On Applying One-Sample Goodness-of-Fit Statistics to Coded FSK Decoding

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***Abstract*—In the previous works we have considered decoding coded FSK modulation (similar to Kautz-Singleton codes) transmitted in a channel with strong interference by using a two-sample goodness-of-fit statistics. In this work we consider the applicability of different one-sample goodness-of-fit statistics for the decoding. Using a computer simulation we show that using these statistics may yield lower error rates relative to the known decoders. The most notable of them are Anderson-Darling statistic that has the lowest error rates for narrow-band interference and Pearson's χ^2 and Kuiper statistics that are more resilient to wide-band interference. We can conclude that one-sample goodness-of-fit statistics can be used for Kautz-Singleton codes decoding with relatively low error rates in case of strong interference in the channel.**

I. Introduction

In this work we consider data transmission in a channel with strong interference. It is very hard to obtain decent channel state information on either receiver or transmitter, so the coded modulation considered does not rely on having any channel state information. Such modulations include Frequency-Shift Keying (FSK) and Differential Phase-Shift Keying. In this work we consider coded FSK, because it is better suited for frequency hopping as we will see later.

There are two main techniques to deal with strong interfering signals: frequency hopping and direct sequence spread spectrum. We only consider frequency hopping in this work. In the context of FSK frequency hopping can be described as pseudo-randomly selecting q FSK carrier frequencies from Q subcarriers.

The uncoded FSK might not be well suited for channels with strong interference so we use coded FSK with best linear codes. In [1] a simple repetition code was used, but in this work we have selected $[72, 2, 64]_8$ best linear code as the code.

In [2], [3] two decoders for this coded modulation were proposed. They are based on different two-sample goodness-of-fit criteria. These decoders are also effective in multiple access channels [1], [4], [5]. In this work we propose a technique to design better decoders using one-sample goodness-of-fit criteria.

II. Coded FSK

A. Channel model

Let us describe the channel in time-frequency domain (with OFDM and frequency hopping):

$$\pi(\mathbf{R}) = \mathbf{H} \circ \pi(\mathbf{S}) + \mathbf{E} + \mathcal{N}$$

where $\mathbf{S}$ is the transmitted signal, $\mathbf{R}$ — received signal, $\mathbf{H}$ — fading coefficients, $\mathcal{N}$ — noise and $\mathbf{E}$ — interference. The rows of the matrices correspond to different subchannels (frequencies), their columns — to different timeslots and $\circ$ denotes Hadamard entrywise matrix product. All matrices have size $Q \times n$ and elements in $\mathbb{C}$. Each column of the interference matrix $\mathbf{E}$ has Hamming weight w_e, $0 \le w_e < Q$. π is a pseudo-random column-wise permutation defined as:

$$\pi\left(\|\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\|\right) = \|\pi_1(\mathbf{s}_1), \pi_2(\mathbf{s}_2), \ldots, \pi_n(\mathbf{s}_n)\|.$$

Let us define Signal to Noise Ratio (SNR) and Signal to Interference Ratio (SIR):

$$\mathrm{SNR} = \frac{\mathrm{tr}(\mathbf{S}^{\mathrm{H}}\mathbf{S})}{\mathrm{tr}(\mathcal{N}^{\mathrm{H}}\mathcal{N})}, \qquad \mathrm{SIR} = \frac{\mathrm{tr}(\mathbf{S}^{\mathrm{H}}\mathbf{S})}{\mathrm{tr}(\mathbf{E}^{\mathrm{H}}\mathbf{E})},$$

where tr is a matrix trace and $\cdot^{\mathrm{H}}$ denotes conjugate transpose.

The probability distribution of the fading coefficients $\mathbf{H}$ is defined by COST 207 standard [6].

B. Coded FSK

The transmitted signal consists of $N \le \frac{Q}{q}$ codewords of a Kautz-Singleton code [7]:

$$\mathbf{S} = \left\|\mathbf{C}_1^{\mathrm{T}}, \mathbf{C}_2^{\mathrm{T}}, \ldots \mathbf{C}_N^{\mathrm{T}}, \mathbf{0}\right\|^{\mathrm{T}}. \tag{1}$$

One can say that codewords of Kautz-Singleton codes with each column modulated with OFDM correspond to coded FSK. In this work we call these codeword “stencil patterns” or just “stencils”. This name has sense as the decoder will use these codewords only to select the received signals at the positions where codeword symbols are nonzero, which is similar to looking through holes of a stencil. The decoder then applies some function to these signal energies and selects a stencil with the best function value.

