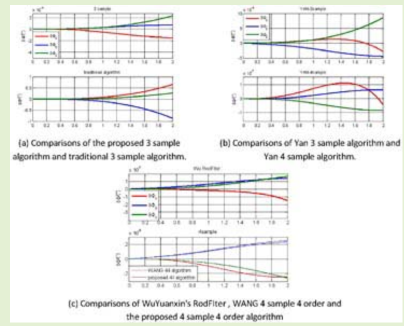
An Improved Attitude Compensation Algorithm in High Dynamic Environment

Pan Jiang, Guochen Wang, Ya Zhang, Lin Zhang, Shiwei Fan, and Dingjie Xu

***Abstract*—With the advent of high-precision gyro laboratory prototypes and the widespread use of gyros in high-dynamic fields, the requirements for navigation calculations will definitely increase. The traditional cone error compensation algorithm is very effective in pure cone motion environment with small half cone angle. However not only the triple cross product term of the noncommutativity error is ignored but also the cross-product terms of the angular increment are simplified in this algorithm for convenience. Thus, the attitude error is extremely huge when the half cone angle is large or in high dynamic environments. Aiming at the above problems, this paper proposes an improved high-precision attitude compensation algorithm based on Taylor series expansion. In this improved algorithm, the neglected and simplified items indicated above are both considered carefully. On this basis, the compensation coefficients are given after deducing in detail. Our algorithm does not need to know the coefficient model in advance, so it is more practical and has advantages in deriving high-order algorithms or sculling compensation algorithms. In order to verify the performance of the algorithm, pure coning and high dynamic environment simulations are performed. And the results show that the improved attitude compensation algorithm can obtain higher precision, which proving its feasibility and effectiveness.**



(a) Comparisons of the proposed 3 sample algorithm and traditional 3 sample algorithm.

(b) Comparisons of Yan 3 sample algorithm and Yan 4 sample algorithm.

(c) Comparisons of WuYuanxin's RodFiter , WANG 4 sample 4 order and the proposed 4 sample 4 order algorithm

***Index Terms*—Noncommutativity error, attitude compensation, equivalent rotation vector, triple cross product.**

I. Introduction

THE precision of the strapdown inertial navigation system mainly depends on the trueness of the inertial sensor components, the accuracy of the navigation algorithm, the processing power of the navigation computer, and the external environment in which the carrier is located. With the rapid development of computer technology, the processing power of navigation computers is no longer the main constraint. Meanwhile, as the manufacturing increases, the error caused by the inertial sensors is getting smaller and smaller. In general the error of the navigation algorithm should be less than 5 percent of the inertial device introduced error [1]–[7]. Therefore, higher requirements are imposed on the accuracy of the navigation algorithm.

The key to the strapdown inertial navigation algorithm is the attitude update algorithm, which is the problem of rigid body fixed axis rotation. It has been extensively and deeply studied by scholars for many years [8]–[14]. At present, the most popular method for solving attitude update is that firstly calculating the equivalent rotation vector using gyro angle increment multi-sample sampling, compensating the rotation noncommutativity error, and then calculating the attitude update quaternion using the equivalent rotation vector. However, in the traditional algorithm, not only the influence of the triple cross product term in the Bortz equation is neglected, but also the equivalent rotation vector in the second order term is approximated as the angular velocity increment [15]–[17]. As a result, the accuracy of traditional algorithms often can not meet the demand for high precision and sometimes the accuracy of the high subsample algorithm is not as good as the low subsample especially in high dynamic environments [18], [19].

Miller first introduced the concept of optimizing the cone algorithm coefficients in pure conical environment and proved the superiority of the coefficient in cone error compensation [20]. Ignagni gave nine conic correction algorithms and derived two important properties about conic integral and angular incremental cross product under conical conditions [21]. Jiang introduced the sum of the angular increments of the previous period as a correction term into the cone compensation operation in the current period, which improved the performance of the attitude compensation algorithm [22], [23]. Park summarized the above work, and gave a formalized method for designing and optimizing the

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coefficients of the attitude compensation algorithm under pure conical conditions [24]. Reference [25] and [26] respectively proposed the display frequency shaping and the extended cone error compensation algorithm, enhanced the error compensation accuracy under the large maneuvering situation. However, they only considered the influence of the second-order term of the noncommutativity error, and the principle error cannot be avoided under large maneuvering conditions. As an improvement of the traditional method, [27] first considered the influence of the third-order term in the Bortz equation, and proposed a higher-precision attitude error compensation algorithm. Nevertheless, the derivation process was cumbersome and only stayed in the verification phase of the simulation experiment. Reference [28], [29] proposed RodFIter, which uses the function iteration of Rodrigues vector to complete the pose reconstruction. RodFIter exhibits better performance than the mainstream rotation vector algorithm, and it also provides gesture results throughout the update interval. Reference [10] proposed the third-order rotation vector algorithm in coning and coexisting environment. Reference [30] further developed more general version high accuracy rotation vector algorithm, in which the fourth-order cross-product terms were considered. As well as that, thorough theoretical error analysis of why the traditional rotation vector is not applicable in large maneuver environment was given in this paper, which represents a breakthrough in the rotation vector algorithm design.

In order to solve the above problems and improve the accuracy of the navigation algorithm, an improved high-precision attitude error compensation algorithm is proposed in this paper. In this algorithm, the triple cross product term of the noncommutativity error and the cross multiplications ignored in the previous attitude compensation algorithm are reconsidered. So that more constraint relationships can be added in the algorithm optimization process. Based on this, we re-derive the Bortz equation and give the three sub-sample and four sub-sample attitude error compensation coefficients. The performance of the proposed algorithm is verified by typical cone motion and high dynamic condition simulation as well as the actual vehicle test. The rest of the paper is arranged as follows: In Section2, some assumptions and approximations in traditional algorithms are pointed out. In response to this problem, an improved high-precision attitude error compensation algorithm is proposed in Section3. Pure coning and high dynamic environment simulations are given in Section 4. Section 5 summarizes some conclusions and major contributions.

II. Error Analysis of Attitude Compensation Algorithm Under High Dynamics

The attitude solution is the key operation in the strapdown inertial navigation system. It is responsible for converting the specific force measured by the accelerometer to the required coordinate system, providing the carrier attitude continuously. And the most popular attitude update method is to calculate the equivalent rotation vector first, compensate the non-exchangeable error, and then calculate the attitude update quaternion using the equivalent rotation vector. In order to study the accuracy of the attitude algorithm in a

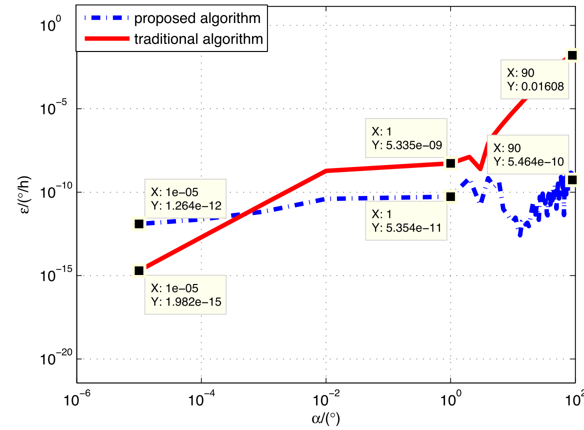

Fig. 1. Comparison of coning drift error.

large maneuvering environment, we give the attitude update algorithm as follows:

The quaternion update equation is:

$$\boldsymbol{Q}(t+h) = \boldsymbol{Q}(t) \otimes \boldsymbol{q}(h) \tag{1}$$

wherein, $\boldsymbol{Q}(t)$ and $\boldsymbol{Q}(t+h)$ are the quaternion at the end of the previous attitude update period and the one at the end of the current update interval. $\otimes$ represents quaternion multiplication and h indicates the attitude update interval. $\boldsymbol{q}(h)$ denotes the update quaternion within the update interval and the relationship with the equivalent rotation vector can be expressed as

$$\boldsymbol{q}(h) = \left[\cos\left(\frac{\phi}{2}\right), \left(\frac{\phi}{\boldsymbol{\phi}}\right)\sin\left(\frac{\phi}{2}\right)\right] \tag{2}$$

where $\boldsymbol{\phi}$ denotes the equivalent rotation vector and its amplitude is $\phi = (\boldsymbol{\phi} \cdot \boldsymbol{\phi})^{1/2}$.

The differential equation of the equivalent rotation vector is

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\phi} \times \boldsymbol{\omega} + \frac{1}{\phi^2}\left[1 - \frac{\phi\sin\phi}{2(1-\cos\phi)}\right]\boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{\omega}) \tag{3}$$

where $\boldsymbol{\omega}$ indicates the angular rate vector obtained by the gyro measurement and symbol $\times$ represents the cross product between vectors.

In Eq.3, $\frac{1}{2}\boldsymbol{\phi} \times \boldsymbol{\omega} + \frac{1}{\phi^2}\left[1 - \frac{\phi\sin\phi}{2(1-\cos\phi)}\right]\boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{\omega})$ represents the noncommutativity error which is the main compensation object of the attitude algorithm. When the rotation angle is small, $\frac{1}{\phi^2}\left[1 - \frac{\phi\sin\phi}{2(1-\cos\phi)}\right]\boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{\omega})$ is approximately zero. So this term is ignored in the traditional attitude compensation algorithm for the sake of convenience. The angular rotation increment is used instead of the equivalent rotation vector and the Eq.3 is integrated to obtain the approximate expression of the equivalent rotation vector:

$$\boldsymbol{\phi}(t) = \boldsymbol{\alpha}(t) + \frac{1}{2}\int_{t_{m-1}}^{t} \boldsymbol{\alpha}(\tau) \times \boldsymbol{\omega}(\tau) d\tau \tag{4}$$

wherein, $\boldsymbol{\alpha}(t)$ is the integral of the angular rate vector $\boldsymbol{\omega}(t)$ in the time interval t_{m-1} to t.

