Output-Feedback Stabilization of Stochastically-Sampled Networked Control Systems Under Packet Dropouts

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***Abstract*—This letter deals with the mean-square output feedback stabilization of sampled-data linear time-invariant (LTI) systems in the presence of sporadically sampled measurement streams and packet dropouts. To address the problem we propose a control structure composed of: a) a hybrid observer, which resets with the arrival of a new measurement sample; and b) a feedback of the latest estimated state and the value of the control signal computed in the previous sampling instant, generating the control to be applied to the continuous-time plant. The control signal is kept constant, by means of a zero-order hold, between two successive sampling instants. The overall closed-loop system exhibits a deterministic behavior except for jumps that occur at random sampling times resulting in a piecewise deterministic Markov process (PDMP). Using Lyapunov-based stability analysis for stochastic systems, we determine sufficient conditions for mean exponential stability (MES) of the overall closed-loop system, which are turned into Linear Matrix Inequalities (LMI) for the design of the proposed hybrid stabilizer. Finally, the effectiveness of the theoretical results is verified by an illustrative example.**

***Index Terms*—Sampled-data control, networked control systems, random sampling, mean exponential stability, output-feedback stabilization, linear matrix inequalities.**

I. Introduction

In real-world industrial applications of networked control systems (NCS), a continuous-time plant is often controlled in a sampled-data fashion by remotely located digital controllers. Sampled-data NCS are generally implemented through a shared digital network of sensors, controllers and actuators for the required data transmission [1], [2]. The presence of communication networks in a feedback control loop suffers of some undesirable phenomena such as sporadic availability of measurements [3], [4] and packet dropouts [5], [6].

The stability analysis and control design of sporadically sampled-data systems with known bounds on transmission intervals, as noted in [7], can be categorized into four major categories- 1) emulation-based approach [4], 2) time-delay method based on Lyapunov-Krasovskii functionals [8], 3) hybrid system approach [3], 4) impulsive systems [9]. In contrast to these works, we rather consider random sampling interval, where the corresponding distribution function has possibly unbounded support.

For such stochastically sampled-data systems, both state and output feedback stabilization are studied respectively in [1], [6], [10], [11], [12] and in [2], [5], [13], [14], [15]. Similar to [1], [16] and [17], we assume that the inter-sampling intervals are independent and identically distributed random variables with exponential distribution. This is a fairly common model in networks and queuing theory [18]. Moreover, the possibility of packet dropouts is explicitly considered and modeled, as in [2], [6], through a Bernoulli distribution. In [10] and [12], inter-sampling intervals are assumed to fluctuate around an ideal sampling period based on certain probability distributions. In contrast, we consider random inter-sampling intervals obeying exponential distribution for which such a deterministic ideal sampling period cannot be obtained.

It should be noted that the majority of these works considers a discrete-time plant model and provides stabilizing design conditions for the underlying discretized model [10], [12], [19]. However, the stability of this discrete-time model is not equivalent to the stability of the corresponding sampled-data system, as pointed out in [20]. To overcome this limitation, in this letter we propose a control design method that guarantees the stabilization in the mean-square exponential sense of the actual sampled-data closed-loop system in the presence of intermittently available measurements and packet dropouts. A stabilizer, consisting of a hybrid observer and a discrete-time controller, is proposed to render the overall closed-loop system mean exponential stable (MES). The discrete controller uses feedback of the estimated state at the current sampling instant and the value of the

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control signal computed in the previous sampling instant. Such a structure of the discrete-time controller, that uses the past value of the control input, leads to non-conservative convex conditions for the computation of controller gains [1].

To derive our stabilizing conditions we use, as in [21], the framework of piecewise deterministic Markov processes (PDMP) [22], which is a subclass of stochastic hybrid systems (SHS) [23]. Based on Lyapunov-like theorems for stochastic systems, we provide sufficient conditions for mean-square exponential stability. These conditions are posed in terms of linear matrix inequalities (LMIs) that can be efficiently exploited to design the stabilizer parameters. The feasibility of these LMIs depends on the packet dropout probability and average sampling intensity. We show that the observer and the controller gains can be designed separately, thereby establishing a kind of separation principle for the proposed stabilizer architecture.

The remainder of the letter is organized as follows. In Section II, we formulate the problem and provide the basic definitions and the objective. Section III presents the main results pertaining to sufficient stability conditions, obtained using Lyapunov-like theorems for stochastic systems, and the design of the stabilizer parameters under sporadic sampling and packet dropouts. An illustrative example is given in Section IV.

