On Iterative LDPC-Based Joint Decoding Scheme for Binary Input Gaussian Multiple Access Channel

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***Abstract*—Non-orthogonal multiple access schemes are of great interest for next generation wireless systems, as such schemes allow to reduce the total number of resources (frequencies or time slots) in comparison to orthogonal transmission (TDMA, FDMA, CDMA). In this paper we consider an iterative LDPC-based joint decoding scheme suggested in [1]. We investigate the most difficult and important problem where all the users have the same power constraint and the same rate. For the case of 2 users we use a known scheme and analyze it by means of simulations. We found the optimal relation between the number of inner and outer iterations. We further extend the scheme for the case of any number of users and investigated the cases of 3 and 4 users by means of simulations. Finally, we showed, that considered non-orthogonal transmission scheme is more efficient (for 2 and 3 users), than orthogonal transmission.**

I. Introduction

The Multiple Access Channel (MAC) is a such type of channel in which a number of transmitters send information to a receiver. In a MAC, the transmitters may also cooperate with each other by exchanging information. The problem of finding the capacity region of the multiple access channel was first studied by Ahlswede in [2], few years after, Cover [3] and Wyner [4] proposed independently an expression for the capacity region of the two-user frame synchronous discrete memoryless Gaussian multiple access channel.

The MAC with correlated sources was first investigated in [5] wherein the capacity region of the discrete two-user MAC with cooperative encoders via common information was established. It was shown that the superposition encoding technique achieves the capacity region.

In another scenario, the transmitters exchange information via conferencing. The two-user MAC with conferencing encoders (MACCE) was first introduced by Willems [6].

In the last years low-density parity-check (LDPC) codes proposed by Gallager [7] have shown their excellent performance near the Shannon limit. In this paper we investigate an iterative LDPC-based joint decoding scheme for Binary Input Gaussian Multiple Access Channel (BI-AWGN-MAC).

Several works have already considered this issue. The main contributions were that of Chayat and Shamai [8] who investigated the convergence of iterative coding on the multiple-access channel, but in their setting the iterative decoder was only able to decode when the users had their powers on different levels. For the equal power case, they proposed

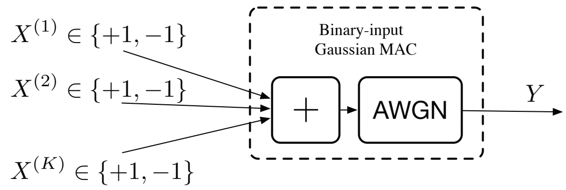

Fig. 1. Binary Input Gaussian Multiple Access Channel (BI-AWGN-MAC)

the use of rate-splitting [9]. We are interested in the most difficult and important problem where all the users have the same power constraint and the same rate and code lengths. An iterative LDPC-based coding scheme for this case was suggested in [1], it was shown, that density evolution threshold of this scheme is only 0.18 dB away from capacity.

We continue investigation of an iterative LDPC-based joint decoding scheme suggested in [1]. For the case of 2 users we use a known scheme and analyze it by means of simulations. We found the optimal relation between the number of inner and outer iterations. We further extend the scheme for the case of any number of users and investigated the cases of 3 and 4 users by means of simulations. Finally, we showed, that suggested non-orthogonal transmission scheme is more efficient (for 2 and 3 users), than orthogonal transmission.

II. Preliminaries

A. Binary Input Gaussian Multiple Access Channel

Consider BI-AWGN-MAC, which is shown in Fig. 1.

Let us denote the number of users by K. For a certain time instant τ the channel inputs are variables $X_\tau^{(1)}, X_\tau^{(2)}, \ldots, X_\tau^{(K)}$, belonging to the alphabet $\{-1,+1\}$, and the channel output at time instant τ is calculated as follows

$$Y_\tau = \sum_{k=1}^{K} X_\tau^{(k)} + Z_\tau,$$

where Z_τ is an i.i.d. sequence of zero mean Gaussian random variables with variance σ^2.

In what follows we omit the index τ. Let $[K] = \{1,2,\ldots,K\}$ and R_i is a rate of i-th user's code. The capacity region of this channel can be written as follows [10]

$$\sum_{i \in U} R_i \le I(\{X^{(i)}\}_{i\in U}; Y | \{X^{(i)}\}_{i \in [K]\setminus U}), \ \forall U \subseteq [K].$$

where $I(A;B|C)$ is a conditional mutual information in between random variables A and B given C.