To describe this decoder scheme in a formal way we need more notations. Let us denote the outer code of the Kautz-Singleton code (the one containing $\mathbf{C}_j$ of (1)) by $\mathcal{C}^{\mathrm{outer}}$ (the inner code is the FSK modulation itself). It is a code of length n over field GF(q). Let us split the matrix $\mathbf{R}$ in the same way as $\mathbf{S}$:

$$\mathbf{R} = \left\| \begin{matrix} \mathbf{R}^{(1)} \\ \mathbf{R}^{(2)} \\ \vdots \\ \mathbf{R}^{(N)} \\ \overline{\mathbf{R}} \end{matrix} \right\|.$$

All $\mathbf{R}^{(j)}$ are decoded in parallel. To do the decoding we first need to select absolute values of symbols of a single pattern. Let us define $\mathbf{x}(\mathbf{R}^{(j)}, \mathbf{c}) = \| |R^{(j)}_{c_1,1}|, \ldots, |R^{(j)}_{c_n,n}| \|^T$, where $\mathbf{c} \in \mathcal{C}^{\text{outer}}$. The decoder having a statistic $g : \mathbb{R}^T \times \mathbb{C}^{Q\times n} \to \mathbb{R}$ will make the following decision for word $\mathbf{R}^{(j)}$:

$$\arg\max_{\mathbf{c}\in\mathcal{C}^{\text{outer}}} g(\mathbf{x}(\mathbf{R}^{(j)}, \mathbf{c}), \mathbf{R})$$

In the most simple case this function g is just a sum of all elements of its first argument. This function is good for decoding only if there is neither fading nor interference in the channel. There are several attempts to introduce a good function for fading channels [8]. In this work we consider Order Statistics-Normalized Envelope Detection Based Diversity Combining (OSN) as the baseline for the error rate comparison.

C. Goodness-of-fit decoders

To make a good decoder for channels with low signal-to-interference ratio (SIR) we need to find a function that is hardly affected by interference.

Let us look at the signals seen through each stencil as samples of different random variables. All these random variables but the one corresponding to the transmitted stencil have the same distribution. This allows us to use statistical goodness-of-fit criteria to determine which stencil was transmitted.

In previous works [1]–[3] goodness-of-fit criteria were used to compare distributions of two data sets: the one seen through gaps of a stencil and the one covered by this stencil. Note that the latter is q times larger than the former. The decoder selected the stencil for which the distributions of these two sets differed the most.

In this work we propose to use other two sets. The first one is the same — the signal energies seen through the gaps of a stencil. But the second set is the same for all stencils and includes all received signal energies. As this set is Q times larger the first one, we will consider its empirical distribution function to be the target theoretical distribution function for goodness-of-fit criteria.

This solution has its downsides as now the distribution of the first set for a “wrong” stencil now differs from the distribution of the second set as it contains different proportion of cells in which the signal was really transmitted.

Let us now describe some one-sample goodness-of-fit criteria statistics. All of them define the function $g(\mathbf{x}, \mathbf{R}) = h(\mathbf{y})$, where $y_i = F_{\mathbf{R}}(x_i)$ and $F_{\mathbf{R}}$ is the empirical cumulative distribution function for the elements of $\mathbf{R}$.

- *Smirnov-Kramer-von Mises criterion* (also known as $n\omega^2$) has the statistic
$$h(\mathbf{y}) = \frac{1}{12n^2} + \sum_{i=1}^{n} \left\{ y_i - \frac{2i-1}{2n} \right\}^2$$
- *Pearson's χ^2 statistic.* Let us select thresholds $\tau_i = \frac{i}{L}$, $i = \overline{0, L}$, to partition the sample into L bins: $\nu_i = |\{j : \tau_{i-1} \le y_j < \tau_i\}|$. The Pearson's χ^2 statistic is defined as [9]
$$h(\mathbf{y}) = \frac{L}{n} \sum_{i=1}^{L} \left(\nu_i - \frac{n}{L}\right)^2$$
- *Number of empty bins statistic.* This simple statistic was described in [10]. It is a empty bins from the previous criterion:
$$h(\mathbf{y}) = |j : \nu_j = 0|$$
- *Kolmogorov-Smirnov statistic.*
$$h(\mathbf{y}) = \max\left\{ \max_{i=\overline{1,n}} \left(y_i - \frac{i-1}{n} \right), \max_{i=\overline{1,n}} \left(\frac{i}{n} - y_i \right) \right\}$$
- *Kuiper statistic [11].*
$$h(\mathbf{y}) = \max_{i=\overline{1,n}} \left(y_i - \frac{i-1}{n} \right) + \max_{i=\overline{1,n}} \left(\frac{i}{n} - y_i \right)$$
- *Anderson-Darling statistic.* Also known as $n\Omega^2$ this statistic is equal to the $n\omega^2$ divided by $F(x)(1 - F(x))$ [12]. A computable form is
$$h(\mathbf{y}) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left(\ln y_i + \ln[1 - y_{n-i+1}] \right)$$
- *Watson statistic.* In computable form the Watson statistic [13] is
$$h(\mathbf{y}) = \sum_{i=1}^{n} \left(y_i - \frac{2i-1}{2n} - \frac{1}{n} \sum_{j=1}^{n} y_j + \frac{1}{2} \right)^2 + \frac{1}{12n}$$