In Eq.4, $\frac{1}{2}\int_{t_{m-1}}^{t} \boldsymbol{\alpha}(\tau) \times \boldsymbol{\omega}(\tau) d\tau$ is called cone correction or cone integration. The core of the attitude compensation algorithm is how to use the numerical integration algorithm

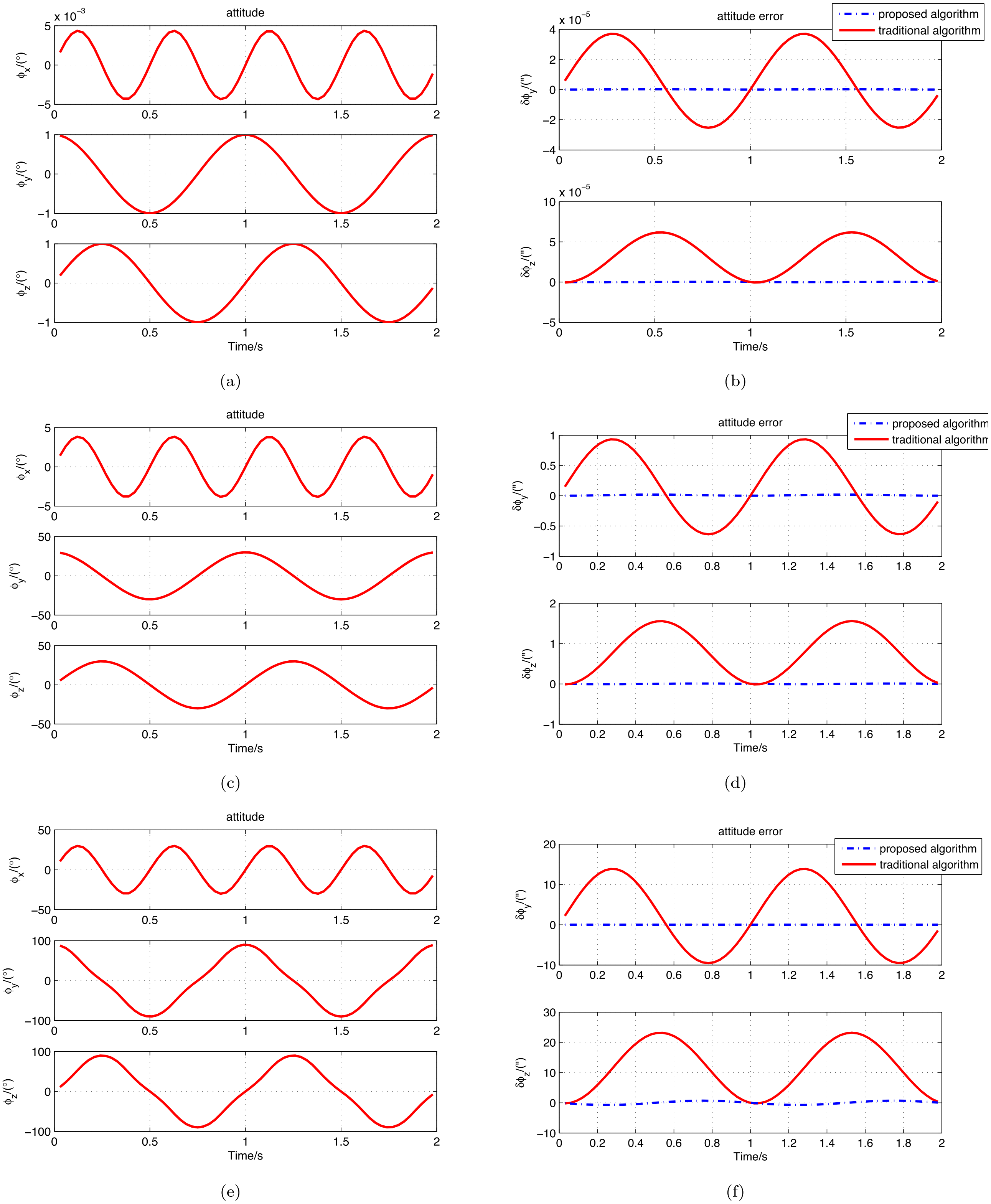

Fig. 2. Non-cone axis drift error comparison at different half cone angles (a) Typical conical motion attitude ($\alpha = 1°$) (b) Drift error comparison for non-coning axis ($\alpha = 1°$) (c) Typical conical motion attitude ($\alpha = 30°$) (d) Drift error comparison for non-coning axis ($\alpha = 30°$) (e) Typical conical motion attitude ($\alpha = 90°$) (f) Drift error comparison for non-coning axis ($\alpha = 90°$).

to accurately describe the cone correction, thereby improving the accuracy of the attitude compensation algorithm in various environments.

Because of the assumption and approximation that the rotation angle is small, the traditional attitude compensation algorithm is only applicable to this environment. Under the conical motion of large cone angle or high dynamic environment, the attitude drift error is serious and can not achieve the expected compensation effect. What's more, in the process of solving the algorithm coefficients, Ignagni and Lee found that the cross product between two angular increments under pure conical motion conditions only depends on their relative time distance and has nothing to do with respective absolute time [21]. Based on this, the compensation coefficient of the algorithm does not affect the compensation performance under pure conical motion conditions and the calculation amount can be reduced inevitably. However, the compensation accuracy will be lost under non-conical motion conditions, which limits the practical application of the attitude compensation algorithm. In view of the above points, this paper proposes an improved high-precision attitude compensation method, in which the triple cross product of non-commutative error is considered.

III. Improved Attitude Compensation Algorithm

A. 3 Sub-Sample 3rd Order Attitude Error Compensation Algorithm

The previously designed attitude compensation algorithm, as described in section2, mostly focuses on the accuracy under pure cone conditions rather than under large maneuver conditions. To this end, we propose a high-precision attitude compensation algorithm and the detailed derivation process is given in this section.

The Bortz equation is the differential equation of equivalent rotation vector, as is shown in Eq.3. As we know, the Taylor function expansion of the trigonometric function in Eq.3 yields a more concise approximation equation within the acceptable range of accuracy. So $\frac{\phi\sin\phi}{2(1-\cos\phi)}$ can be expanded as:

$$\begin{aligned}\frac{\phi\sin\phi}{2(1-\cos\phi)} &= \frac{\phi\cdot 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\cdot 2\sin^2\frac{\phi}{2}} \\ &= \frac{\phi}{2}\cot\frac{\phi}{2} \\ &= \frac{\phi}{2}\left[\frac{2}{\phi}-\frac{1}{3}\frac{\phi}{2}-\frac{1}{45}\left(\frac{\phi}{2}\right)^3-\frac{2}{945}\left(\frac{\phi}{2}\right)^5 - \frac{1}{4725}\left(\frac{\phi}{2}\right)^7-\cdots\right] \\ &= 1-\frac{\phi^2}{12}-\frac{\phi^4}{720}-\frac{\phi^6}{30240}-\frac{\phi^8}{1209600}-\cdots \end{aligned} \tag{5}$$

Since that the noncommutativity error is $\frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega}+\frac{1}{\phi^2}\left[1-\frac{\phi\sin\phi}{2(1-\cos\phi)}\right]\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega})$, the noncommutativity error is at least 5 orders, if we take the 4th order approximation $\frac{\phi\sin\phi}{2(1-\cos\phi)}=1-\frac{\phi^2}{12}-\frac{\phi^4}{720}$. For the three sample attitude compensation algorithm, theoretical derivation shows that 5th order error has little effect. Therefore, this article only take the approximation $\frac{\phi\sin\phi}{2(1-\cos\phi)}=1-\frac{\phi^2}{12}$. And the approximate equation commonly used in engineering is obtained:

$$\dot{\boldsymbol{\phi}}=\boldsymbol{\omega}+\frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega}+\frac{1}{12}\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega}) \tag{6}$$

Eq.6 shows that in the attitude update, it is only necessary to solve the equivalent rotation vector corresponding to the rotation of the body coordinate system from t_k to t_{k+1} time, without having to know the evolution process of the rotation vector in the t_k to t_{k+1} period. Therefore, the Taylor series expansion method can be used to solve the rotation vector.