A. Notations

The symbols $P[x]$ and $\mathbb{E}[x]$ denote, respectively, the probability and expectation of a random variable x. The set $\mathbb{N}$ denotes the set of positive integers including zero, $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$, $\mathbb{R}_+ = (0, \infty), \mathbb{R}_{\geq 0} = \mathbb{R}_+ \cup \{0\}$, I and 0 represent respectively the identity matrix and a zero matrix with appropriate dimensions. The symbol $\mathbb{R}^{n\times m}$ denotes the space $n \times m$ of real matrices. For a symmetric matrix A, $A \succ 0$ (or $A \succeq 0$) denotes that matrix A is positive definite (or semi-definite). The symbol $\bullet$ denotes a symmetric block in symmetric matrices. The symbol $\circ$ denotes the composition of functions. For square matrices A_i, $i = 1, 2, \ldots, N$, $A = \text{diag}(A_1, A_2, \ldots, A_N)$ is a block-diagonal matrix with diagonal elements A_i. The notation $\lambda_{\max}(A)$ (or $\lambda_{\min}(A)$) represents the largest (or smallest) eigenvalue of a symmetric matrix A, $\text{He}(A) = A + A^{\mathrm{T}}$. For $x, y \in \mathbb{R}^N$, $\|x\|$ is the Euclidean norm of x, $\text{col}(x, y) = \left[x^{\mathrm{T}}, y^{\mathrm{T}}\right]^{\mathrm{T}}$. A shorthand notation x^+ is used to denote the value of x after an instantaneous jump.

II. Problem Formulation

Consider a continuous-time plant described by the following linear model

$$\mathcal{P} \begin{cases} \dot{x}_p = A_p x_p + B_p u, \\ y_p = C_p x_p, \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^n, u \in \mathbb{R}^m, y_p \in \mathbb{R}^p$ respectively represent the plant state, the control input, and the measured output which is available only at some isolated time instants t_k, $k \in \mathbb{N}$ with inter-sampling intervals $\delta_k = t_{k+1} - t_k$. Let y_p^+ denote that the measured value of y_p is impulsively available only at $t = t_k$. Therefore,

$$\begin{aligned} y_p^+ &= C_p x_p, \text{ if no packet dropout at } t = t_k, \\ y_p^+ &= y_p, \text{ if packet dropout at } t = t_k, \end{aligned} \tag{2}$$

where the measured value of y_p at $t = t_k$ is $C_p x_p(t_k)$ unless there is a packet dropout event in which case y_p is the measured output at the preceding sampling instant $y_p(t_{k-1})$. The matrix triplet (A_p, B_p, C_p) is assumed to be both stabilizable and detectable. With no loss of generality, let us take $t_0 = 0$. In this letter, we assume that $\{\delta_k\}$ is a sequence of independent and identically distributed random variables with exponential distribution $\text{Exp}(\lambda)$ of intensity $\lambda > 0$ and $\mathbb{E}[\delta_k] = \frac{1}{\lambda}$. In particular, the cumulative distribution function takes the form

$$F(s) \triangleq P[\delta_k \leq s] = 1 - e^{-\lambda s}, \;\; k \in \mathbb{N}, \;\; s \geq 0. \tag{3}$$

Since $\delta_k \sim \text{Exp}(\lambda)$, the number of samples k occurred until the current time t is modeled by a Poisson process. Let

$$N_t \triangleq \sup\{k \in \mathbb{N} | t_k \leq t\}, \tag{4}$$

then the probability that $N_t = n$ under this Poisson process is given by

$$P[N_t = n] \;=\; e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \tag{5}$$

Inter-sampling events have been successfully modeled with a Poisson process in [1], [16] and [17]. Since $P[N_t = \infty] = 0$ and $F(0) = 0$, there is zero probability of infinite sampling events occurring in a finite time (i.e., zero probability of Zeno behavior). The control input is updated only at the sampling times t_k and is kept constant in between by means of a zero-order hold device according to the law:

$$\mathcal{K} \begin{cases} \dot{u} = 0, & \text{if } t \in [t_k, t_{k+1}), \\ u^+ = K_x \hat{x}_p + K_u u, & \text{if no packet dropout at } t = t_k, \\ u^+ = u, & \text{if packet dropout at } t = t_k, \end{cases} \tag{6}$$

where $K_x \in \mathbb{R}^{m\times n}$, $K_u \in \mathbb{R}^{m\times m}$ are control gains to be designed and $\hat{x}_p \in \mathbb{R}^n$ is the estimation of x_p from sporadic measurements of y_p. The control law u depends on the samples of $\hat{x}_p$ and also on the last control input applied, $u(t_{k-1})$. Next, to generate $\hat{x}_p$, the following hybrid observer dynamics is considered:

$$\mathcal{L} \begin{cases} \dot{\hat{x}}_p = A_p \hat{x}_p + B_p u, & \text{if } t \in [t_k, t_{k+1}), \\ \hat{x}_p^+ = \hat{x}_p + L\big(C_p \hat{x}_p - y_p\big), & \text{if no packet dropout at } t = t_k, \\ \hat{x}_p^+ = \hat{x}_p, & \text{if packet dropout at } t = t_k, \end{cases} \tag{7}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain to be designed. Unless there is a packet dropout, the estimated state is updated at the sampling instants t_k with the measured error estimates.