III. Simulation

To compare the error correction efficiencies of different decoders we have selected the following coded modulation parameters:

- $Q = 1663$ frequencies were used.
- Each stencil has $q = 8$ frequencies.
- Each OFDM symbol contained $N = 16$ codewords.
- The code of the coded modulation is $[72, 2, 64]_8$ best linear code.

We used COST 207 fading model in the following way:

- Each column of the matrix $\mathbf{S}$ (its height is $Q = 1663$) was transmitted on central frequencies of an OFDM symbol with 2048 subchannels.
- These OFDM used 31 MHz of bandwidth, but only 25 MHz contained any signal.
- Signal to interference ratio (SIR) is -30 dB.
- The channel model is COST 207 hilly terrain with relative speed 120 km/h.

We considered four different values for w_e (number of symbols in each column affected by interference): 0, 266, 532 and 1064. They correspond to the interference bandwidth 0, 4, 8 and 16 MHz (0%, 16%, 32% and 64% of the signal bandwidth respectively).

At first let us look at the error rates of different decoders in a channel without interference. These results are show on Fig. 1.

In this scenario OSN decoder has the lowest error rates. Its "win" might be due to OSN be the only considered decoder that utilizes the fact that the correct stencil has higher net energy. Among the "statistical" decoders the best ones are the Anderson-Darling and the two-sample sum-of-ranks decoders. As we will see later Anderson-Darling decoder is better that the sum-of-ranks decoder in every scenario which makes it a good substitution.

Also we can already remove the Watson and the Number of empty bins decoders from consideration as their error rates are significantly worse than the error rate of any other decoder. As we will see later their performance in other scenarios is also among "outsiders".

The narrowband (4 MHz) interference scenario is shown on Fig. 2. In this scenario OSN decoder changes from the best to the worst. Otherwise these results are similar to the ones of the no-interference scenario. The narrowband interference has made most decoders "lose" 1–2 dB

In the 8 MHz interference scenario shown on Fig. 3 OSN decoders is still among the worst, while Anderson-Darling decoder is still the best. Although, the Kuiper and two-sample Kolmogorov-Smirnov decoders are very close. The sum-of-ranks decoder is 2 dB behind them in this scenario.

In the 16 MHz interference scenario shown on Fig. 4 the error rates change drastically. Every decoder has lost at least 4 dB relative to 8 MHz case. Half of decoders got an error floor above 10^{-2}. OSN decoder has energy loss of 7 dB relative to Pearson and Kuiper decoders. Strangely this is the only scenario where Pearson;s χ^2 decoder has the lowest error rate. In every other scenario it has energy loss at least 2 dB relative to Anderson-Darling decoder. And Kuiper decoder has 1 dB energy gain relative to Kolmogorov-Smirnov decoder.

Another important result is that the decoder based on the one-sample Kolmogorov-Smirnov criterion beats its two-sample version proposed in [1] by 1 dB in every scenario. Therefore the technique introduced in this paper does improve the error correction efficiency even for the decoders based on two-sample goodness-of-fit criteria. This result was not evident before the simulation as though the proposed decoder scheme makes use of more data to estimate the second distribution most of this data is not tied to any decoder decision.