Let $\boldsymbol{\phi}(h)$ be the equivalent rotation vector in $[t_k, t_{k+1}]$, where $h=t_{k+1}-t_k$. Taking the three sub-cone error compensation as an example, the angular velocity of the carrier is fitted by a parabola:

$$\boldsymbol{\omega}(t_k+\tau)=\boldsymbol{a}+\boldsymbol{2b}\tau+\boldsymbol{3c}\tau^2, 0\le\tau\le h \tag{7}$$

With the Taylor series expansion, $\boldsymbol{\phi}(h)$ can be represented as:

$$\boldsymbol{\phi}(h)=\boldsymbol{\phi}(0)+h\dot{\boldsymbol{\phi}}(0)+\frac{h^2}{2!}\ddot{\boldsymbol{\phi}}(0)+\frac{h^3}{3!}\boldsymbol{\phi}^{(3)}(0)+\cdots \tag{8}$$

The increment of the angle is

$$\Delta\boldsymbol{\theta}(\tau)=\int_0^\tau\boldsymbol{\omega}(t_k+\tau)d\tau \tag{9}$$

Then from Eq.7 and Eq.9, we can obtain the derivatives of angular velocity and angular velocity increment:

$$\begin{aligned}\boldsymbol{\omega}(t_k+\tau) &= \boldsymbol{\omega}(t_k+\tau)|_{\tau=0}=\boldsymbol{a}\\ \dot{\boldsymbol{\omega}}(t_k+\tau) &= \dot{\boldsymbol{\omega}}(t_k+\tau)|_{\tau=0}=2\boldsymbol{b}\\ \ddot{\boldsymbol{\omega}}(t_k+\tau) &= \ddot{\boldsymbol{\omega}}(t_k+\tau)|_{\tau=0}=6\boldsymbol{c}\\ \boldsymbol{\omega}^{(i)}(t_k+\tau) &= \boldsymbol{\omega}^{(i)}(t_k+\tau)|_{\tau=0}=\boldsymbol{0},\\ & \qquad i=3,4,5\cdots\end{aligned} \tag{10}$$

$$\begin{aligned}\Delta\boldsymbol{\theta}(0) &= \Delta\boldsymbol{\theta}(\tau)|_{\tau=0}=\boldsymbol{0}\\ \Delta\dot{\boldsymbol{\theta}}(0) &= \Delta\dot{\boldsymbol{\theta}}(\tau)|_{\tau=0}=\boldsymbol{\omega}(t_k+\tau)|_{\tau=0}=\boldsymbol{a}\\ \Delta\ddot{\boldsymbol{\theta}}(0) &= \Delta\ddot{\boldsymbol{\theta}}(\tau)|_{\tau=0}=\dot{\boldsymbol{\omega}}(t_k+\tau)|_{\tau=0}=2\boldsymbol{b}\\ \Delta\boldsymbol{\theta}^{(3)}(0) &= \Delta\boldsymbol{\theta}^{(3)}(\tau)|_{\tau=0}=\ddot{\boldsymbol{\omega}}(t_k+\tau)|_{\tau=0}=6\boldsymbol{c}\\ \Delta\boldsymbol{\theta}^{(i)}(0) &= \Delta\boldsymbol{\theta}^{(i)}(\tau)|_{\tau=0}=\boldsymbol{\omega}^{(i-1)}(t_k+\tau)|_{\tau=0}=\boldsymbol{0},\\ & \qquad i=4,5,6,\cdots\end{aligned} \tag{11}$$

Substitute $\boldsymbol{\phi}(t)=\boldsymbol{\alpha}(t)+\frac{1}{2}\int_{t_{m-1}}^{t}\boldsymbol{\phi}\times\boldsymbol{\omega}d\tau+\frac{1}{12}\int_{t_{m-1}}^{t}\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega})d\tau$ into Eq.6 and discarding four and above orders cross multiplications because this section only considers three order terms, Eq.6 can be expressed as

$$\dot{\boldsymbol{\phi}}=\boldsymbol{\omega}+\frac{1}{2}\left[\Delta\boldsymbol{\theta}+\frac{1}{2}\int_0^\tau(\Delta\boldsymbol{\theta}\times\boldsymbol{\omega})dt\right]\times\boldsymbol{\omega} +\frac{1}{12}[\Delta\boldsymbol{\theta}\times](\Delta\boldsymbol{\theta}\times\boldsymbol{\omega}) \tag{12}$$

In order to make the derivation process more concise and clear, we make the following definition

$$\begin{aligned}\boldsymbol{A} &= \Delta\dot{\boldsymbol{\theta}}\times\boldsymbol{\omega}+\Delta\boldsymbol{\theta}\times\dot{\boldsymbol{\omega}}\\ \boldsymbol{B} &= \Delta\ddot{\boldsymbol{\theta}}\times\boldsymbol{\omega}+2\Delta\dot{\boldsymbol{\theta}}\times\dot{\boldsymbol{\omega}}+\Delta\boldsymbol{\theta}\times\ddot{\boldsymbol{\omega}}\\ \boldsymbol{C} &= \Delta\boldsymbol{\theta}^{(3)}\times\boldsymbol{\omega}+3\Delta\ddot{\boldsymbol{\theta}}\times\dot{\boldsymbol{\omega}}+3\dot{\boldsymbol{\Delta}}\theta\times\ddot{\boldsymbol{\omega}}\\ \boldsymbol{D} &= 4\Delta\boldsymbol{\theta}^{(3)}\times\dot{\boldsymbol{\omega}}+6\Delta\ddot{\boldsymbol{\theta}}\times\ddot{\boldsymbol{\omega}}\\ \boldsymbol{E} &= 10\Delta\boldsymbol{\theta}^{(3)}\times\ddot{\boldsymbol{\omega}}\end{aligned} \tag{13}$$

Then the derivative of each order for Eq.12 can be indicated in (14), as shown at the top of the next.

$$
\begin{aligned}
\ddot{\boldsymbol{\phi}} &= \dot{\omega} + \frac{1}{2}\boldsymbol{A} + \frac{1}{4}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) \times \boldsymbol{\omega} + \frac{1}{4}\int_0^{\tau}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega})\,dt \times \dot{\boldsymbol{\omega}} + \frac{1}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times (\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{A} \\
\boldsymbol{\phi}^{(3)} &= \ddot{\boldsymbol{\omega}} + \frac{1}{2}\boldsymbol{B} + \frac{1}{4}\boldsymbol{A} \times \boldsymbol{\omega} + \frac{1}{4}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) \times \dot{\boldsymbol{\omega}} + \frac{1}{4}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) \times \dot{\boldsymbol{\omega}} \\
&\quad + \frac{1}{4}\int_0^{\tau}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega})\,dt \times \ddot{\boldsymbol{\omega}} + \frac{1}{12}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times (\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) + \frac{1}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{A} + \frac{1}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{A} + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{B} \\
&= \ddot{\boldsymbol{\omega}} + \frac{1}{2}\boldsymbol{B} + \frac{1}{4}\boldsymbol{A} \times \boldsymbol{\omega} + \frac{1}{2}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) \times \dot{\boldsymbol{\omega}} + \frac{1}{4}\int_0^{\tau}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega})\,dt \times \ddot{\boldsymbol{\omega}} + \frac{1}{12}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times (\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) + \frac{1}{6}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{A} + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{B} \\
\boldsymbol{\phi}^{(4)} &= \frac{1}{2}\boldsymbol{C} + \frac{1}{4}\boldsymbol{B} \times \boldsymbol{\omega} + \frac{1}{4}\boldsymbol{A} \times \dot{\boldsymbol{\omega}} + \frac{1}{2}\boldsymbol{A} \times \dot{\boldsymbol{\omega}} + \frac{1}{2}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) \times \ddot{\boldsymbol{\omega}} + \frac{1}{4}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) \times \ddot{\boldsymbol{\omega}} \\
&\quad + \frac{1}{12}\boldsymbol{\Delta\theta}^{(3)} \times (\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) + \frac{1}{12}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{A} + \frac{1}{6}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{A} + \frac{1}{6}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{B} + \frac{1}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{B} + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{C} \\
&= \frac{1}{2}\boldsymbol{C} + \frac{1}{4}\boldsymbol{B} \times \boldsymbol{\omega} + \frac{3}{4}\boldsymbol{A} \times \dot{\boldsymbol{\omega}} + \frac{3}{4}(\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) \times \ddot{\boldsymbol{\omega}} + \frac{1}{12}\boldsymbol{\Delta\theta}^{(3)} \times (\boldsymbol{\Delta\theta} \times \boldsymbol{\omega}) + \frac{1}{4}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{A} + \frac{1}{4}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{B} + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{C} \\
\boldsymbol{\phi}^{(5)} &= \frac{1}{2}\boldsymbol{D} + \frac{1}{4}\boldsymbol{C} \times \boldsymbol{\omega} + \frac{1}{4}\boldsymbol{B} \times \dot{\boldsymbol{\omega}} + \frac{3}{4}\boldsymbol{B} \times \dot{\boldsymbol{\omega}} + \frac{3}{4}\boldsymbol{A} \times \ddot{\boldsymbol{\omega}} + \frac{3}{4}\boldsymbol{A} \times \ddot{\boldsymbol{\omega}} \\
&\quad + \frac{1}{12}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{A} + \frac{1}{4}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{A} + \frac{1}{4}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{B} + \frac{1}{4}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{B} + \frac{1}{4}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{C} + \frac{1}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{C} + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{D} \\
&= \frac{1}{2}\boldsymbol{D} + \frac{1}{4}\boldsymbol{C} \times \boldsymbol{\omega} + \boldsymbol{B} \times \dot{\boldsymbol{\omega}} + \frac{3}{2}\boldsymbol{A} \times \ddot{\boldsymbol{\omega}} + \frac{1}{3}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{A} + \frac{1}{2}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{B} + \frac{1}{3}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{C} + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{D} \\
\boldsymbol{\phi}^{(6)} &= \frac{1}{2}\boldsymbol{E} + \frac{1}{4}\boldsymbol{D} \times \boldsymbol{\omega} + \frac{1}{4}\boldsymbol{C} \times \dot{\boldsymbol{\omega}} + \boldsymbol{C} \times \dot{\boldsymbol{\omega}} + \boldsymbol{B} \times \ddot{\boldsymbol{\omega}} + \frac{3}{2}\boldsymbol{B} \times \ddot{\boldsymbol{\omega}} \\
&\quad + \frac{1}{3}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{B} + \frac{1}{2}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{B} + \frac{1}{2}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{C} + \frac{1}{3}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{C} + \frac{1}{3}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{D} + \frac{1}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{D} + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{E} \\
&= \frac{1}{2}\boldsymbol{E} + \frac{1}{4}\boldsymbol{D} \times \boldsymbol{\omega} + \frac{5}{4}\boldsymbol{C} \times \dot{\boldsymbol{\omega}} + \frac{5}{2}\boldsymbol{B} \times \ddot{\boldsymbol{\omega}} + \frac{5}{6}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{B} + \frac{5}{6}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{C} + \frac{5}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{D} + \frac{1}{12}\boldsymbol{\Delta\theta} \times \boldsymbol{E} \\
\boldsymbol{\phi}^{(7)} &= \frac{1}{4}\boldsymbol{E} \times \boldsymbol{\omega} + \frac{1}{4}\boldsymbol{D} \times \dot{\boldsymbol{\omega}} + \frac{5}{4}\boldsymbol{D} \times \dot{\boldsymbol{\omega}} + \frac{5}{4}\boldsymbol{C} \times \ddot{\boldsymbol{\omega}} + \frac{5}{2}\boldsymbol{C} \times \ddot{\boldsymbol{\omega}} \\
&\quad + \frac{5}{6}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{C} + \frac{5}{6}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{C} + \frac{5}{6}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{D} + \frac{5}{12}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{D} + \frac{5}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{E} + \frac{1}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{E} \\
&= \frac{1}{4}\boldsymbol{E} \times \boldsymbol{\omega} + \frac{3}{2}\boldsymbol{D} \times \dot{\boldsymbol{\omega}} + \frac{15}{4}\boldsymbol{C} \times \ddot{\boldsymbol{\omega}} + \frac{5}{3}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{C} + \frac{5}{4}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{D} + \frac{1}{2}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}} \times \boldsymbol{E} \\
\boldsymbol{\phi}^{(8)} &= \frac{1}{4}\boldsymbol{E} \times \dot{\boldsymbol{\omega}} + \frac{3}{2}\boldsymbol{E} \times \dot{\boldsymbol{\omega}} + \frac{3}{2}\boldsymbol{D} \times \ddot{\boldsymbol{\omega}} + \frac{15}{4}\boldsymbol{D} \times \ddot{\boldsymbol{\omega}} + \frac{5}{3}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{D} + \frac{5}{4}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{D} + \frac{5}{4}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{E} + \frac{1}{2}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{E} \\
&= \frac{7}{4}\boldsymbol{E} \times \dot{\boldsymbol{\omega}} + \frac{21}{4}\boldsymbol{D} \times \ddot{\boldsymbol{\omega}} + \frac{35}{12}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{D} + \frac{7}{4}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}} \times \boldsymbol{E} \\
\boldsymbol{\phi}^{(9)} &= \frac{7}{4}\boldsymbol{E} \times \ddot{\boldsymbol{\omega}} + \frac{21}{4}\boldsymbol{E} \times \ddot{\boldsymbol{\omega}} + \frac{35}{12}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{E} + \frac{7}{4}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{E} = 7\boldsymbol{E} \times \ddot{\boldsymbol{\omega}} + \frac{14}{3}\boldsymbol{\Delta\theta}^{(3)} \times \boldsymbol{E} \\
\boldsymbol{\phi}^{(10)} &= \boldsymbol{\phi}^{(11)} = \boldsymbol{\phi}^{(12)} = \cdots = \boldsymbol{0}
\end{aligned} \tag{14}
$$

