Complex Unit Vector for the Complex Wave Constant $\tilde{k}$ in a Lossy Medium

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In a 2011 article in *IEEE Antennas and Propagation Magazine*, it was suggested that Snell's law and Fresnel coefficients in their standard forms are incorrect when a wave is incident from a lossless medium to a lossy medium [1]. However, this article finds why those conclusions were reached and shows that those equations in their standard forms are all correct.

BACKGROUND ON SNELL'S LAW AND FRESNEL COEFFICIENTS

Snell's law and Fresnel coefficients in their "standard" forms were once suggested to be incorrect when light is incident from a lossless to a lossy medium [1]. The incidence and transmission of this case are illustrated in Figure 1 (our article uses different notations from those in [1]). We find that the conclusions in [1] were reached based on the approach to the unit vector of $\vec{\tilde{k}}$, in Figure 1, as being a purely real vector.

Equations (1a) and (1b) are Snell's law in two different forms: (1a) being the so-called standard form and (1b) in terms of the modified propagation constants of the lossy medium. The article [1] suggests that (1a) is incorrect while (1b) is correct: [1] also says that the term "$\tilde{k}\cos\tilde{\theta}_t$" in the standard form of Fresnel coefficients should be replaced by "$k_r\cos\theta_r + ik_I$" as "$\tilde{k}\cos\tilde{\theta}_t$" is not identical to "$k_r\cos\theta_r + ik_I$", where k_r and k_I are the modified propagation constants of the lossy medium responsible for the phase propagation and amplitude, respectively [2], [3]. Equation (2) shows this suggestion.

$$k_i \sin\theta_i = \tilde{k}\sin\tilde{\theta}_t, \tag{1a}$$

$$k_i \sin\theta_i = k_r \sin\theta_r, \tag{1b}$$

$$\tilde{k}\cos\tilde{\theta}_t \to k_r\cos\theta_r + ik_I. \tag{2}$$

This article is partly to say that the conclusions made in [1] are reasonable if we limit the unit vector of $\vec{\tilde{k}}$ to only be a purely real vector. However, we extend the unit vector of $\vec{\tilde{k}}$ to generally be a complex vector, written as "$\hat{\tilde{m}}$" in this article. In the "Derivation" section, the complex unit vector $\hat{\tilde{m}}$ is identified, and the standard forms of Snell's law and Fresnel coefficients are demonstrated to be valid with use of the complex unit

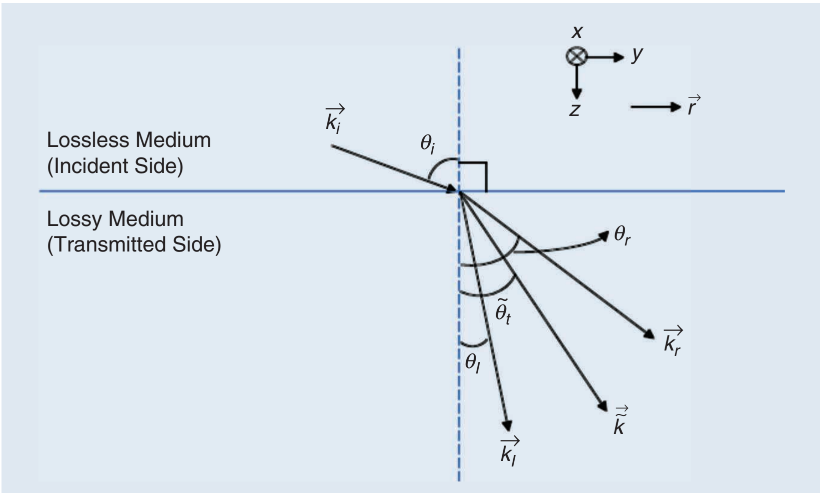

FIGURE 1. The incident and transmitted waves from a lossless medium to a lossy medium. The vector $\vec{r}$ is aligned such that $\vec{r} = (0, y, 0)$. The complex-valued angle $\tilde{\theta}_t$ is visualized as if it were a real angle. However, the angle $\tilde{\theta}_t$ is just algebraic, and no geometrical meaning is put into this.