As in [2], [5] and [6], the probability of packet dropout is modeled by a Bernoulli process $\{\alpha_k\}$, $k \in \mathbb{N}$ with a probability distribution

$$P[\alpha_k = 1] = \mu_1 \in \;(0, 1), \;\; P[\alpha_k = 0] = 1 - \mu_1, \tag{8}$$

where $\alpha_k = 0$ indicates an event of packet dropout at the sampling instants $t = t_k$. The sequence $\{\alpha_k\}$ in (8) is a sequence of independent and identically distributed Bernoulli random variables and thus the events of packet dropout for each t_k are mutually independent. Let us define $e \triangleq \hat{x}_p - x_p$,

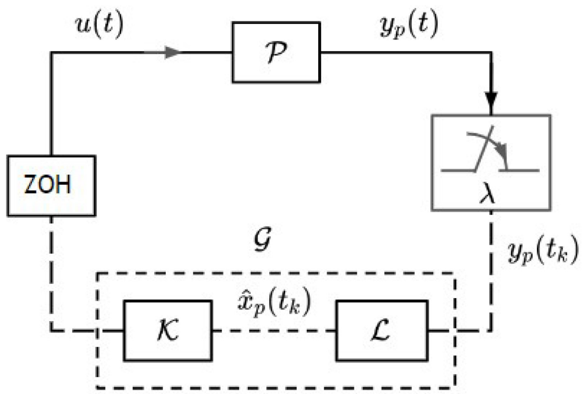


Fig. 1. Schematic diagram of the NCS architecture with sporadically sampled output measurements.

and $z \triangleq \mathrm{col}(x_p, u)$. Then, by using (1), (6), (7) and (8), the overall closed-loop dynamics with state vector $\Psi \triangleq \mathrm{col}(z, e)$ can be described more compactly by the following impulsive system $\mathcal{H}$:

$$\mathcal{H}\begin{cases} \dot{\Psi} = f(\Psi), & \forall t \geq 0,\ t \neq t_k, \\ \Psi^+ = g(\Psi), & \text{if } \alpha_k = 1 \text{ at } t = t_k, \\ \Psi^+ = \Psi, & \text{if } \alpha_k = 0 \text{ at } t = t_k, \end{cases} \tag{9}$$

where $\Psi(0) = \Psi_0 \in \mathbb{R}^{2n+m}$, $f(\Psi) \triangleq A\Psi$, $g(\Psi) \triangleq N\Psi$,

$$A \triangleq \mathrm{diag}\big(A_c, A_p\big),\ A_c \triangleq \begin{bmatrix} A_p & B_p \\ 0 & 0 \end{bmatrix}, N \triangleq \begin{bmatrix} A_d & K_d \\ 0 & I + LC_p \end{bmatrix},$$

$$A_d \triangleq A_r + B_r K,\ A_r \triangleq \mathrm{diag}(I, 0),\ B_r \triangleq \begin{bmatrix} 0 & I \end{bmatrix}^{\mathrm{T}},$$

$$K \triangleq \begin{bmatrix} K_x & K_u \end{bmatrix},\ \text{and } K_d \triangleq \begin{bmatrix} 0 & K_x^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

The schematic diagram of the networked control system (NCS) $\mathcal{H}$ consisting of the plant model $\mathcal{P}$, controller $\mathcal{K}$ and observer $\mathcal{L}$, sampler with the average sampling rate λ and a zero-order hold (ZOH) device is shown in Figure 1. Since the estimation error dynamics in (9) are decoupled from the dynamics of z, system $\mathcal{H}$ with state Ψ can be viewed as a cascade of two hybrid systems Σ_1 and Σ_2 as follows:

$$\Sigma_1\begin{cases} \dot{e} = A_p e, \\ e^+ = \big(I + LC_p\big)e, & \text{if } t = t_k,\ \alpha_k = 1, \\ e^+ = e, & \text{if } t = t_k,\ \alpha_k = 0, \end{cases} \tag{10}$$

$$\Sigma_2\begin{cases} \dot{z} = A_c z, \\ z^+ = A_d z + K_d e, & \text{if } t = t_k,\ \alpha_k = 1, \\ z^+ = z, & \text{if } t = t_k,\ \alpha_k = 0. \end{cases} \tag{11}$$

Definition 1: The origin $\Psi = 0$ of (9) is *mean exponentially stable* (MES) if there exist constants c, $\gamma > 0$ such that for every initial condition $\Psi_0 \in \mathbb{R}^{2n+m}$:

$$\mathbb{E}\Big[\|\Psi(t)\|^2\Big] \leq ce^{-\gamma t}\|\Psi_0\|^2,\ \forall t \geq 0, \tag{12}$$

where $\gamma > 0$ is referred to as the decay rate of the system (9).

It is clear that (9) presents a deterministic behavior except for the jumps that occur at random sampling times. Due to the memorylessness property of the Poisson process, $\Psi(t)$ in (9) is a PDMP [24], which is a subclass of SHS [22]. The hybrid observer $\mathcal{L}$ and the discrete-time feedback law $\mathcal{K}$ constitute the hybrid controller $\mathcal{G}$ that should stabilize the closed-loop system under sporadic output measurement and control signal update. The corresponding time-driven stochastic hybrid automaton, with all the transitional probabilities, is shown in Figure 2. The stochastic output feedback stabilization problem we aim to solve can now be stated as follows.