For the case of equal rates (i.e. $R_i = R,\ i = 1 \dots K$) the capacity is dominated by the last inequality [10], i.e

$$\sum_{u=1}^{K} R_u = KR \le I(X^{(1)}, X^{(2)}, \dots, X^{(K)}; Y).$$

Particularly for BI-AWGN-MAC and the case of equal rates we have (see [1] for more details)

$$KR \le \int_{-\infty}^{+\infty} \Phi(K,x)\log_2(\Phi(K,x))dx - \frac{1}{2}\log_2(2\pi e\sigma^2), \quad (1)$$

where

$$\Phi(2,x) = \frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{1}{4}e^{\frac{-(x-2)^2}{2\sigma^2}} + \frac{1}{2}e^{\frac{-x^2}{2\sigma^2}} + \frac{1}{4}e^{\frac{-(x+2)^2}{2\sigma^2}}\right),$$

$$\Phi(3,x) = \frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{1}{8}e^{\frac{-(x-3)^2}{2\sigma^2}} + \frac{3}{8}e^{\frac{-(x-1)^2}{2\sigma^2}} + \frac{3}{8}e^{\frac{-(x+1)^2}{2\sigma^2}} + \frac{1}{8}e^{\frac{-(x+3)^2}{2\sigma^2}}\right)$$

and

$$\Phi(4,x) = \frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{1}{16}e^{\frac{-(x-4)^2}{2\sigma^2}} + \frac{1}{4}e^{\frac{-(x-2)^2}{2\sigma^2}} + \frac{3}{8}e^{\frac{-x^2}{2\sigma^2}} + \frac{1}{4}e^{\frac{-(x+2)^2}{2\sigma^2}} + \frac{1}{16}e^{\frac{-(x+4)^2}{2\sigma^2}}\right)$$

Note, that BI-AWGN-MAC can be seen as a degraded version of the binary adder channel and its capacity region for K users is upper bounded by the capacity of the corresponding binary adder channel, i.e

$$\begin{aligned} 2R &\le 1.5 & K=2 \\ 3R &\le 1.81... & K=3 \\ 4R &\le 2.03... & K=4 \end{aligned}$$

At the end of this section we present (see Table I) Shannon limits (in terms of dB) for the cases which we consider below. Let us give some explanations. The first column contains of a number of users K and the row is a code rate R (we consider the case when all users' codes have an equal rates). The numbers in the cells of the table are the signal-to-noise ratio (SNR) values obtained from inequality (1). Here and in what follows we calculate SNR as follows

$$\text{SNR} = \frac{E_s}{N_0} \text{ (dB)},$$

where E_s is an average signal energy ($E_s = 1$ for BPSK constellation) and N_0 is the noise power spectral density ($N_0 = \sigma^2$).

B. LDPC codes

A binary LDPC code $\mathcal{C}$ of length N is a null-space of an $M \times N$ sparse binary parity-check matrix $\mathbf{H} = [h_{i,j}]$, $1 \le i \le M$, $1 \le j \le N$. By ℓ_j, $j = 1, \dots, N$, we denote the weight of j-th column, by Δ_i, $i = 1, \dots, M$, we denote the weight of i-th row. Here and in what follows by weight we mean the Hamming weight, i.e. the number of non-zero elements in a vector.

TABLE I
SHANNON LIMITS FOR BI-AWGN-MAC, DB

$K \setminus R$	0.1	0.2	0.3	0.4	0.5	0.6	0.7
2	-7.96	-4.29	-1.83	0.17	1.99	3.87	6.39
3	-7.64	-3.62	-0.77	1.66	6.51	9.32	–
4	-7.31	-2.92	0.33	3.33	8.35	–	–

The constructed code $\mathcal{C}$ can be described with use of bipartite graph, which is called the Tanner graph [11] (see Fig. 2). The vertex set of the graph consists of the set of variable nodes $V = \{v_1, v_2, \dots, v_N\}$ and the set of check nodes $C = \{c_1, c_2, \dots, c_M\}$. The variable node v_j and the check node c_i are connected with an edge if and only if $h_{i,j} \neq 0$.