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vector $\hat{\tilde{m}}$. To begin with, we represent the incident (E_i) and transmitted (E_t) electric fields in Figure 1 by

$$E_i(r,t) = E_{oi}\, e^{i(\vec{k}_i\cdot\vec{r}-\omega t)}, \quad (3)$$

$$E_t(r,t) = E_{ot}\, e^{i((\vec{k}_r+i\vec{k}_I)\cdot\vec{r}-\omega t)}, \quad (4a)$$

$$E_t(r,t) = E_{ot}\, e^{i(\vec{\tilde{k}}\cdot\vec{r}-\omega t)}, \quad = E_{ot}\, e^{i(\tilde{k}\hat{\tilde{m}}\cdot\vec{r}-\omega t)}. \quad (4b)$$

DERIVATION

COMPLEX UNIT VECTOR $\hat{\tilde{M}}$

Based on their approach to the unit vector of interest, [1] states that the form of the electric field in (4b) is incorrect while (4a) is correct. However, in this work, let us start the derivation by extending our view of vectors to complex such that (4a) and (4b) are identical. To identify the complex unit vector $\hat{\tilde{m}}$, write the wave vector $\vec{\tilde{k}}$ in Figure 1 as

$$\begin{aligned}\vec{\tilde{k}} &= \vec{k}_r + i\vec{k}_I \\ &= (0, k_{ry} + ik_{Iy}, k_{rz} + ik_{Iz}) \\ &= \tilde{k}\hat{\tilde{m}} \\ &= (k' + ik'')(0, \tilde{m}_y, \tilde{m}_z) \\ &= (0, \tilde{k}_y, \tilde{k}_z), \end{aligned} \quad (5)$$

where $\tilde{k} = \sqrt{(0)^2 + (\tilde{k}_y)^2 + (\tilde{k}_z)^2}$ and k' and k'' are the intrinsic propagation constants, respectively. From the relationship in (5), each component of $\hat{\tilde{m}}$ is obtained as follows:

$$\tilde{m}_x = \tilde{k}_x = 0, \quad (6a)$$

$$\tilde{m}_y = \frac{k_{ry} + ik_{Iy}}{\tilde{k}} = \frac{k_r \sin\theta_r + ik_I \sin\theta_I}{\tilde{k}}, \quad (6b)$$

$$\tilde{m}_z = \frac{k_{rz} + ik_{Iz}}{\tilde{k}} = \frac{k_r \cos\theta_r + ik_I \cos\theta_I}{\tilde{k}}. \quad (6c)$$

SNELL'S LAW DERIVATION BASED ON (3) AND (4a)

From (3) and (4a), the boundary condition says

$$\vec{k}_i \cdot \vec{r} = (\vec{k}_r + i\vec{k}_I)\cdot\vec{r}. \quad (7)$$

The left-hand side (LHS) of (7) $= (k_i \sin\theta_i)(y)$. The right-hand side (RHS) of (7) $= (k_r \sin\theta_r + ik_I \sin\theta_I)(y)$.

From the equality of the LHS and RHS of (7), we obtain $k_i \sin\theta_i = k_r \sin\theta_r + ik_I \sin\theta_I$. Since the imaginary part of the equation must be zero, θ_I is 0. Therefore, this leads to

$$k_i \sin\theta_i = k_r \sin\theta_r. \quad (8)$$

Equation (8) is Snell's law in terms of the modified propagation constants of the lossy medium, which is equal to (1b). Note that, as described well in [2] and [3], the modified propagation constants, k_r and k_I, are not necessarily identical to the intrinsic propagation constants, k' and k'', respectively, when the medium is lossy.