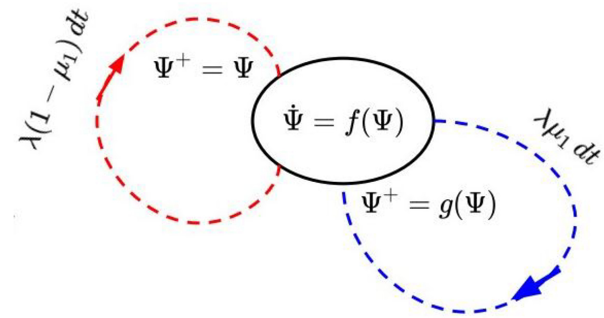


Fig. 2. Stochastic hybrid automata for system $\mathcal{H}$ (9).

Problem 1 (Mean Square Exponential Stabilization): Given the parameters of the exponential and of the Bernoulli distributions (i.e., λ, μ_1), provide convex design conditions for the controller gain K and the observer gain L such that the resulting closed-loop system (9) is MES.

III. Main Results

The overall closed-loop system (9) with the state space $\mathbb{R}^{2n+m}$ is a PDMP, as noted above. Moreover, the closed-loop system (9) exhibits the following characteristics:

- The flow vector field $\Psi \mapsto f(\Psi)$ in (9) is globally Lipschitz, which yields complete maximal solutions[1] to $\dot{\Psi} = f(\Psi)$ for every initial conditions $\Psi_0 \in \mathbb{R}^{2n+m}$;
- Constant average jump rate $\lambda \in \mathbb{R}_+$;
- Locally bounded transition intensity and reset maps $g(\Psi)$, i.e., for every bounded set $\mathcal{B} \subset \mathbb{R}^{2n+m}$, $\|g(\Psi)\| \leq \bar{g}$ when $\Psi \in \mathcal{B}$ with $\bar{g} = \max\{1, \|N\|\}$;
- For every initial conditions $\Psi_0 \in \mathbb{R}^{2n+m}$, $\mathbb{E}[N_t] = \lambda t < \infty$ where N_t counts the number of samples under Poisson process (4).

These facts allow us to establish the following key result, which is closely related to [23, Th. 2]. The proof of this result follows from [24, Th. 26.14 and Remark 26.16].

Theorem 1: Consider a continuously differentiable function $V : \mathbb{R}^{2n+m} \to \mathbb{R}$ such that

$$\mathbb{E}\left[\sum_{k=0}^{N_T} \big|V\big(\Psi^+\big) - V(\Psi)\big|\right] < \infty, \tag{13}$$

$\forall N_T \in \mathbb{N}$, $\Psi_0 \in \mathbb{R}^{2n+m}$. Then, $\forall t \geq 0$, and $\Psi_0 \in \mathbb{R}^{2n+m}$:

$$\mathbb{E}[V(\Psi)] = V(\Psi_0) + \mathbb{E}\left[\int_0^t \mathscr{U}V(\Psi(s))\ ds\right], \tag{14}$$

$$\mathscr{U}V(\Psi) \triangleq \frac{\partial V(\Psi)}{\partial \Psi} f(\Psi) + \lambda\big(\bar{V}(\Psi) - V(\Psi)\big), \tag{15}$$

$$\bar{V}(\Psi) \triangleq \mu_1 V(g(\Psi)) + (1 - \mu_1)V(\Psi).$$

The relation in (14) is known as Dykin's formula [24, P. 33], and it can be intuitively interpreted as a stochastic version of

[1] A solution is said to be maximal if its domain cannot be extended and it is said to be complete if its domain is unbounded.

the fundamental theorem of calculus. The possibility of packet dropout also causes a stochastic transition that depends on the Bernoulli distribution. The expected value of V after a jump is the weighted average with all possible values of Ψ^+ and is given by the term $\bar{V}(\Psi)$.

Next, we derive sufficient conditions for MES of (9) from Theorem 1 which will lead to a computationally tractable design of the parameters of $\mathcal{K}$ and $\mathcal{L}$. Since the SHS $\mathcal{H}$ is a cascade of two subsystems, namely Σ_1 in (10) and Σ_2 in (11), proving MES of the overall system $\mathcal{H}$ is pursued here by showing MES of these individual subsystems, analogously to the classical "input-to-state stability" philosophy. Therefore, to derive MES conditions of $\mathcal{H}$, let us first consider Lyapunov functions $V_1(e) = e^{\mathrm{T}} P_1 e,\ P_1 \succ 0$, and $V_2(z) = z^{\mathrm{T}} Q z,\ Q = P_2^{-1} \succ 0$, corresponding to the subsystems Σ_1 and Σ_2. The sufficient MES conditions for Σ_1 will lead to the design of the observer gain L, while the MES conditions of Σ_2 will give K. The following result then shows that

$$V(\Psi) \triangleq V_1(e) + \alpha V_2(z), \alpha \in \mathbb{R}_+ \tag{16}$$

can be used as a Lyapunov function to prove MES of $\mathcal{H}$.