When $\tau = 0$, substituting Eq.10, and Eq.11 into Eq.14, we can know that:

$$
\begin{aligned}
\dot{\boldsymbol{\phi}}(0) &= \boldsymbol{a}, \quad \ddot{\boldsymbol{\phi}}(0) = 2\boldsymbol{b}, \quad \boldsymbol{\phi}^{(3)}(0) \\
&= 6\boldsymbol{c} + \frac{1}{2}\cdot 2\boldsymbol{a} \times \boldsymbol{b} = 6\boldsymbol{c} + \boldsymbol{a} \times \boldsymbol{b} \\
\boldsymbol{\phi}^{(4)}(0) &= 6\boldsymbol{a} \times \boldsymbol{c} + \frac{1}{4} \times 2(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a} + \frac{1}{4} \times \boldsymbol{a} \times 2(\boldsymbol{a} \times \boldsymbol{b}) \\
&= 6\boldsymbol{a} \times \boldsymbol{c} \\
\boldsymbol{\phi}^{(5)}(0) &= 12\boldsymbol{b} \times \boldsymbol{c} + \frac{1}{4} \times 12(\boldsymbol{a} \times \boldsymbol{c}) \times \boldsymbol{a} + 2(\boldsymbol{a} \times \boldsymbol{b}) \times 2\boldsymbol{b} \\
&\quad + \frac{1}{3}\boldsymbol{a} \times 12(\boldsymbol{a} \times \boldsymbol{c}) \\
&= 12\boldsymbol{b} \times \boldsymbol{c} + 4(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{b} + \boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{c}) \\
\boldsymbol{\phi}^{(6)}(0) &= \frac{1}{4} \times 24(\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{a} + \frac{5}{4} \times 12(\boldsymbol{a} \times \boldsymbol{c}) \times 2\boldsymbol{b} \\
&\quad + \frac{5}{2} \times 2(\boldsymbol{a} \times \boldsymbol{b}) \times 6\boldsymbol{c} + \frac{5}{6} \times 6\boldsymbol{c} \times 2(\boldsymbol{a} \times \boldsymbol{b}) \\
&\quad + \frac{5}{6} \times 2\boldsymbol{b} \times 12(\boldsymbol{a} \times \boldsymbol{c}) + \frac{5}{12}\boldsymbol{a} \times 24(\boldsymbol{b} \times \boldsymbol{c}) \\
&= 4a \times (\boldsymbol{b} \times \boldsymbol{c}) + 10(\boldsymbol{a} \times \boldsymbol{c}) \times \boldsymbol{b} + 20(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \\
\boldsymbol{\phi}^{(7)}(0) &= \frac{3}{2} \times 24(\boldsymbol{b} \times \boldsymbol{c}) \times 2\boldsymbol{b} + \frac{15}{4} \times 12(\boldsymbol{a} \times \boldsymbol{c}) \times 6\boldsymbol{c} \\
&\quad + \frac{5}{3} \times 6\boldsymbol{c} \times 12(\boldsymbol{a} \times \boldsymbol{c}) + \frac{5}{4} \times 2\boldsymbol{b} \times 24(\boldsymbol{b} \times \boldsymbol{c}) \\
&= 12(\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{b} + 150(\boldsymbol{a} \times \boldsymbol{c}) \times \boldsymbol{c} \\
\boldsymbol{\phi}^{(8)}(0) &= \frac{21}{4} \times 24(\boldsymbol{b} \times \boldsymbol{c}) \times 6\boldsymbol{c} + \frac{35}{12}6\boldsymbol{c} \times 24(\boldsymbol{b} \times \boldsymbol{c}) \\
&= 336(\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{c} \\
\boldsymbol{\phi}^{(9)}(0) &= \boldsymbol{0}, \quad \boldsymbol{\phi}^{(10)}(0) = \boldsymbol{\phi}^{(11)}(0) = \boldsymbol{\phi}^{(12)}(0) = \cdots = \boldsymbol{0}
\end{aligned}
$$

Combining the above, the equivalent rotation vector $\boldsymbol{\phi}(h)$ is:

$$\begin{aligned}\boldsymbol{\phi}(h) &= \boldsymbol{\phi}(0)+\dot{\boldsymbol{\phi}}(0)h+\frac{1}{2!}\ddot{\boldsymbol{\phi}}(0)h^2+\frac{1}{3!}\boldsymbol{\phi}^{(3)}(0)h^3\\&\quad+\frac{1}{4!}\boldsymbol{\phi}^{(4)}(0)h^4+\frac{1}{5!}\boldsymbol{\phi}^{(5)}(0)h^5+\frac{1}{6!}\boldsymbol{\phi}^{(6)}(0)h^6\\&\quad+\frac{1}{7!}\boldsymbol{\phi}^{(7)}(0)h^7+\frac{1}{8!}\boldsymbol{\phi}^{(8)}(0)h^8\\&= \boldsymbol{a}h+\boldsymbol{b}h^2+\boldsymbol{c}h^3+\frac{1}{3!}\boldsymbol{a}\times\boldsymbol{b}h^3\\&\quad+\frac{1}{4}\boldsymbol{a}\times\boldsymbol{c}h^4+\frac{1}{10}\boldsymbol{b}\times\boldsymbol{c}h^5+\frac{1}{30}(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{b}h^5\\&\quad+\frac{1}{5!}\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{c})h^5+\frac{1}{180}\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})h^6\\&\quad+\frac{1}{72}(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{b}h^6\\&\quad+\frac{1}{36}(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}h^6+\frac{1}{420}(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{b}h^7\\&\quad+\frac{5}{168}(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{c}h^7+\frac{1}{120}(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{c}h^8\end{aligned}\tag{15}$$

The variables $\boldsymbol{a}h$,$\boldsymbol{b}h^2$ and $\boldsymbol{c}h^3$ in the Eq.15 are as follows:

$$\begin{aligned}\boldsymbol{a}h &= \boldsymbol{\Delta\theta_3}-\frac{7}{2}\boldsymbol{\Delta\theta_2}+\frac{11}{2}\boldsymbol{\Delta\theta_1}\\\boldsymbol{b}h^2 &= \frac{9}{2}(-\boldsymbol{\Delta\theta_3}+3\boldsymbol{\Delta\theta_2}-2\boldsymbol{\Delta\theta_1})\\\boldsymbol{c}h^3 &= \frac{9}{2}(\boldsymbol{\Delta\theta_3}-2\boldsymbol{\Delta\theta_2}+\boldsymbol{\Delta\theta_1})\end{aligned}\tag{16}$$

wherein, $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_2}$ and $\boldsymbol{\theta_3}$ are the angular increments in the $\left[t_k, t_k+\frac{h}{3}\right]$, $\left[t_k+\frac{h}{3}, t_k+\frac{2h}{3}\right]$ and $\left[t_k+\frac{2h}{3}, t_k+h\right]$ time periods, respectively.