SNELL'S LAW DERIVATION BASED ON (3) AND (4b)

By definition, the magnitude of the complex unit vector $\hat{\tilde{m}}$ is equal to 1. That is, from (6),

$$\begin{aligned}\tilde{m} &= \sqrt{(\tilde{m}_x)^2 + (\tilde{m}_y)^2 + (\tilde{m}_z)^2} \\ &= \sqrt{0 + \left(\frac{\tilde{k}_y}{\tilde{k}}\right)^2 + \left(\frac{\tilde{k}_z}{\tilde{k}}\right)^2} \\ &= \sqrt{\left(\frac{\tilde{k}}{\tilde{k}}\right)^2} = 1. \end{aligned} \quad (9)$$

Thus, the y-component of $\hat{\tilde{m}}$ can be written as

$$\tilde{m}_y = \tilde{m}\sin\tilde{\theta}_t = \sin\tilde{\theta}_t. \quad (10)$$

From (10) and (6b),

$$\begin{aligned}\sin\tilde{\theta}_t = \tilde{m}_y &= \frac{k_{ry} + ik_{Iy}}{(k' + ik'')} \\ &= \left(\frac{k_r \sin(\theta_r)k' + k_I \sin(\theta_I)k''}{k'^2 + k''^2}\right) \\ &\quad + i\left(\frac{k_I \sin(\theta_I)k' - k_r \sin(\theta_r)k''}{k'^2 + k''^2}\right) \\ &\equiv a + ib. \end{aligned} \quad (11)$$

From (3) and (4b), the boundary condition says

$$\vec{k}_i \cdot \vec{r} = (\tilde{k}\hat{\tilde{m}})\cdot\vec{r}. \quad (12)$$

The LHS of $(12) = (k_i \sin\theta_i)(y)$.
The RHS of $(12) = (\tilde{k}\tilde{m}_y)(y) = (\tilde{k}\sin\tilde{\theta}_t)(y)$.

From the equality of the LHS and RHS of (12),

$$\begin{aligned}k_i \sin\theta_i &= \tilde{k}\sin\tilde{\theta}_t \\ &= (k' + ik'')(a + ib) \\ &= (k'a - k''b) + i(k'b + k''a). \end{aligned} \quad (13)$$

In (13), the imaginary part of the RHS must be zero. Therefore,

$$k'b = -k''a. \quad (14)$$

The LHS of (14)

$$= k'\left(\frac{k_I \sin(\theta_I)k' - k_r \sin(\theta_r)k''}{k'^2 + k''^2}\right) = \frac{k_I \sin(\theta_I)k'^2 - k_r \sin(\theta_r)k'k''}{k'^2 + k''^2}.$$

The RHS (14)

$$= -k''\left(\frac{k_r \sin(\theta_r)k' + k_I \sin(\theta_I)k''}{k'^2 + k''^2}\right) = -\frac{k_r \sin(\theta_r)k'k'' + k_I \sin(\theta_I)k''^2}{k'^2 + k''^2}.$$

From the equality of the LHS and RHS of (14) displayed previously, we obtain $\theta_I = 0$. By plugging this result into (13) again, (13) becomes $k_i \sin\theta_i = \tilde{k}\sin\tilde{\theta}_t = (k'a - k''b) = k_r \sin\theta_r$. Therefore, from this relationship, the standard form of Snell's law in (1a) is identical to (1b).

THE "$\tilde{k}\cos\tilde{\theta}_t$" TERM IN THE STANDARD FORM OF FRESNEL COEFFICIENTS

From (9), the z-component of $\hat{\tilde{m}}$ can be written as

$$\tilde{m}_z = \tilde{m}\cos\tilde{\theta}_t = \cos\tilde{\theta}_t. \quad (15)$$

From (15) and (6c), $\tilde{k}\cos\tilde{\theta}_t$ can be written as

$$\tilde{k}\cos\tilde{\theta}_t = \tilde{k}\tilde{m}_z = k_r \cos\theta_r + ik_I \cos\theta_I. \quad (16)$$

Because $\theta_I = 0$, (16) reduces to

$$\tilde{k}\cos\tilde{\theta}_t = k_r \cos\theta_r + ik_I. \quad (17)$$

Therefore, the term "$\tilde{k}\cos\tilde{\theta}_t$" in the standard form of Fresnel coefficients in [1] is identical to "$k_r \cos\theta_r + ik_I$".