Lemma 1: Given the average Poisson sampling rate $\lambda > 0$, and the packet dropout probability $\mu_1 \in (0, 1)$, if there exist three positive scalars γ_e, γ_c, σ, and matrices $P_1 \in \mathbb{R}^{n\times n} \succ 0$, $P_2 \in \mathbb{R}^{n+m} \succ 0$, $Y \in \mathbb{R}^{n\times p}$, $M \in \mathbb{R}^{m\times(n+m)}$ such that for some $\sigma \in \mathbb{R}_+$,

$$\begin{bmatrix} \mathrm{He}\left(P_1 A_p\right) - (\lambda\mu_1 - \gamma_e) P_1 & \bullet \\ \lambda\mu_1\left(P_1 + Y C_p\right) & -\lambda\mu_1 P_1 \end{bmatrix} \prec 0, \tag{17}$$

$$\begin{bmatrix} \mathrm{He}\,(A_c P_2) - (\lambda\mu_1 - \gamma_c) P_2 & \bullet \\ \lambda\mu_1 (A_r P_2 + B_r M) & -\dfrac{\lambda\mu_1}{1+\sigma} P_2 \end{bmatrix} \prec 0, \tag{18}$$

then $L = P_1^{-1} Y,\ K = M P_2^{-1}$ render the systems Σ_1 in (10), and Σ_2 in (11) MES with respective decay rates γ_e and $\gamma_c \in \mathbb{R}_+$. Consequently, the origin of the overall closed-loop system $\mathcal{H}$ can be shown MES with a decay rate $\gamma \in \mathbb{R}_+$.

Proof: The proof is based on Theorem 1. Consider a time-varying function of the form

$$W(\Psi, t) = e^{\gamma t} V(\Psi) \triangleq e^{\gamma t}(V_1(e) + \alpha V_2(z)), \tag{19}$$

where $\gamma \in \mathbb{R}_+,\ \alpha \in (0, \alpha^*)$ with

$$\alpha^* = \frac{1}{\lambda\mu_1}\left(\frac{\sigma}{\sigma+1}\right)\frac{\gamma_e \lambda_{\min}(P_1)\lambda_{\min}(P_2)}{\|K_x\|^2}. \tag{20}$$

Since Ψ in (9) is a PDMP, so is $(\Psi(t), t)$, as noted in [24, p. 84]. Furthermore, for an augmented process $(\Psi(t), t)$, as noted in the proof of [1, Lemma 2], Dykin's formula is analogous to (14) with

$$\mathscr{U} W(\Psi, t) = e^{\gamma t}(\gamma V(\Psi) + \mathscr{U} V(\Psi)) \tag{21}$$

as long as the condition (13) is satisfied. To show that (13) holds, from the PDMP of the trajectories Ψ in (9), for any $t \geq 0$ we have that

$$\begin{aligned} \Psi(t) = \phi_{t-t_{N_t}} \circ g \circ \phi_{t_{N_t} - t_{N_t - 1}} \circ g, \ldots \\ , \ldots g \circ \phi_{t_2 - t_1} \circ g \circ \phi_{t_1}(\Psi_0), \end{aligned} \tag{22}$$

where $\phi_t(\Psi) = e^{At}\Psi$. With $\bar{c} \triangleq \max\{\|A_c\|, \|A_p\|\}$ and $\bar{g} = \max\{1, \|N\|\}$, we thus obtain

$$\|\Psi(t)\| \leq e^{\bar{c}t}\|g\|^{N_t}\|\Psi_0\| \leq e^{\bar{c}t}\bar{g}^{N_t}\|\Psi_0\|, \tag{23}$$

where N_t counts the number of jumps up to time t. Since $\|\Psi^+\| \leq \bar{g}\|\Psi\|, \forall t = t_k$, then

$$\|\Psi^+\| \leq e^{\bar{c}t}\bar{g}^{N_t+1}\|\Psi_0\|. \tag{24}$$

Furthermore, by construction, one has from (19):

$$W(\Psi, t) = e^{\gamma t}\Psi^{\mathrm{T}}\left[\mathrm{diag}(\alpha Q, P_1)\right]\Psi \leq \bar{\lambda}_V e^{\gamma t}\|\Psi\|^2, \tag{25}$$

where $\bar{\lambda}_V = \max(\lambda_{\max}(P_1), \alpha\lambda_{\max}(Q))$ and $Q = P_2^{-1} \succ 0$. Next, at the jump instant $t = t_k$, it follows that

$$W\left(\Psi^+, t_k^+\right) \leq \bar{\lambda}_V e^{(\gamma+2\bar{c})t_k}\bar{g}^{2(k+1)}\|\Psi_0\|^2, \tag{26}$$

$$W(\Psi, t_k) \leq \bar{\lambda}_V e^{(\gamma+2\bar{c})t_k}\bar{g}^{2k}\|\Psi_0\|^2. \tag{27}$$