In summary, we can get the three sub-sample attitude compensation coefficient as follows

$$\begin{aligned}\boldsymbol{\phi}(h) &= \boldsymbol{\Delta\theta_1}+\boldsymbol{\Delta\theta_2}+\boldsymbol{\Delta\theta_3}\\&\quad+\frac{57}{80}\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2}+\frac{33}{80}\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3}\\&\quad+\frac{57}{80}\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3}\\&\quad-0.1165(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2})\times\boldsymbol{\Delta\theta_1}\\&\quad+0.1848(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2})\times\boldsymbol{\Delta\theta_2}\\&\quad+0.4380(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2})\times\boldsymbol{\Delta\theta_3}\\&\quad-0.0603(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_1}\\&\quad-0.1688(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_2}\\&\quad+0.0603(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_3}\\&\quad-0.1005(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_1}\\&\quad-0.1848(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_2}\\&\quad+0.1165(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_3}\end{aligned}\tag{17}$$

The Eq.2 of 2018/2019 WANG and Eq.17 of our manuscript have exactly the same essence, just in different forms. They are derived from the same equivalent rotational vector differential equation, but with different derivation processes. In Wang's paper, it is believed that $(t-t_{n-1})^7$ has a very small impact on the algorithm accuracy and can be ignored, so 7th-order Taylor expansion is derived. However, since that the 10th and above order terms are 0, the 9th-order Taylor expansion is executed in our manuscript. So there are some differences between the coefficients of these two equations.

From Eq.17 we can see that all angle increment cross-multiplier within the third order are used to describe the non-commutative error in this proposed algorithm. Obviously, the improved attitude compensation algorithm is more accurate than the traditional algorithm. Besides, the noncommutativity error is the main compensation object of the attitude algorithm and its accuracy will directly determine the precision of the attitude compensation algorithm. Therefore, the attitude error compensation algorithm proposed in this paper has higher precision in principle.

B. 4th Order Attitude Error Compensation Algorithm

In the previous section, we introduced the attitude error compensation algorithm that considered the third-order term. In this section, we further deduced to consider the 4th order. Substitute $\boldsymbol{\phi}(t) = \boldsymbol{\alpha}(t)+\frac{1}{2}\int_{t_{m-1}}^{t}\boldsymbol{\phi}\times\boldsymbol{\omega}d\tau+\frac{1}{12}\int_{t_{m-1}}^{t}\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega})d\tau$ into Eq.6 and discarding five and above orders cross multiplications because this section only considers fourth order terms. So $\Delta\dot{\phi}_4$ should be expressed as:

$$\Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4}} = \Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4A}}+\Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4B}}+\Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4C}}+\Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4D}}\tag{18}$$

wherein,

$$\begin{aligned}\Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4A}} &= \frac{1}{12}\left[\frac{1}{2}\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})dt\right]\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\\\Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4B}} &= \frac{1}{12}\boldsymbol{\Delta\theta}\times\left[\frac{1}{2}\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})dt\right]\times\boldsymbol{\omega}\\\Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4C}} &= \frac{1}{2}\left[\frac{1}{12}\int_0^\tau\boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})dt\right]\times\boldsymbol{\omega}\\\Delta\dot{\boldsymbol{\phi}}_{\boldsymbol{4D}} &= \frac{1}{8}\left[\int_0^\tau\left(\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})dt\times\boldsymbol{\omega}\right)dt\right]\times\boldsymbol{\omega}\end{aligned}$$

In the same way, using Taylor's expansion, we can get equations (19)–(23), as shown on the next pages.

Similarly, the four-sample fourth-order attitude error compensation algorithm can also be derived. The different is that the Eq.7 should be as follows:

$$\boldsymbol{\omega}(t_k+\tau) = \boldsymbol{a}+2\boldsymbol{b}\tau+3\boldsymbol{c}\tau^2+4\boldsymbol{d}\tau^3,\quad 0\le\tau\le h\tag{24}$$

Then the 4 sub-sample 4th-order attitude error compensation algorithm can be obtained as equation (25):
where ϕ_4 can be expressed as shown at the bottom of page 10.

It can be seen that our noncommutativity error compensation coefficient is almost the same as WANG. But we used a completely different approach to achieve WANG's algorithms, and we think it is also innovative and meaningful.

IV. Experimental Verification

To design an improved attitude compensation algorithm, we must first ensure that its accuracy is constant with respect to the classical conic algorithm under pure cone conditions, and then improve its accuracy under high dynamics. Therefore, in order to verify the effectiveness of the proposed high-precision attitude compensation algorithm, we performed simulations under pure cone and high dynamic conditions.

