal{L}_p$ strictly for a given $p \in (2, \infty)$. For the double integrator controlled by saturated linear feedback, these scenarios are relevant since the existing literature only reports exponential results that hold when the control magnitude is always smaller than the saturation level [8], [9] and only demonstrates global $\mathcal{L}_p$ performance for $p \in [1, 2]$ (see [3], [5], [12]). To illustrate the practical application of this letter, the simulation tests considered the position tracking problem [10, Sec. IV] in one-dimensional space. Within this context, the tracking dynamics are controlled by a smooth saturated linear static state feedback, yielding a closed-loop system described by (2), where x_1 and x_2 denote the position and velocity tracking errors, respectively.

The first simulation test studied the behavior in the absence of the external disturbance $d(t)$ and considered the following values: $M = 2$, $k_1 = 2$, $k_2 = 1$, and $\mathbf{x_0} = (1, 1)$. The saturation function used was $\sigma(s) = M\tanh(sM^{-1})$. From the response depicted in Fig. 2, one can conclude that the resulting

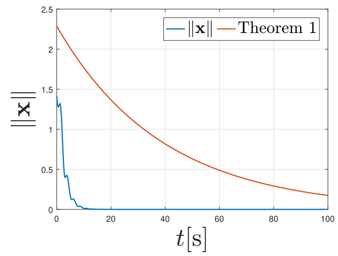
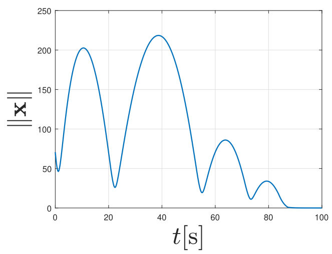
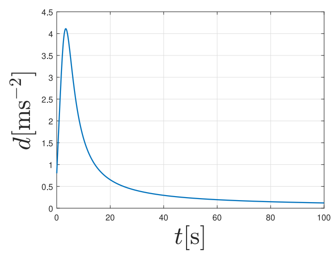


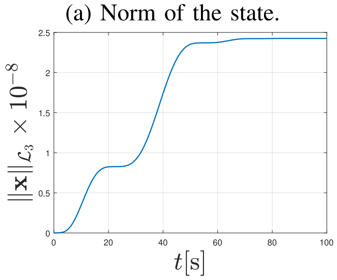
Fig. 2. Norm of the state $\mathbf{x}(t)$ obtained in the first test.



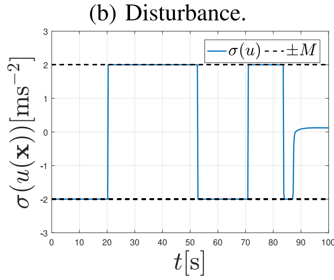
(a) Norm of the state.



(b) Disturbance.



(c) $\mathcal{L}_3$-norm of the state.



(d) Saturated linear feedback.

Fig. 3. Responses obtained in the second test.

trajectory $\mathbf{x}(t)$, verifying $u(\mathbf{x_0}) > M$, complied with the exponential bound presented in (7), being, thereby, coherent with the stability result of Theorem 1.

The $\mathcal{L}_p$ performance evaluation resorted to the same saturation function, featured the parameters $M = 2$, $k_1 = 6$, $k_2 = 12$, and $\mathbf{x_0} = (50, -50)$, and involved the piecewise continuous external disturbance

$$d(t) = \begin{cases} 0.2a, & \text{for} \quad t = 0 \\ a\left(\left(\frac{t}{b}\right)^{\frac{1}{p^*}}\left(\log\left(\frac{t}{b}\right)^2 + 1\right)\right)^{-1}, & \text{for} \quad t > 0 \end{cases}$$

with $a = 4$, $b = 4$, and $p^* = 3$. It is worth emphasizing the particularity that $d(t) \in \mathcal{L}_p$ if and only if $p = p^*$. To provide some insight to the reader, this external disturbance is presented in Fig. 3(b). The outcome of this second test is exhibited in Fig. 3. The joint effect of the challenging initial condition and the external disturbance satisfying $\|d(t)\|_{\mathcal{L}_\infty} > M$ led to the saturation of the linear feedback during the majority of the experiment, as one can observe in Fig. 3(d). Despite this constraint, the state $\mathbf{x}$ did not grow unbounded, which is in line with Corollary 1, and ultimately converged to zero. Furthermore, as displayed in Fig. 3(c), the L_3-norm of the state approached a finite value, thereby corroborating the external stability result presented in Theorem 4.

VII. Conclusion

This letter addressed the simultaneous global internal and external stabilization problem for the double integrator controlled by a saturated static state linear feedback. By capitalizing on the characteristics of the smooth strictly increasing saturation function considered, this letter yields noteworthy improvements in internal stability results and $\mathcal{L}_p$ performance. Specifically, by pairing the linear feedback with this saturation function, the resulting closed-loop system is globally exponentially stable in the absence of disturbances. Furthermore, in the presence of a non-input additive disturbance, this approach significantly extends the existing $\mathcal{L}_p$ stability results from the literature: for any given initial condition and disturbance $d \in \mathcal{L}_p$ with $p \in [1, \infty)$, the saturated double integrator is $\mathcal{L}_p$ stable and its states remain bounded. The simulation results highlighted these contributions.

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