IV. Conclusion

We have described a coded FSK system to be used in systems having fading channels and strong interference. In previous work two decoders based on two-sample goodness-of-fit criteria were introduced. In this work we have introduced a way to one-sample goodness-of-fit criteria for decoding. We have implemented six such decoders and estimated their error correction performance with a computer simulation. Two of them based on the number of empty bins and the Watson criteria didn't show any good results. The performance of the decoder based on the Smirnov-Kramer-von Mises was worse than that of the Kolmogorov-Smirnov decoder introduced in previous works. The decoder based on Anderson-Darling criterion has proved to be a good substitution to the previously known sum-of-ranks decoder as it had performance in every simulated scenario. In the wideband interference scenario the decoders based on Pearson's χ^2 criterion and Kuiper criterion had 1-2 dB energy gain relative to the previous "leader", Kolmogorov-Smirnov decoder. The decoder based on the Kuiper criterion only has significant energy loss relative to the Kolmogorov-Smirnov decoder in a no-interference case.

We also compared the results for the decoder based on the Kolmogorov-Smirnov proposed earlier and the one designed with the technique described in this work. We can conclude that the proposed technique substantially improves the error rate relative to the older construction.

Generalizing all above we can say that one-sample goodness-of-fit criteria can be used for decoding coded FSK. The selection of the criterion heavily depends on the channel such characteristics as interference bandwidth and signal to interference ratio.

Acknowledgment

This work has been supported by RScF, research project No. 14-50-00150.

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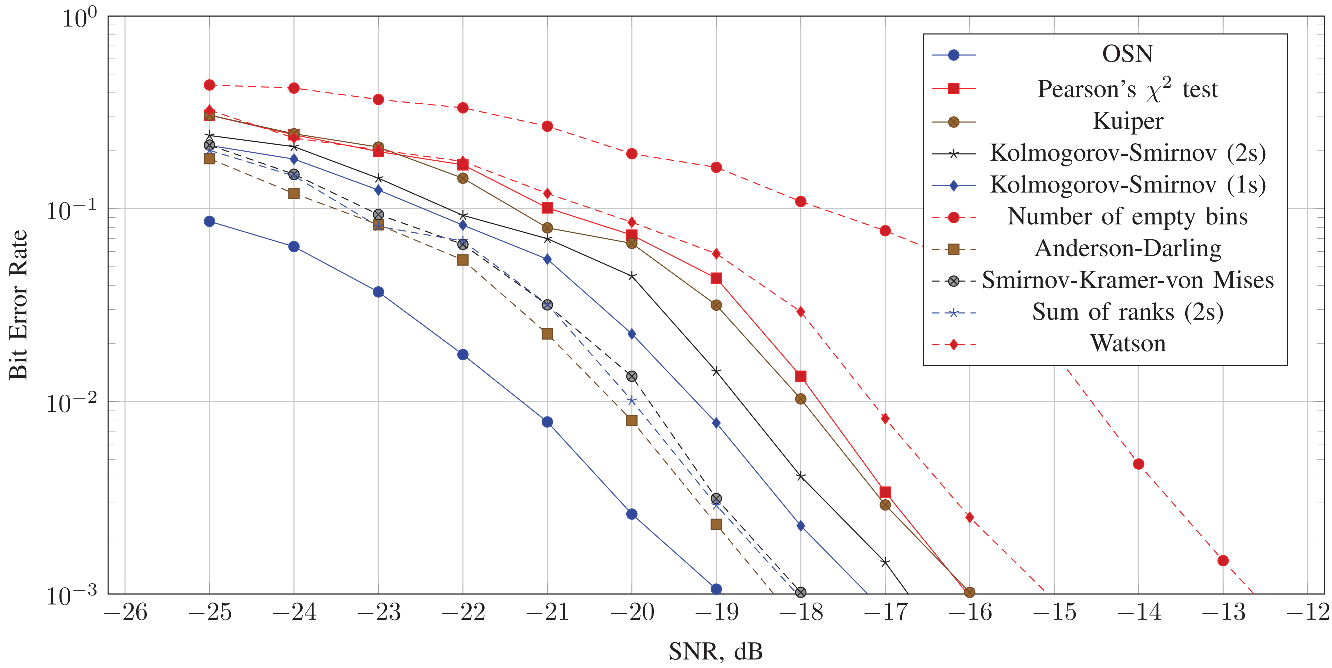