$$
\begin{aligned}
\Delta\dot{\boldsymbol{\phi}}_{4A} &= \frac{1}{24}\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt \times (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\\
\Delta\ddot{\boldsymbol{\phi}}_{4A} &= \frac{1}{24}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega}) + \frac{1}{24}\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt \times \boldsymbol{A}\\
\Delta\boldsymbol{\phi}_{4A}^{(3)} &= \frac{1}{24}\boldsymbol{A}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega}) + \frac{1}{24}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{A} + \frac{1}{24}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{A} + \frac{1}{24}\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt \times \boldsymbol{B}\\
\Delta\boldsymbol{\phi}_{4A}^{(4)} &= \frac{1}{24}\boldsymbol{A}\times\boldsymbol{A} + \frac{1}{24}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{B} + \frac{1}{24}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{B} + \frac{1}{24}\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt \times \boldsymbol{C}\\
\Delta\boldsymbol{\phi}_{4A}^{(5)} &= \frac{1}{12}\boldsymbol{A}\times\boldsymbol{B} + \frac{1}{12}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{C} + \frac{1}{24}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{C} + \frac{1}{24}\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt \times \boldsymbol{D}\\
\Delta\boldsymbol{\phi}_{4A}^{(6)} &= \frac{1}{12}\boldsymbol{B}\times\boldsymbol{B} + \frac{1}{12}\boldsymbol{A}\times\boldsymbol{C} + \frac{1}{8}\boldsymbol{A}\times\boldsymbol{C} + \frac{1}{8}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{D} + \frac{1}{24}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{D} + \frac{1}{24}\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt \times \boldsymbol{E}\\
\Delta\boldsymbol{\phi}_{4A}^{(7)} &= \frac{5}{24}\boldsymbol{B}\times\boldsymbol{C} + \frac{5}{24}\boldsymbol{A}\times\boldsymbol{D} + \frac{1}{6}\boldsymbol{A}\times\boldsymbol{D} + \frac{1}{6}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{E} + \frac{1}{24}(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{E} = 5(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c}) \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\dot{\boldsymbol{\phi}}_{4B} &= \frac{1}{12}\boldsymbol{\Delta\theta}\times\left[\frac{1}{2}\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\boldsymbol{\omega}\\
\Delta\ddot{\boldsymbol{\phi}}_{4B} &= \frac{1}{24}\boldsymbol{\Delta\dot\theta}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\boldsymbol{\omega} + \frac{1}{24}\boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega} + \frac{1}{24}\boldsymbol{\Delta\theta}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\dot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4B}^{(3)} &= \frac{1}{24}\boldsymbol{\Delta\ddot\theta}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\boldsymbol{\omega} + \frac{1}{12}\boldsymbol{\Delta\dot\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega} + \frac{1}{24}\boldsymbol{\Delta\theta}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\ddot{\boldsymbol{\omega}}\\
&\quad + \frac{1}{12}\boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}} + \frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{A}\times\boldsymbol{\omega} + \frac{1}{12}\boldsymbol{\Delta\dot\theta}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\dot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4B}^{(4)} &= \frac{1}{24}\boldsymbol{\Delta\theta}^{(3)}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\boldsymbol{\omega} + \frac{1}{8}\boldsymbol{\Delta\ddot\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega} + \frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{B}\times\boldsymbol{\omega} + \frac{1}{8}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{A}\times\boldsymbol{\omega}\\
&\quad + \frac{1}{4}\boldsymbol{\Delta\dot\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}} + \frac{1}{8}\boldsymbol{\Delta\dot\theta}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\ddot{\boldsymbol{\omega}} + \frac{1}{8}\boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}}\\
&\quad + \frac{1}{8}\boldsymbol{\Delta\ddot\theta}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\dot{\boldsymbol{\omega}} + \frac{1}{8}\boldsymbol{\Delta\theta}\times\boldsymbol{A}\times\dot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4B}^{(5)} &= \frac{1}{6}\boldsymbol{\Delta\theta}^{(3)}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega} + \frac{1}{6}\boldsymbol{\Delta\theta}^{(3)}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\dot{\boldsymbol{\omega}} + \frac{1}{4}\boldsymbol{\Delta\ddot\theta}\times\boldsymbol{A}\times\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\Delta\ddot\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}}\\
&\quad + \frac{1}{4}\boldsymbol{\Delta\ddot\theta}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\ddot{\boldsymbol{\omega}} + \frac{1}{6}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{B}\times\boldsymbol{\omega} + \frac{1}{4}\boldsymbol{\Delta\theta}\times\boldsymbol{A}\times\ddot{\boldsymbol{\omega}} + \frac{1}{2}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{A}\times\dot{\boldsymbol{\omega}} + \frac{1}{2}\boldsymbol{\Delta\dot\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}}\\
&\quad + \frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{C}\times\boldsymbol{\omega} + \frac{1}{6}\boldsymbol{\Delta\theta}\times\boldsymbol{B}\times\dot{\boldsymbol{\omega}} = \frac{1}{3}\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}\\
\Delta\boldsymbol{\phi}_{4B}^{(6)} &= \frac{5}{12}\boldsymbol{\Delta\theta}^{(3)}\times\boldsymbol{A}\times\boldsymbol{\omega} + \frac{5}{6}\boldsymbol{\Delta\theta}^{(3)}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}} + \frac{5}{12}\boldsymbol{\Delta\theta}^{(3)}\times\left[\int_0^\tau (\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\ddot{\boldsymbol{\omega}} + \frac{5}{12}\boldsymbol{\Delta\ddot\theta}\times\boldsymbol{B}\times\boldsymbol{\omega}\\
&\quad + \frac{5}{4}\boldsymbol{\Delta\ddot\theta}\times\boldsymbol{A}\times\dot{\boldsymbol{\omega}} + \frac{5}{4}\boldsymbol{\Delta\ddot\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}} + \frac{5}{24}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{C}\times\boldsymbol{\omega} + \frac{5}{6}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{B}\times\dot{\boldsymbol{\omega}} + \frac{5}{4}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{A}\times\ddot{\boldsymbol{\omega}}\\
&\quad + \frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{D}\times\boldsymbol{\omega} + \frac{5}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{C}\times\dot{\boldsymbol{\omega}} + \frac{5}{12}\boldsymbol{\Delta\theta}\times\boldsymbol{B}\times\ddot{\boldsymbol{\omega}} = \frac{5}{3}\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a} + \frac{5}{2}\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{a}\\
&\quad + \frac{10}{3}\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{b}\\
\Delta\boldsymbol{\phi}_{4B}^{(7)} &= \frac{20}{24}\boldsymbol{\Delta\theta}^{(3)}\times\boldsymbol{B}\times\boldsymbol{\omega} + \frac{60}{24}\boldsymbol{\Delta\theta}^{(3)}\times\boldsymbol{A}\times\dot{\boldsymbol{\omega}} + \frac{60}{24}\boldsymbol{\Delta\theta}^{(3)}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}} + \frac{15}{24}\boldsymbol{\Delta\ddot\theta}\times\boldsymbol{C}\times\boldsymbol{\omega} + \frac{60}{24}\boldsymbol{\Delta\ddot\theta}\times\boldsymbol{B}\times\dot{\boldsymbol{\omega}}\\
&\quad + \frac{90}{24}\boldsymbol{\Delta\ddot\theta}\times\boldsymbol{A}\times\ddot{\boldsymbol{\omega}} + \frac{6}{24}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{D}\times\boldsymbol{\omega} + \frac{30}{24}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{C}\times\dot{\boldsymbol{\omega}} + \frac{60}{24}\boldsymbol{\Delta\dot\theta}\times\boldsymbol{B}\times\ddot{\boldsymbol{\omega}} + \frac{6}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{D}\times\dot{\boldsymbol{\omega}}\\
&\quad + \frac{15}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{C}\times\ddot{\boldsymbol{\omega}} + \frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{E}\times\boldsymbol{\omega}\\
&= 10\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a} + 15\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{a} + 10\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{b} + 6\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}\\
&\quad + 30\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{b} + 30\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} \qquad (20)
\end{aligned}
$$

A. Error Analysis and Verification Under Pure Cone Environment

The cone motion simulation parameters are set to: cone frequency $f = 1Hz$, half cone angle range $\alpha = 0.05'' \sim 90^\circ$ and angular incremental sampling interval $h = 10ms$. The improved high-precision attitude compensation algorithm proposed in this paper is compared with the traditional algorithm. Fig.1 shows the attitude drift error of these two algorithms. The red solid line is the attitude drift error of the traditional three-subject attitude compensation algorithm,

$$
\begin{aligned}
\Delta\dot{\boldsymbol{\phi}}_{4C} &= \frac{1}{2}\left[\frac{1}{12}\int_0^\tau \boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\right]\times\boldsymbol{\omega}\\
\Delta\ddot{\boldsymbol{\phi}}_{4C} &= \frac{1}{24}\boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega}+\frac{1}{24}\int_0^\tau \boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\dot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4C}^{(3)} &= \frac{1}{24}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega}+\frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{A}\times\boldsymbol{\omega}+\frac{1}{12}\boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}}+\frac{1}{24}\int_0^\tau \boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\ddot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4B}^{(4)} &= \frac{1}{24}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega}+\frac{1}{12}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{A}\times\boldsymbol{\omega}+\frac{1}{8}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}}+\frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{B}\times\boldsymbol{\omega}\\
&\quad+\frac{1}{8}\boldsymbol{\Delta\theta}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}}+\frac{1}{8}\boldsymbol{\Delta\theta}\times\boldsymbol{A}\times\dot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4B}^{(5)} &= \frac{1}{24}\boldsymbol{\Delta\theta}^{(3)}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega}+\frac{1}{8}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times\boldsymbol{A}\times\boldsymbol{\omega}+\frac{1}{6}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}}+\frac{1}{8}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{B}\times\boldsymbol{\omega}+\frac{1}{4}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}}\\
&\quad+\frac{1}{8}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{A}\times\dot{\boldsymbol{\omega}}+\frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{C}\times\boldsymbol{\omega}+\frac{1}{6}\boldsymbol{\Delta\theta}\times\boldsymbol{B}\times\dot{\boldsymbol{\omega}}+\frac{1}{4}\boldsymbol{\Delta\theta}\times\boldsymbol{A}\times\ddot{\boldsymbol{\omega}}=\frac{1}{4}\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}\\
\Delta\boldsymbol{\phi}_{4B}^{(6)} &= \frac{1}{6}\boldsymbol{\Delta\theta}^{(3)}\times\boldsymbol{A}\times\boldsymbol{\omega}+\frac{7}{24}\boldsymbol{\Delta\theta}^{(3)}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}}+\frac{1}{4}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times\boldsymbol{B}\times\boldsymbol{\omega}+\frac{5}{8}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times\boldsymbol{A}\times\dot{\boldsymbol{\omega}}+\frac{5}{12}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}}\\
&\quad+\frac{1}{6}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{C}\times\boldsymbol{\omega}+\frac{5}{6}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{A}\times\ddot{\boldsymbol{\omega}}+\frac{5}{8}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{B}\times\dot{\boldsymbol{\omega}}+\frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{D}\times\boldsymbol{\omega}+\frac{5}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{C}\times\dot{\boldsymbol{\omega}}+\frac{5}{12}\boldsymbol{\Delta\theta}\times\boldsymbol{B}\times\ddot{\boldsymbol{\omega}}\\
&= \boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}+2\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{a}+\frac{5}{2}\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{b}\\
\Delta\boldsymbol{\phi}_{4B}^{(7)} &= \frac{5}{12}\boldsymbol{\Delta\theta}^{(3)}\times\boldsymbol{B}\times\boldsymbol{\omega}+\frac{13}{12}\boldsymbol{\Delta\theta}^{(3)}\times\boldsymbol{A}\times\dot{\boldsymbol{\omega}}+\frac{17}{24}\boldsymbol{\Delta\theta}^{(3)}\times(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}}+\frac{5}{12}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times\boldsymbol{C}\times\boldsymbol{\omega}+\frac{3}{2}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times\boldsymbol{B}\times\dot{\boldsymbol{\omega}}\\
&\quad+\frac{15}{8}\boldsymbol{\Delta}\ddot{\boldsymbol{\theta}}\times\boldsymbol{A}\times\ddot{\boldsymbol{\omega}}+\frac{5}{24}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{D}\times\boldsymbol{\omega}+\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{C}\times\dot{\boldsymbol{\omega}}\\
&\quad+\frac{15}{8}\boldsymbol{\Delta}\dot{\boldsymbol{\theta}}\times\boldsymbol{B}\times\ddot{\boldsymbol{\omega}}+\frac{1}{24}\boldsymbol{\Delta\theta}\times\boldsymbol{E}\times\boldsymbol{\omega}+\frac{1}{4}\boldsymbol{\Delta\theta}\times\boldsymbol{D}\times\dot{\boldsymbol{\omega}}+\frac{5}{8}\boldsymbol{\Delta\theta}\times\boldsymbol{C}\times\ddot{\boldsymbol{\omega}}\\
&= 5\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}+10\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{a}+12\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{b}+5\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+24\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{b}\\
&\quad+22.5\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}
\end{aligned}\tag{21}
$$