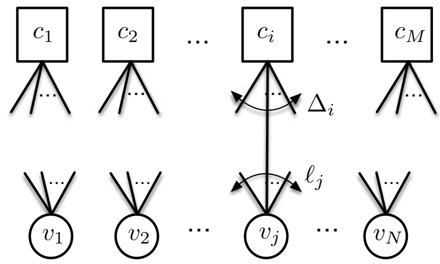

Fig. 2. Tanner graph

In this paper we need to construct LDPC codes for high ($R \ge 0.8$), medium ($0.3 \le R \le 0.8$) and low rates ($R < 0.3$). The usual procedure to construct an LDPC code of rate $R \ge 0.3$ is as follows:

- use density evolution method to find optimal row and column weight distributions (see [12]);
- use progressive-edge grows (PEG) algorithm [13] to construct parity-check matrix with big girth (the length of the shortest cycle in the Tanner graph) from found distributions.

If we apply these 2 steps to construct low-rate LDPC code we will fail with probability very close to 1. To construct LDPC codes with $R < 0.3$ we use a Raptor-like technique [14]. The idea is to start with high- or medium-rate matrix and enlarge it with a special method.

We use these techniques to construct all the matrices in the paper. We do not know optimal distributions for BI-AWGN-MAC (and it is the further research task), so we used regular codes with 3 ones in column for $K = 2, 3, 4$ and $R \ge 0.3$. The matrices for $R = 0.1$ and 0.2 are constructed with use of a Raptor-like technique. For $K = 1$ we use optimal degree distributions.

III. ITERATIVE JOINT DECODING ALGORITHM

By $\mathcal{C}^{(k)}$, $k \in [K]$, we denote the codes used by users (the codes are binary). Recall, that N and R are accordingly the

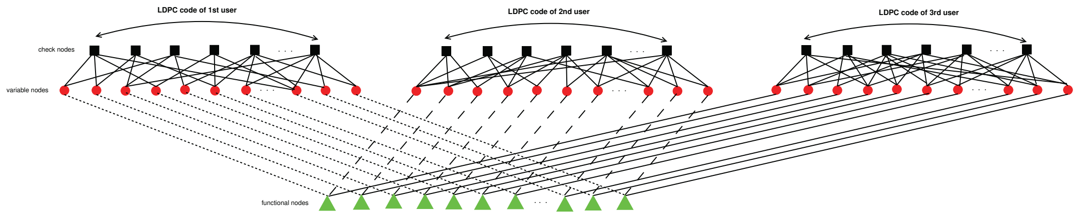

Fig. 3. Joint decoder graph representation for $K = 3$

length and the rate of $\mathcal{C}^{(k)}$, $k \in [K]$. For the simplicity of explanation, we assume frame synchronization to present in the system. This is usual for current wireless systems (e.g. LTE). At the same time, we would like to point out, that all the techniques still work in the case where there is no frame synchronization.

Thus K users send codewords $\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(K)}$. After BPSK modulator we have the sequences $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(K)}$, $\mathbf{x}^{(i)} \in \{-1, +1\}^N$. The channel output ($\mathbf{y}$) is the element-wise sum of the sequences affected by Gaussian noise.

The aim of joint multi-user decoder is to recover the all the codewords based on received vector $\mathbf{y}$. The decoder employs a low-complexity iterative belief propagation (BP) decoder that deals with a received soft information presented in LLR (log likelihood ratio) form. The decoding system can be represented as a graph (factor graph), which is shown in Fig. 3. User LDPC codes are presented with use of Tanner graphs with variable and check nodes. At the same time there is a third kind of nodes in the figure – functional nodes (marked with green color). These nodes correspond to the elements of the received sequence $\mathbf{y}$.

Following the by now standard methodology of factor graphs, see [12], [15], we can write down the corresponding message passing decoding algorithm.

1) initialize the LLR values of variable nodes for each user code with zero values assuming equal probability for 1 and -1 values;
2) perform I_O outer iterations, where each iteration consists of the following steps:
   a) perform maximum likelihood decoding of functional nodes (i.e. calculate update messages for variable nodes);
   b) perform I_I inner iterations of BP decoder for users' LDPC codes and update LLR values of variable nodes (this is done in parallel);