Then, for any $T \in \mathbb{R}_+$, and $\Psi_0 \in \mathbb{R}^{2n+m}$, we have

$$\mathbb{E}\left[\sum_{k=0}^{N_T}\left|W\left(\Psi^+, t_k^+\right) - W(\Psi, t_k)\right|\right] \tag{28}$$

$$\begin{aligned} &\leq \bar{\lambda}_V e^{(\gamma+2\bar{c})T}\|\Psi_0\|^2\mathbb{E}\left[\sum_{k=0}^{N_T}\bar{g}^{2k+2} + \bar{g}^{2k}\right] \\ &= \bar{\lambda}_V e^{(\gamma+2\bar{c})T}\left(\bar{g}^2+1\right)\|\Psi_0\|^2\mathbb{E}\left[\sum_{k=0}^{N_T}\bar{g}^{2k}\right] \triangleq \bar{C}\mathbb{E}\left[\sum_{k=0}^{N_T}\bar{g}^{2k}\right] \\ &= \bar{C}\mathbb{E}\left[\sum_{k=0}^{\infty}\bar{g}^{2k}\sum_{j=k}^{\infty}P[N_T = j]\right] \\ &= \bar{C}\mathbb{E}\left[\sum_{k=0}^{\infty}\bar{g}^{2k}P[N_T \geq k]\right] = \bar{C}\mathbb{E}\left[\sum_{k=0}^{\infty}\bar{g}^{2k}P[t_k \leq T]\right] \\ &= \bar{C}\mathbb{E}\left[\sum_{k=0}^{N_T}\bar{g}^{2k}\right] < \infty, \end{aligned} \tag{29}$$

and thus the condition (13) of Theorem 1 holds.

Now, to prove MES of (9), we first show that $\mathscr{U} W(\Psi, t) \leq 0$, i.e., $\mathscr{U} V(\Psi) \leq -\gamma V(\Psi)$ per (21). In this regard, we evaluate the jump term from (15) as

$$\bar{V}(\Psi) - V(\Psi) = \mu_1(V(g(\Psi)) - V(\Psi)), \tag{30}$$

with

$$\begin{aligned} V(g(\Psi)) &= e^{\mathrm{T}}\left(I + LC_p\right)^{\mathrm{T}}P_1\left(I + LC_p\right)e \\ &\quad + \alpha\left[z^{\mathrm{T}}A_d^{\mathrm{T}}QA_d z + 2z^{\mathrm{T}}A_d^{\mathrm{T}}QK_d e + e^{\mathrm{T}}K_d^{\mathrm{T}}QK_d e\right] \\ &\leq \alpha\left[(1+\sigma)z^{\mathrm{T}}A_d^{\mathrm{T}}QA_d z + \left(1 + \frac{1}{\sigma}\right)e^{\mathrm{T}}K_d^{\mathrm{T}}QK_d e\right] \\ &\quad + e^{\mathrm{T}}\left(I + LC_p\right)^{\mathrm{T}}P_1\left(I + LC_p\right)e, \end{aligned} \tag{31}$$

where the Young inequality is used. Then,

$$\mathscr{U} V(\Psi) \leq \Psi^{\mathrm{T}}\left[\mathrm{diag}\left(\alpha\Pi_1, \Pi_2 + \alpha\lambda\mu_1(1 + \frac{1}{\sigma})K_d^{\mathrm{T}}QK_d\right)\right]\Psi,$$

$$\Pi_1 \triangleq \mathrm{He}(QA_c) + \lambda\mu_1\left[(1+\sigma)A_d^{\mathrm{T}}QA_d - Q\right], \tag{32}$$

$$\Pi_2 \triangleq \mathrm{He}(P_1A_p) + \lambda\mu_1\left[\left(I + LC_p\right)^{\mathrm{T}}P_1\left(I + LC_p\right) - P_1\right]. \tag{33}$$

Using Schur complement and a congruence transformation, from (32), $\Pi_1 \prec -\gamma_c Q$ is equivalent to

$$\begin{bmatrix} \mathrm{He}(QA_c) - (\lambda\mu_1 - \gamma_c)Q & \bullet \\ QA_d & -\dfrac{1}{(1+\sigma)\lambda\mu_1}Q \end{bmatrix} \prec 0. \quad (34)$$

Since $P_2 = Q^{-1}$, by left and right multiplying (34) by $\mathrm{diag}(P_2, P_2)$, one gets

$$\begin{bmatrix} \mathrm{He}(A_c P_2) - (\lambda\mu_1 - \gamma_c)P_2 & \bullet \\ A_d P_2 & -\dfrac{1}{(1+\sigma)\lambda\mu_1}P_2 \end{bmatrix} \prec 0. \quad (35)$$