$$
\begin{aligned}
\Delta\dot{\boldsymbol{\phi}}_{4D} &= \frac{1}{8}\left[\int_0^\tau\left(\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\boldsymbol{\omega}\right)dt\right]\times\boldsymbol{\omega}\\
\Delta\ddot{\boldsymbol{\phi}}_{4D} &= \frac{1}{8}\left(\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\boldsymbol{\omega}\right)\times\boldsymbol{\omega}+\frac{1}{8}\left[\int_0^\tau\left(\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\boldsymbol{\omega}\right)dt\right]\times\dot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4D}^{(3)} &= \frac{1}{8}\left((\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega}+\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\dot{\boldsymbol{\omega}}\right)\times\boldsymbol{\omega}+\frac{1}{4}\left(\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\boldsymbol{\omega}\right)\times\dot{\boldsymbol{\omega}}\\
&\quad+\frac{1}{8}\left[\int_0^\tau\left(\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\boldsymbol{\omega}\right)dt\right]\times\ddot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4D}^{(4)} &= \frac{1}{8}\left(\boldsymbol{A}\times\boldsymbol{\omega}+2(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}}+\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\ddot{\boldsymbol{\omega}}\right)\times\boldsymbol{\omega}+\frac{3}{8}\left((\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega}+\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\dot{\boldsymbol{\omega}}\right)\times\dot{\boldsymbol{\omega}}\\
&\quad+\frac{3}{8}\left(\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\boldsymbol{\omega}\right)\times\ddot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4D}^{(5)} &= \frac{1}{8}(\boldsymbol{B}\times\boldsymbol{\omega}+3\boldsymbol{A}\times\dot{\boldsymbol{\omega}}+3(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}})\times\boldsymbol{\omega}+\frac{1}{2}\left(\boldsymbol{A}\times\boldsymbol{\omega}+2(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}}+\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\ddot{\boldsymbol{\omega}}\right)\times\dot{\boldsymbol{\omega}}\\
&\quad+\frac{3}{4}\left((\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\boldsymbol{\omega}+\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\dot{\boldsymbol{\omega}}\right)\times\ddot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4D}^{(6)} &= \frac{1}{8}(\boldsymbol{C}\times\boldsymbol{\omega}+4\boldsymbol{B}\times\dot{\boldsymbol{\omega}}+6\boldsymbol{A}\times\ddot{\boldsymbol{\omega}})\times\boldsymbol{\omega}+\frac{5}{8}(\boldsymbol{B}\times\boldsymbol{\omega}+3\boldsymbol{A}\times\dot{\boldsymbol{\omega}}+3(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}})\times\dot{\boldsymbol{\omega}}\\
&\quad+\frac{5}{4}\left(\boldsymbol{A}\times\boldsymbol{\omega}+2(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\dot{\boldsymbol{\omega}}+\int_0^\tau(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\,dt\times\ddot{\boldsymbol{\omega}}\right)\times\ddot{\boldsymbol{\omega}}\\
\Delta\boldsymbol{\phi}_{4D}^{(7)} &= \frac{1}{8}(\boldsymbol{D}\times\boldsymbol{\omega}+5\boldsymbol{C}\times\dot{\boldsymbol{\omega}}+10\boldsymbol{B}\times\ddot{\boldsymbol{\omega}})\times\boldsymbol{\omega}+\frac{3}{4}(\boldsymbol{C}\times\boldsymbol{\omega}+4\boldsymbol{B}\times\dot{\boldsymbol{\omega}}+6\boldsymbol{A}\times\ddot{\boldsymbol{\omega}})\times\dot{\boldsymbol{\omega}}\\
&\quad+\frac{15}{8}(\boldsymbol{B}\times\boldsymbol{\omega}+3\boldsymbol{A}\times\dot{\boldsymbol{\omega}}+3(\boldsymbol{\Delta\theta}\times\boldsymbol{\omega})\times\ddot{\boldsymbol{\omega}})\times\ddot{\boldsymbol{\omega}}=24\left[(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{b}\right]\times\boldsymbol{b}
\end{aligned}\tag{22}
$$

$$\begin{aligned}\Delta\dot{\boldsymbol{\phi}}_4 &= \Delta\dot{\boldsymbol{\phi}}_{4A} + \Delta\dot{\boldsymbol{\phi}}_{4B} + \Delta\dot{\boldsymbol{\phi}}_{4C} + \Delta\dot{\boldsymbol{\phi}}_{4D} \\ &= \begin{bmatrix} -0.001877\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_1}) + 0.066102\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_2}) + 0.047474\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_3}) \\ -0.023244\,(\boldsymbol{\Delta\theta_2}\cdot\boldsymbol{\Delta\theta_2}) + 0.117330\,(\boldsymbol{\Delta\theta_2}\cdot\boldsymbol{\Delta\theta_3}) - 0.004890\,(\boldsymbol{\Delta\theta_3}\cdot\boldsymbol{\Delta\theta_3}) \end{bmatrix} (\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2}) \\ &\quad + \begin{bmatrix} 0.002547\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_1}) + 0.073853\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_2}) + 0.032214\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_3}) \\ -0.135795\,(\boldsymbol{\Delta\theta_2}\cdot\boldsymbol{\Delta\theta_2}) - 0.007508\,(\boldsymbol{\Delta\theta_2}\cdot\boldsymbol{\Delta\theta_3}) + 0.002547\,(\boldsymbol{\Delta\theta_3}\cdot\boldsymbol{\Delta\theta_3}) \end{bmatrix} (\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3}) \\ &\quad + \begin{bmatrix} -0.086252\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_1}) + 0.117329\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_2}) + 0.12883\,(\boldsymbol{\Delta\theta_1}\cdot\boldsymbol{\Delta\theta_3}) \\ -0.023245\,(\boldsymbol{\Delta\theta_2}\cdot\boldsymbol{\Delta\theta_2}) + 0.066101\,(\boldsymbol{\Delta\theta_2}\cdot\boldsymbol{\Delta\theta_3}) - 0.066101\,(\boldsymbol{\Delta\theta_3}\cdot\boldsymbol{\Delta\theta_3}) \end{bmatrix} (\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3}) \end{aligned} \tag{23}$$

$$\begin{aligned}\boldsymbol{\phi}(h) &= \boldsymbol{\Delta\theta_1} + \boldsymbol{\Delta\theta_2} + \boldsymbol{\Delta\theta_3} + \boldsymbol{\Delta\theta_4} \\ &\quad + \frac{232}{315}\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2} + \frac{46}{105}\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3} + \frac{18}{35}\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_4} + \frac{178}{315}\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3} + \frac{46}{105}\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_4} + \frac{232}{315}\boldsymbol{\Delta\theta_3}\times\boldsymbol{\Delta\theta_4} \\ &\quad - 0.11411\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2})\times\boldsymbol{\Delta\theta_1} + 0.24641\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2})\times\boldsymbol{\Delta\theta_2} + 0.44119\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2})\times\boldsymbol{\Delta\theta_3} \\ &\quad + 0.33516\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_2})\times\boldsymbol{\Delta\theta_4} - 0.06736\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_1} - 0.22590\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_2} \\ &\quad + 0.02931\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_3} + 0.25831\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_4} - 0.09930\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_1} \\ &\quad - 0.01909\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_2} - 0.17279\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_3} + 0.09930\,(\boldsymbol{\Delta\theta_1}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_4} \\ &\quad - 0.10836\,(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_1} - 0.13487\,(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_2} + 0.13487\,(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_3} \\ &\quad + 0.48085\,(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_3})\times\boldsymbol{\Delta\theta_4} - 0.06642\,(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_1} - 0.02931\,(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_2} \\ &\quad - 0.14659\,(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_3} + 0.06736\,(\boldsymbol{\Delta\theta_2}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_4} - 0.14327\,(\boldsymbol{\Delta\theta_3}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_1} \\ &\quad - 0.06871\,(\boldsymbol{\Delta\theta_3}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_2} - 0.24640\,(\boldsymbol{\Delta\theta_3}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_3} + 0.11411\,(\boldsymbol{\Delta\theta_3}\times\boldsymbol{\Delta\theta_4})\times\boldsymbol{\Delta\theta_4} + \boldsymbol{\phi_4} \end{aligned} \tag{25}$$

while the blue dotted line is the attitude drift error of the proposed three sub-sample attitude compensation algorithm proposed in this paper.

From Fig.1, we can see that the red curve fluctuates greatly with the angle of the half cone, and the blue curve is less affected. The attitude drift error of the traditional algorithm reaches $0.01°/h$ when the half cone angle is 90 degrees while the attitude drift error of the proposed algorithm is only $10^{-10°}/h$. What's more when the half cone angle is greater than 0.01 degrees, the accuracy of the proposed algorithm is significantly better than the traditional algorithm. And when the half cone angle is small, the proposed algorithm also has a high enough precision ($10^{-12°}/h$). Therefore the error is negligible for practical applications. In general, the proposed algorithm is less affected by the half cone angle and has greater practicability.