CONCLUSIONS

The standard forms of Snell's law and Fresnel coefficients were previously suggested to be incorrect when the wave is incident from a lossless medium to a lossy medium [1]. These conclusions were drawn from the approach that the unit vector of $\vec{\tilde{k}}$ is a purely real vector and, therefore, the electric field representations in (4a) and (4b) cannot be identical. However, in this work, the complex unit vector of $\vec{\tilde{k}}$ is shown to exist and identified such that (4a) and (4b) are identical. Therefore, as a result, both forms of Snell's law in (1a) and (1b) are correct and identical. Also, as the term "$\tilde{k}\cos\tilde{\theta}_t$" is demonstrated to be identical to "$k_r\cos\theta_r + ik_I$," the Fresnel coefficients in their standard form in [1] are correct as well. The equations written in terms of the modified propagation constants, k_r and k_I, may deliver their physical meanings more directly than the so-called standard form equations do. However, apart from that, both types of equations are mathematically correct and valid in representing the wave behaviors.

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AUTHOR'S RESPONSE

In July 2011, I presented the article "Corrected Fresnel Coefficients for Lossy Materials" at the IEEE Antennas and Propagation Symposium in Spokane, Washington [1]. In it, I considered the case of the propagation of a plane wave from a lossless isotropic medium across a planar boundary into a lossy isotropic medium. The study came to a conclusion that certainly would be expected to be controversial.

That article considered the "standard" derivation of the Fresnel coefficients as given in standard texts, including Stratton [2], Jackson [3], and Feynman et al. [4], and concluded that there was an error in all these texts both in the derivation and in the final result. Up until now, I have not discussed in print how I believe that this is possible given that the "standard" Fresnel coefficients have been in the literature for 80 years. However, it is time to do so as it will help shed light on a previous comment [5] on the article [1] and also on Minsu Oh's current comment to which I am responding

It is my opinion that the reason an incorrect formula has been accepted all these years is that electrical engineers have been able to use that formula to get correct results by an unexpected method. It is also my opinion that the "standard" Fresnel equations are so flawed that it is not possible to put numbers into them to perform a calculation. However, the "standard" Fresnel equations for both polarizations scream of a physical meaning. Intelligent electrical engineers see that physical meaning and use numerical values consistent with it when performing calculations. Thus, it appears that calculations have actually been unknowingly performed for decades with the "corrected" Fresnel coefficients given in [1], even though they produce a different numerical result than would be obtained with the "standard" Fresnel coefficients. In other words, it appears that most electrical engineers look at the "standard" Fresnel equations and then use the numbers for the "corrected" Fresnel equations in them. This is described in more detail at the end of my response.

The previous comment on [1] (see [5]) and the current comment both raise the issue of whether the "corrected" Fresnel coefficients in [1] are different from or equal to the "standard" Fresnel coefficients. That is, do the "corrected" coefficients predict the same scattering and transmission coefficients as the Fresnel coefficients that have been accepted for 80 years? Later, I argue that if the "standard" Fresnel coefficients are applied EXACTLY AS WRITTEN, they give different values for the reflection and transmission coefficients than the "corrected" Fresnel coefficients.

This magazine previously published the comment by Ionis Besieris [5] on my article [1]. The abstract of that comment states in full, "It is shown that the 'corrected' Fresnel reflection and transmission coefficients derived recently by Canning [1] using a complex transmission wave vector approach and involving a real true angle of reflection are *identical* to the traditional coefficients based on a complex angle of refraction." In the response [6] to that comment, I assert that in deriving his equation (12), an assumption is made that is equivalent to assuming the "standard" and "corrected" forms are equal (see the first full paragraph after (21) in the reply [6]). Another way of saying this is that the way electrical engineers have been modifying the "standard" form of the Fresnel equations for decades by replacing one term with what appears to equal it and that changes it into the "corrected" form of the Fresnel equations was used in deriving his equation (12).

The current comment by Minsu Oh is in regard to my reply [6] to the comment by Ioanis Besieris [5] on my article [1]. The title of the current comment is "Complex Unit Vector for the Complex Wave Constant $\tilde{k}$ in a Lossy Medium." That is, unit vectors are used that are in a direction of one of the three Cartesian coordinate directions and that have a unit magnitude and also have a phase factor. My first contribution on this subject [1] and those that followed stressed that in the lossy medium, the phase changed in an oblique direction to the interface between refractive media while the decay in magnitude occurred in a perpendicular direction. The proofs presented in [1] that the derivations of the "standard" Fresnel coefficients are incorrect used the fact that those "standard" derivations incorrectly assumed that both changes occurred in the same spatial direction.