Since $A_d = A_r + B_r K$, the above inequality corresponds to (18) with $M = KP_2$. By using again Schur complement and a congruence transformation, from (33), we analogously obtain $\Pi_2 \prec -\gamma_e P_1$ when (17) holds with $Y = P_1 L$. Consequently,

$$\begin{aligned} \mathscr{U}V(\Psi) \le\ & \Psi^{\mathrm{T}}[\mathrm{diag}(-\alpha\gamma_c P_2^{-1}, -\gamma_e P_1 \\ & + \alpha\lambda\mu_1(1+\frac{1}{\sigma})K_d^{\mathrm{T}} P_2^{-1} K_d)]\Psi \prec 0, \end{aligned} \quad (36)$$

when $\alpha \in (0, \alpha^*)$ with α^* given in (20). Furthermore, for any such α,

$$\mathscr{U}V \le -\gamma V,\ \ \gamma = \min\{\gamma_1, \gamma_2\},\ \ \gamma_1 \le \gamma_c, \quad (37)$$

$$\gamma_2 \le \gamma_e - \frac{\alpha\lambda\mu_1}{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}\left(1 + \frac{1}{\sigma}\right)\|K_x\|^2, \quad (38)$$

which also implies $\mathscr{U}W(\Psi, t) \le 0$ for all $\Psi \in \mathbb{R}^{2n+m}$ in (21), and thus from (14),

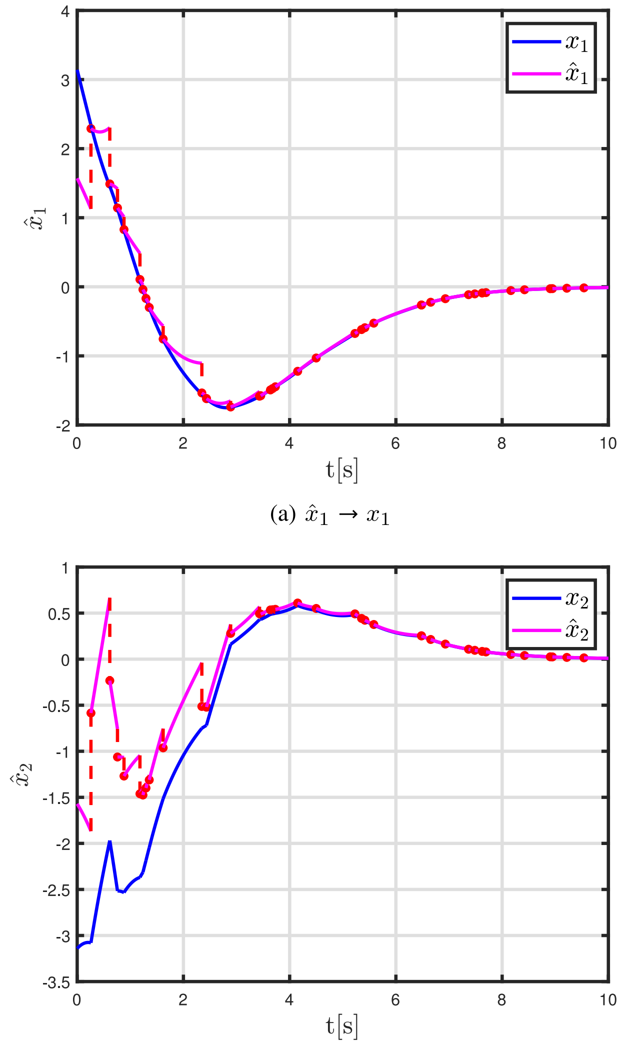
$$\begin{aligned} &\mathbb{E}[W(\Psi, t)] = \mathbb{E}\big[e^{\gamma t}V(\Psi)\big] \le V(\Psi_0), \\ &\mathbb{E}[V(\Psi)] \le e^{-\gamma t}V(\Psi_0) \le \bar{\lambda}_V e^{-\gamma t}\|\Psi_0\|^2, \\ &\mathbb{E}\Big[\|\Psi(t)\|^2\Big] \le \left(\frac{\bar{\lambda}_V}{\underline{\lambda}_V}\right)e^{-\gamma t}\|\Psi_0\|^2 \triangleq ce^{-\gamma t}\|\Psi_0\|^2, \end{aligned} \quad (39)$$

with $\underline{\lambda}_V = \min\{\lambda_{\min}(P_1), \alpha\lambda_{\min}(P_2^{-1})\}$. Thus the origin of system (9) is MES with a decay rate γ. This concludes the proof. ∎

The conditions (17) and (18) correspond respectively to the MES of the estimation error e and the state feedback control for the state z. The decoupled design of the hybrid observer gain L and the discrete controller gain K for the stochastic sampled-data system (9) resembles the "separation principle" approach in the sense that these gains can be computed from the decoupled LMIs (17) and (18).