Fig. 1. Bit error rate without interference

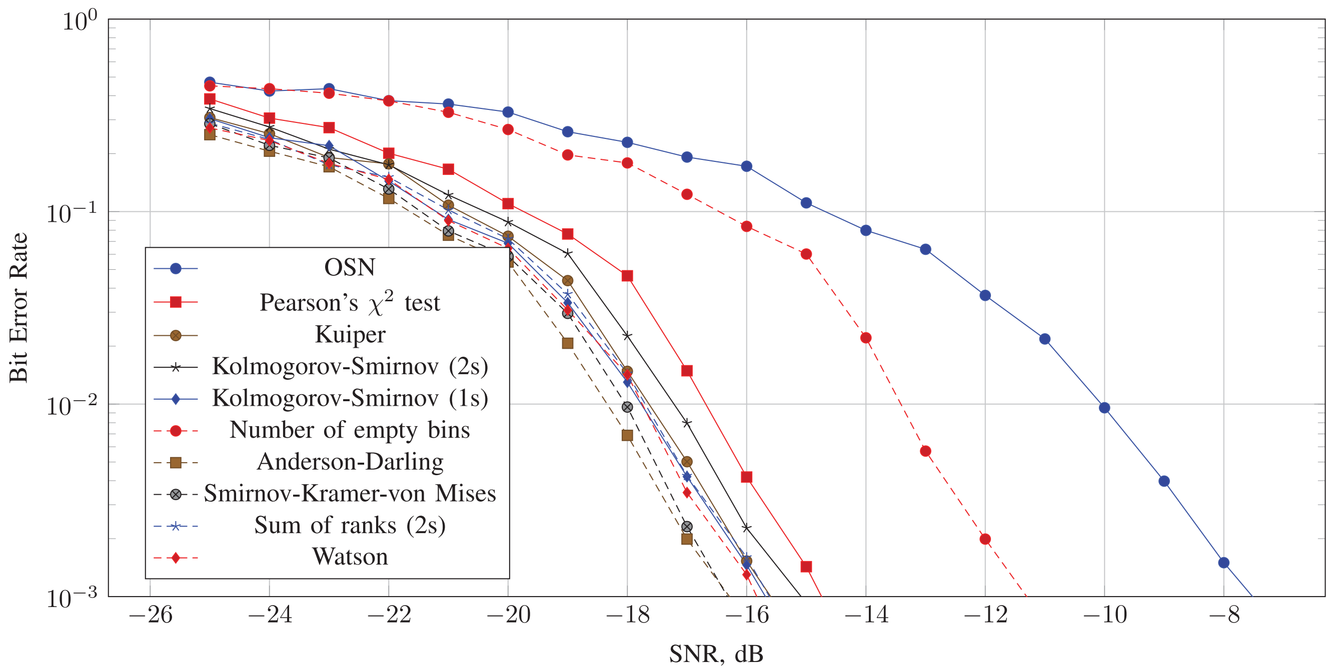

Fig. 2. Bit error rate with interference bandwidth 4 MHz (16% of signal bandwidth)