The BP part is standard, i.e. each user utilizes standard BP decoding algorithm (Sum-Product or Min-Sum) to decode an LDPC code. The most interesting part is the decoding of functional nodes. Following the principles of message-passing algorithms, the update rule to compute the message (μ) sent to i-th variable node of k-th user ($k = 1, \ldots, K,\ i = 1, \ldots, N$) from a functional node $\mathcal{F}_i$ is the following:

$$\begin{aligned} \mu_{\mathcal{F}_i \to \text{user } k}\left(x_i^{(k)}\right) &= \sum_{\sim x_i^{(k)}} \prod_{j \neq k} \mu_{\text{user } j \to \mathcal{F}_i}\left(x_i^{(j)}\right) \\ &\times \ \Pr\left(y_i | x_i^{(1)}, \ldots, x_i^{(K)}\right). \end{aligned}$$

and

$$LLR_{\mathcal{F}_i \to \text{user } k}\left(x_i^{(k)}\right) = \log\left(\frac{\mu_{\mathcal{F}_i \to \text{user } k}\left(x_i^{(k)} = 1\right)}{\mu_{\mathcal{F}_i \to \text{user } k}\left(x_i^{(k)} = -1\right)}\right)$$

The number of computations necessary to obtain the outgoing messages from the node $\mathcal{F}_i$ grows exponentially with the number of users, nevertheless, this number of users usually remains small, and we will therefore not be concerned with this fact.

IV. Simulation results

In this section we present the simulation results. For all the simulations in the section the length of users' LDPC codes N is equal to 2000 bits. Clear, that for the scheme to work users should use different codes. This can be explained as follows. If the codes are the same and the code of the first user cannot correct errors, so does (because of the code linearity) the code for the second user.

We generate different codes as follows: we start with one LDPC code and each user utilizes different permutation of code bits. In the scheme we have outer iterations and inner ones (iterations of BP decoding of LDPC code or just LDPC iterations). In all the simulations the product of inner and outer iterations is equal to 50.

First we investigate the dependence of performance (in terms of Block Error Rate (BLER)) on the number of outer iterations (see Fig. 4) and inner iterations. As a result we present a number of curves that represent a dependency between BLER and SNR for the case when number of users $K = 2$ and $R = 0.5$. We see that the best performance is achieved if we use 10 outer iterations and 5 inner iterations. The explanation of this fact is as follows. First of all it is clear, that the number of inner iterations cannot be big as in this case we have K single users' decoders rather that joint decoder. At the same time, small number of inner iterations is not sufficient for LDPC decoder. Thus, we have an optimal value in the middle ($I_I = 5$).

Now we present the simulation results for $K = 2, 3$ and 4 users. We see the scheme for $K = 2$ (Fig. 5) is stable up to

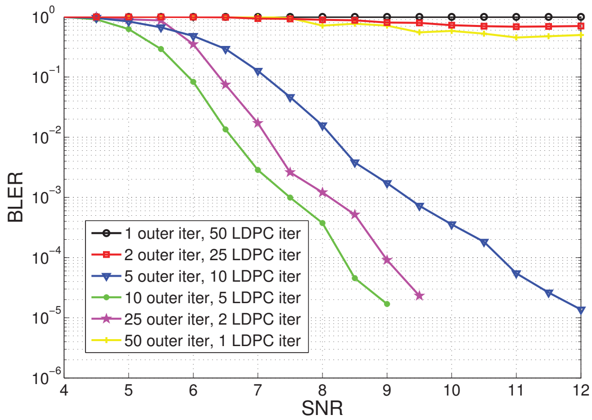

Fig. 4. BLER versus SNR for the various number of outer and LDPC (inner) iterations, $K = 2$, $R = 0.5$

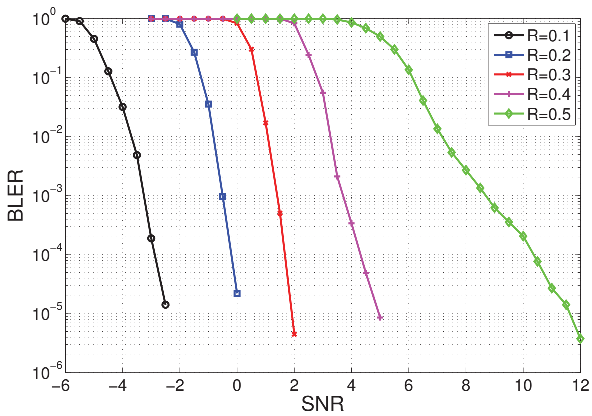

Fig. 5. Result for $K = 2$

$R = 0.5$, for $K = 3$ (Fig. 6) – up to $R = 0.3$ and for $K = 4$ (Fig. 7) – up to $R = 0.2$

We also note, that the results for finite length are rather far from Shannon limits (see Table I).