When $\sigma = 0$ and $\mu_1 = 1$ (i.e., zero packet dropout probability), the LMI condition (18) resembles the one for the state feedback MES in [1, eq. (18)]. Since (A_p, B_p) is stabilizable, there always exists some $\lambda \ge \lambda_0 \in \mathbb{R}_+$ for which the inequality (18) holds. For such a λ, the appropriate value of σ is then selected through a simple line search.

Furthermore, if λ_0 is the minimum average sampling rate for a non-packet dropout case, then the proposed hybrid stabilizer $\mathcal{G}$ can offer MES to closed-loop system (9) in the packet dropout case with $\mu_1 \in [\lambda_0/\lambda, 1]$.



(a) $\hat{x}_1 \to x_1$

(b) $\hat{x}_2 \to x_2$

Fig. 3. Convergence of the estimated states $\hat{x}_p$ to x_p.

IV. Numerical Example

In this section, we consider a numerical example to illustrate the theoretical developments presented in this letter. Let us take the following system matrices from the example given in [1]:

$$A_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ -5 \end{bmatrix},\ C_p = [1\ 0], \quad (40)$$

where the unstable open-loop plant model has eigenvalues located at 1 and -1, and its output y_p is available sporadically with the mean sampling rate $\lambda = 4$. Let us first consider the case when there is no packet dropout, i.e., $\mu_1 = 1$. Since the triplet (A_p, B_p, C_p) is both controllable and observable, we obtain feasible solutions to the LMIs in (17) and (18) for given decay rate estimates $\gamma_e = 1.9$, and $\gamma_c = 0.3$. With $\sigma = 0.09$, the gain matrices for the hybrid stabilizer

$$K = \begin{bmatrix} 0.319 & 0.319 & -0.016 \end{bmatrix},\ L = \begin{bmatrix} -1.01 & -1.09 \end{bmatrix}^{\mathrm{T}},$$

obtained from solving LMIs (17) and (18) render the closed-loop system (9) MES with the decay rate estimate $\gamma = 0.8$. The convergence of the estimated state component $\hat{x}_i,\ i = 1, 2$

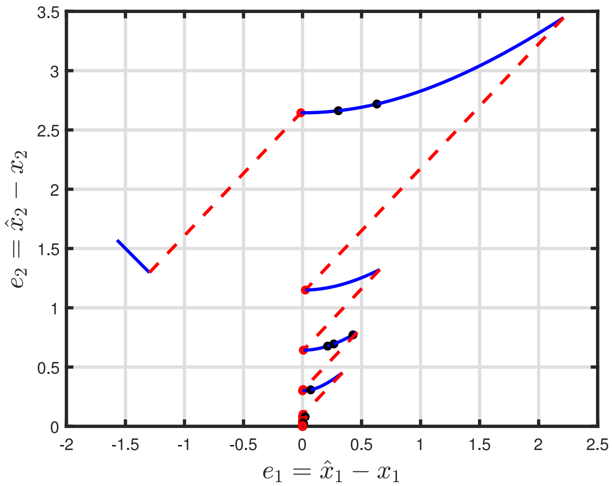


Fig. 4. Phase portrait of the estimation error under packet dropouts (such instants are marked with black circles).

to the plant state component x_i are captured in Figure 3 with solid lines indicating the flow and dashed lines the jumps. Given Ψ_0, this result shows the evolution of system states for a single realization of the sequence $\{\delta_k\}$ with mean sampling rate $\lambda = 4$. For a different sampling sequence with identical intensity λ, the trajectories converge to the origin analogously.

Let us now consider that there is about a 25% chance of packets being lost. Therefore, $\mu_1 = 1 - 0.25 = 0.75$, and solving the LMIs (17) and (18) yields the following feedback gain matrices

$$K = \begin{bmatrix} 0.262 & 0.264 & -0.0082 \end{bmatrix}, \; L = \begin{bmatrix} -1.13 & -1.13 \end{bmatrix}^{\mathrm{T}},$$

which lead to the MES of (9). This can be seen in Figure 4 where the error $e = \hat{x} - x$ converges to zero despite packet dropouts. Given $\lambda = 4$, by using a linear search, we can additionally compute from (17) and (18) that the maximal probability of packet dropout is 29.1%, i.e., $\mu_1 = 0.71$. For a given λ, the decay rate of convergence γ (in MES sense) decreases with an increasing packet dropout probability.

V. Conclusion

In this letter, we studied the output feedback stabilization of a stochastic sampled-data system where the outputs of the system are only available sporadically and are subject to packet dropouts. Unlike [10] and [19], which focus on the discrete-time trajectories of the system, we formally guarantee the stability of the continuous-time system.

The proposed stability analysis is built on the Dykin's theorem. Using a Lyapunov-like stability analysis method for such a SHS, we obtain sufficient stability conditions, which also lead to a numerically simple design of the hybrid stabilizer. The proposed analysis extends the notion of the "separation principle" from the classical to the stochastic case. In light of [6], the proposed results can also be trivially extended to the case when the inter-sampling intervals are Erlang-distributed. In the future, we would also like to extend this work to account for input and output nonlinearities such as actuator saturation and sensor quantization.

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