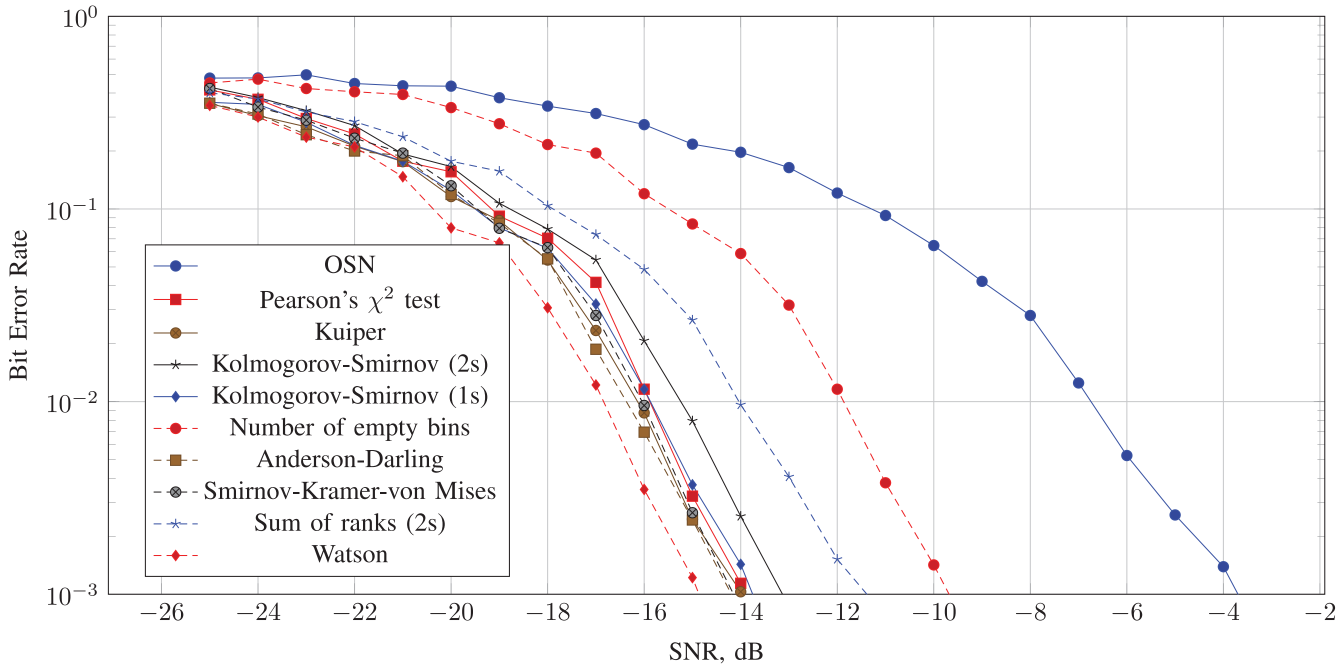

Fig. 3. Bit error rate with interference bandwidth 8 MHz (32% of signal bandwidth)

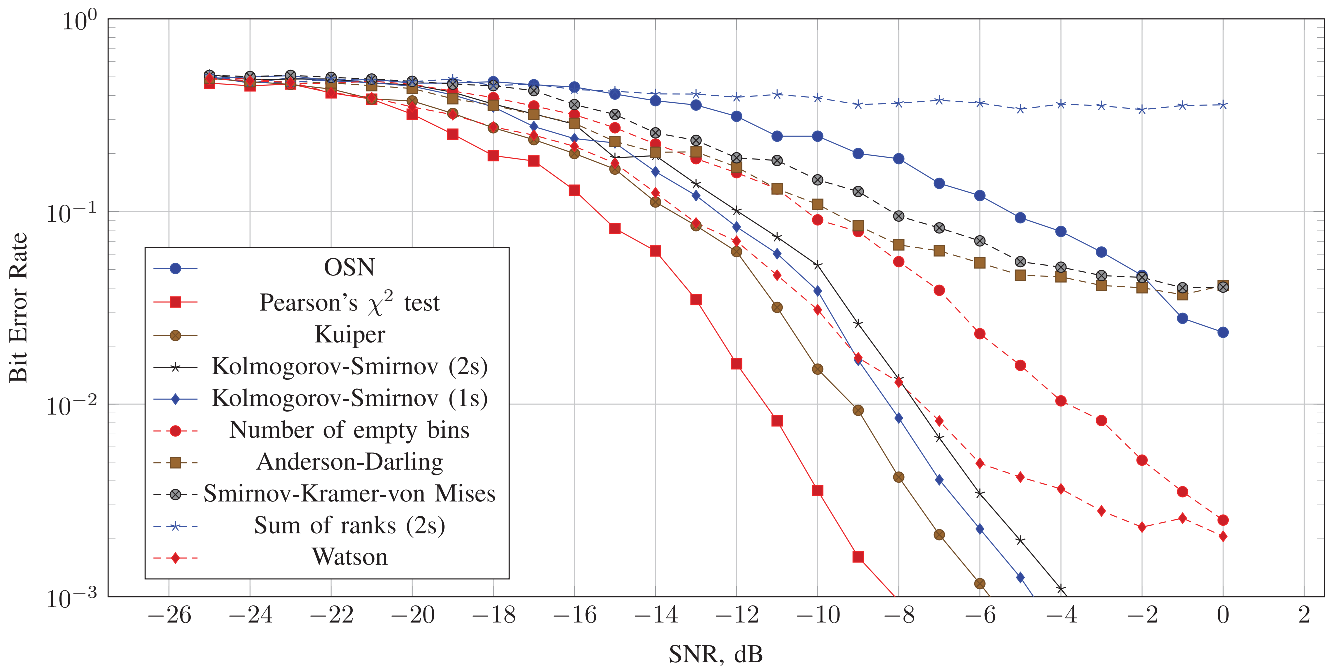

Fig. 4. Bit error rate with interference bandwidth 16 MHz (64% of signal bandwidth)