The current comment uses a different notation to handle the two different directions than that used in [1]. However, the use of a "complex unit vector" allows the necessary degrees of freedom to accommodate the correct physics. Indeed, it is gratifying that the steps in the derivation in the current comment all mirror those in [1], albeit in a different notation. Thus, the current comment computes exactly the "corrected" Fresnel coefficients found in [1].

The next question is whether or not these "corrected" Fresnel coefficients are equal to the "standard" Fresnel coefficients. I presented an article at the International Conference on

Electromagnetics in Advanced Applications meeting in Torino, Italy, in 2015 [7] that demonstrates [see (5) in that article] that for the E perpendicular polarization, one may change the equation for the "standard" Fresnel coefficients into that for the "corrected" Fresnel coefficients by making the substitution

$$k_1 \cos(\theta_c) \Rightarrow k_r \cos(\theta_1) + ik_1. \quad (1)$$

The current comment makes this same observation (in its notation) as its (2). Note that the angle on the left-hand side (LHS) of (1) is a complex number. In terms of the complex unit vector notation used in the current comment, an equation is derived there having the same form as (1), where the LHS and right-hand side (RHS) of (1) are equal [see (17) in the current comment]. This result is then used to claim the "standard" and "corrected" Fresnel equations are equivalent.

I respectfully disagree with that conclusion. The angle on the LHS of the current comment's (17) was calculated from the complex unit vector notation used there. The complex angle on the LHS of (1) is the complex angle in the "standard" form of the Fresnel equations and is calculated from very different equations than those used in the current comment's complex unit vector approach. Thus, there is no reason to think these two complex angles would be equal.

It is only by assuming that these two complex angles are equal that one may conclude that the "standard" and "corrected" Fresnel equations are equivalent. In the current comment, it is only shown that these angles occur in equations of the same form, and not that they are equal. Thus, the current comment's conclusion that both the "standard" and "corrected" Fresnel equations are correct and equivalent is not supported by the arguments in the current comment.

It should be noted that [1] and [7] described an error in the published derivations of the "standard" Fresnel equations. Of course, just because there is an error in deriving an equation doesn't necessarily mean the derived equation is incorrect. For that reason, the second half of [7] is devoted to demonstrating that the "standard" and "corrected" Fresnel equations give different numerical results.

PERMISSION TO SPEAK FREELY, PLEASE

It is clear that there is an apparent paradox here that has lasted since the publication of [1] in 2011, if not since Stratton's 1941 book [2]. That apparent paradox must end here. The explanation takes only a few sentences. Until now I have been reluctant to impute motives to electrical engineers in general, but now it is clear this is the only way to dispel the paradox. I believe that there is an error that has been made by the great majority of electrical engineers who use the "standard" form of the Fresnel equations, and this error has permitted them to get a correct answer from an incorrect equation.

If I may speak freely, I believe that the great majority of electrical engineers that use the Fresnel equations do so without first making a significant effort to review how they were derived. Don't we all do that? I know that I often do! Consider an electrical engineer attempting to put numbers into the "standard" Fresnel equations.

One of the quantities that he or she must give a numerical value to is the LHS of (1), presented previously. It is clear that k_1 is a wavenumber for the field transmitted into the lossy region. It is also clear that θ_c is a complex number representing the direction of propagation in the lossy region. Also, θ_1 is defined as a real angle describing the direction of propagation of the phase while the decay occurs perpendicular to the interface. With that information, it is fairly simple to GUESS that the LHS of (1) that should be put into the calculation must be given by the RHS of (1). Indeed, doing so produces the correct answer, so this behavior is reinforced.

The problem here is that the LHS of (1) is not equal to the RHS If one were to go back to the derivation of the "standard" Fresnel equations, then the equations that define θ_c may be identified. Any attempt to use those equations will either be aborted because something seems not right or will end in a result that one's intuition might question as incorrect. For this reason, it appears that it is an almost universal practice to not carefully compute the LHS of (1), but rather to replace it with the RHS of (1) when using the "standard" Fresnel equations. This practice allows one to obtain a correct result from an incorrect formula.

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