At last we want to compare the best schemes for $K = 1, 2, 3$ and 4. We note, that the scheme for $K = 1$ corresponds to orthogonal transmission (e.g. TDMA). There are no collisions in the scheme and users just utilize single users LDPC decoders (there are no outer iterations and the number of inner iterations is equal to 50). To compare these schemes we introduce a general E_b/N_0 (SNR per information bit) value as follows

$$E_b/N_0 = \frac{E_s}{N_0 R_\Sigma} \text{ (dB)},$$

where R_Σ is a sum rate, i.e. the total number of bits sent by K users per channel use. Recall, that E_s is an average signal energy ($E_s = 1$ for BPSK modulation) and $N_0 = \sigma^2$ is noise power spectral density.

The results are shown in Fig. 8, we see that the best schemes for $K = 2$ and 3 have a gain approx. 1 dB in comparison to single user case.

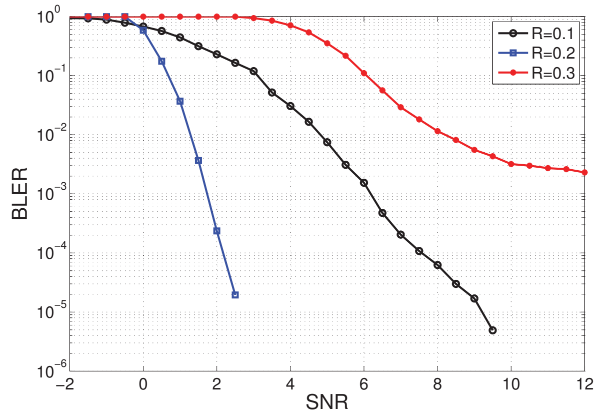

Fig. 6. Result for $K = 3$

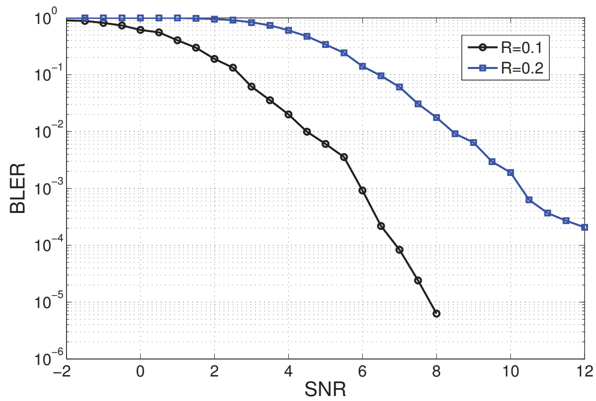

Fig. 7. Result for $K = 4$

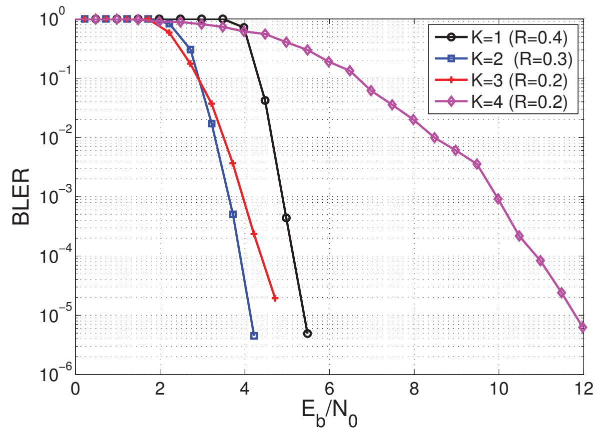

Fig. 8. Comparison of the best schemes

V. Conclusion

In this paper we considered an iterative LDPC-based joint decoding scheme for BI-AWGN-MAC. We investigated the most difficult and important problem where all the users have the same power constraint and the same rate. For the case of 2 users we use a known scheme and analyze it by means of simulations. We found the optimal relation between the number of inner and outer iterations. We further extend the scheme for the case of any number of users and investigated the cases of 3 and 4 users by means of simulations. Finally, we showed, that considered non-orthogonal transmission scheme is more efficient (for 2 and 3 users), than orthogonal transmission.

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