In addition, the simulation also shows that the error of the traditional algorithm on the non-conical axis is large while the error of the proposed algorithm is small. Taking the half cone angle $\alpha = 1°$, $\alpha = 30°$ and $\alpha = 90°$ as examples, the typical cone motion posture is shown in Fig.2 a,c and e. The angle update error on the non-cone angle is shown in Fig.2 b,d and f. In these figures the red line is the attitude error curve of the traditional three sub-sample algorithm while the blue line is the proposed three sub-sample algorithm attitude error curve proposed in this paper.

It can be seen from Fig.2 that the error of the traditional algorithm reaches $20''$ at the maximum, and the error of the proposed algorithm is always small, only about $10^{-2''}$.

Combined with the above two simulation experiments, it can be seen that the accuracy of the proposed algorithm is better than that of the traditional algorithm in the typical conical motion environment.

B. Error Analysis and Verification Under High Dynamic Environment

In order to fully test the performance of the proposed algorithm, simulation in high dynamic environment was carried out. The high dynamic simulation conditions use the 2s large-angle maneuvering environment represented by polynomial in [19], [25] and [26]. The typical 2s internal angular rate increment $\alpha(t)$is :

$$\begin{aligned}\alpha_x &= \frac{5}{2^3}\frac{t^2}{T_m} + \frac{0.75}{2^4}\frac{t^3}{T_m} + \frac{2.5}{2^5}\frac{t^4}{T_m} - \frac{5}{2^6}\frac{t^5}{T_m} \\ \alpha_y &= -\frac{1}{2^3}\frac{t^2}{T_m} + \frac{1.2}{2^4}\frac{t^3}{T_m} - \frac{2.2}{2^5}\frac{t^4}{T_m} - \frac{0.2}{2^6}\frac{t^5}{T_m} \\ \alpha_z &= -\frac{5.1}{2^3}\frac{t^2}{T_m} + \frac{6}{2^4}\frac{t^3}{T_m} - \frac{2.7}{2^5}\frac{t^4}{T_m} + \frac{1.2}{2^6}\frac{t^5}{T_m} \end{aligned} \tag{26}$$

where $T_m = t_m - t_{m-1}$indicates the sampling period while t is the time.

After calculated the derivative of t for $\alpha(t)$, the angular rate and angular acceleration are obtained, as shown in Fig.3.

The 100Hz attitude update results calculated by Yan's 6 subsample 10 iteration times algorithm are used as the attitude reference in this simulation. And the attitude error comparison in high dynamic motion environment is shown in Fig.4. The red, blue, and green curves represent attitude drift errors in x, y and z direction, respectively. The upper part of Fig.4(a) is the proposed three-subsample algorithm (named 3 sample for short), and the lower part is the traditional 3 sample algorithm (named traditional algorithm for short); the upper part of the Fig.4(b) is the three subsample algorithm of YAN (named Yan-3sample for short) [18], [19], and the lower half is the four subsample algorithm of Yan (named Yan-4sample for short); the upper part of Fig.4(c) is WuYuanxin's RodFIter algorithm (named RodFlter for short), the lower part of the solid line is the four-subsample fourth-order algorithm of WANG (named WANG 44 algorithm for short), and the dotted line is the four-subsample fourth-order algorithm obtained in this paper (named proposed 44 algorithm for short).

The traditional algorithm is the optimized 3 sample cone error compensation algorithm proposed by Miller [20]. Yan's 3 sample and 4 sample algorithms are accurate numerical solution for strapdown attitude algorithm based on picard iteration [19]. The samples and iteration times used in the RodFIter algorithm are 4 and 6, respectively. If you are interested, you can refer to the RodFIter application and test by yourself: https://www.researchgate.net/project/Motion-Representation-and-Inertial-Computation-Inertial-Navigation-and-Beyond. WANG 44 algorithm is Eq.30-32 in [30] which the fourth-order cross-product terms were first considered.

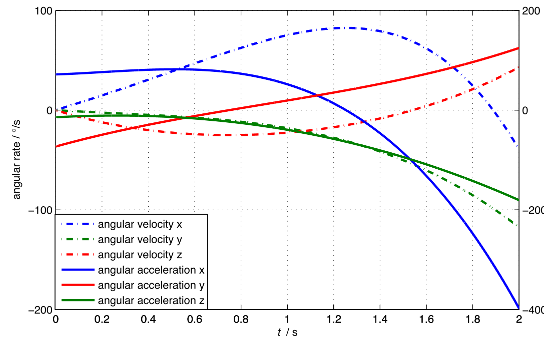

Fig. 3. Angular rate and angular acceleration versus time.

From Fig.4, we can see that the accuracy of the three subsamples of YAN and the three subsamples proposed in this paper is significantly better than the one of the traditional three subsamples. YAN's four-sample algorithm is better than

$$\begin{aligned}
\boldsymbol{\phi}_4 = & \begin{bmatrix} a11(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_1)+a12(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_2)+a13(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_3)+a14(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_4)\\ +a15(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_2)+a16(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_3)+a17(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_4)\\ +a18(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_3)+a19(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_4)+a10(\boldsymbol{\Delta\theta}_4\cdot\boldsymbol{\Delta\theta}_4)\end{bmatrix}(\boldsymbol{\Delta\theta}_1\times\boldsymbol{\Delta\theta}_2)\\
&+\begin{bmatrix} a21(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_1)+a22(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_2)+a23(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_3)+a24(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_4)\\ +a25(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_2)+a26(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_3)+a27(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_4)\\ +a28(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_3)+a29(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_4)+a20(\boldsymbol{\Delta\theta}_4\cdot\boldsymbol{\Delta\theta}_4)\end{bmatrix}(\boldsymbol{\Delta\theta}_1\times\boldsymbol{\Delta\theta}_3)\\
&+\begin{bmatrix} a31(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_1)+a32(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_2)+a33(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_3)+a34(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_4)\\ +a35(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_2)+a36(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_3)+a37(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_4)\\ +a38(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_3)+a39(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_4)+a30(\boldsymbol{\Delta\theta}_4\cdot\boldsymbol{\Delta\theta}_4)\end{bmatrix}(\boldsymbol{\Delta\theta}_1\times\boldsymbol{\Delta\theta}_4)\\
&+\begin{bmatrix} a41(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_1)+a42(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_2)+a43(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_3)+a44(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_4)\\ +a45(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_2)+a46(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_3)+a47(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_4)\\ +a48(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_3)+a49(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_4)+a40(\boldsymbol{\Delta\theta}_4\cdot\boldsymbol{\Delta\theta}_4)\end{bmatrix}(\boldsymbol{\Delta\theta}_2\times\boldsymbol{\Delta\theta}_3)\\
&+\begin{bmatrix} a51(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_1)+a52(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_2)+a53(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_3)+a54(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_4)\\ +a55(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_2)+a56(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_3)+a57(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_4)\\ +a58(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_3)+a59(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_4)+a50(\boldsymbol{\Delta\theta}_4\cdot\boldsymbol{\Delta\theta}_4)\end{bmatrix}(\boldsymbol{\Delta\theta}_2\times\boldsymbol{\Delta\theta}_4)\\
&+\begin{bmatrix} a61(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_1)+a62(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_2)+a63(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_3)+a64(\boldsymbol{\Delta\theta}_1\cdot\boldsymbol{\Delta\theta}_4)\\ +a65(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_2)+a66(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_3)+a67(\boldsymbol{\Delta\theta}_2\cdot\boldsymbol{\Delta\theta}_4)\\ +a68(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_3)+a69(\boldsymbol{\Delta\theta}_3\cdot\boldsymbol{\Delta\theta}_4)+a60(\boldsymbol{\Delta\theta}_4\cdot\boldsymbol{\Delta\theta}_4)\end{bmatrix}(\boldsymbol{\Delta\theta}_3\times\boldsymbol{\Delta\theta}_4)
\end{aligned}$$

$$\begin{aligned}
&a11=0.000798,\quad a12=0.076565,\quad a13=0.057156,\quad a14=0.072077,\quad a15=-0.052933\\
&a16=0.136235,\quad a17=0.025011,\quad a18=-0.028135,\quad a19=-0.046021,\quad a10=-0.002076\\
&a21=-0.000260,\quad a22=0.051696,\quad a23=0.026941,\quad a24=0.034546,\quad a25=-0.157276\\
&a26=0.034933,\quad a27=0.184724,\quad a28=0.006690,\quad a29=0.010357,\quad a20=0.007517\\
&a31=-0.002200,\quad a32=0.049461,\quad a33=0.024497,\quad a34=0.047786,\quad a35=-0.092599\\
&a36=-0.162411,\quad a37=-0.027322,\quad a38=-0.057501,\quad a39=0.005158,\quad a30=0.001018\\
&a41=0.079650,\quad a42=0.170350,\quad a43=0.112433,\quad a44=-0.122331,\quad a45=-0.010313\\
&a46=0.058643,\quad a47=0.022299,\quad a48=-0.010311,\quad a49=0.132553,\quad a40=0.005693\\
&a51=-0.036778,\quad a52=0.102338,\quad a53=0.370460,\quad a54=0.086366,\quad a55=0.006192\\
&a56=0.125138,\quad a57=0.026938,\quad a58=-0.119481,\quad a59=-0.033644,\quad a50=0.00021\\
&a61=-0.040611,\quad a62=-0.231752,\quad a63=-0.010103,\quad a64=0.122628,\quad a65=-0.118264\\
&a66=0.098439,\quad a67=0.142499,\quad a68=0.043230,\quad a69=0.076565,\quad a60=0.000556
\end{aligned}$$

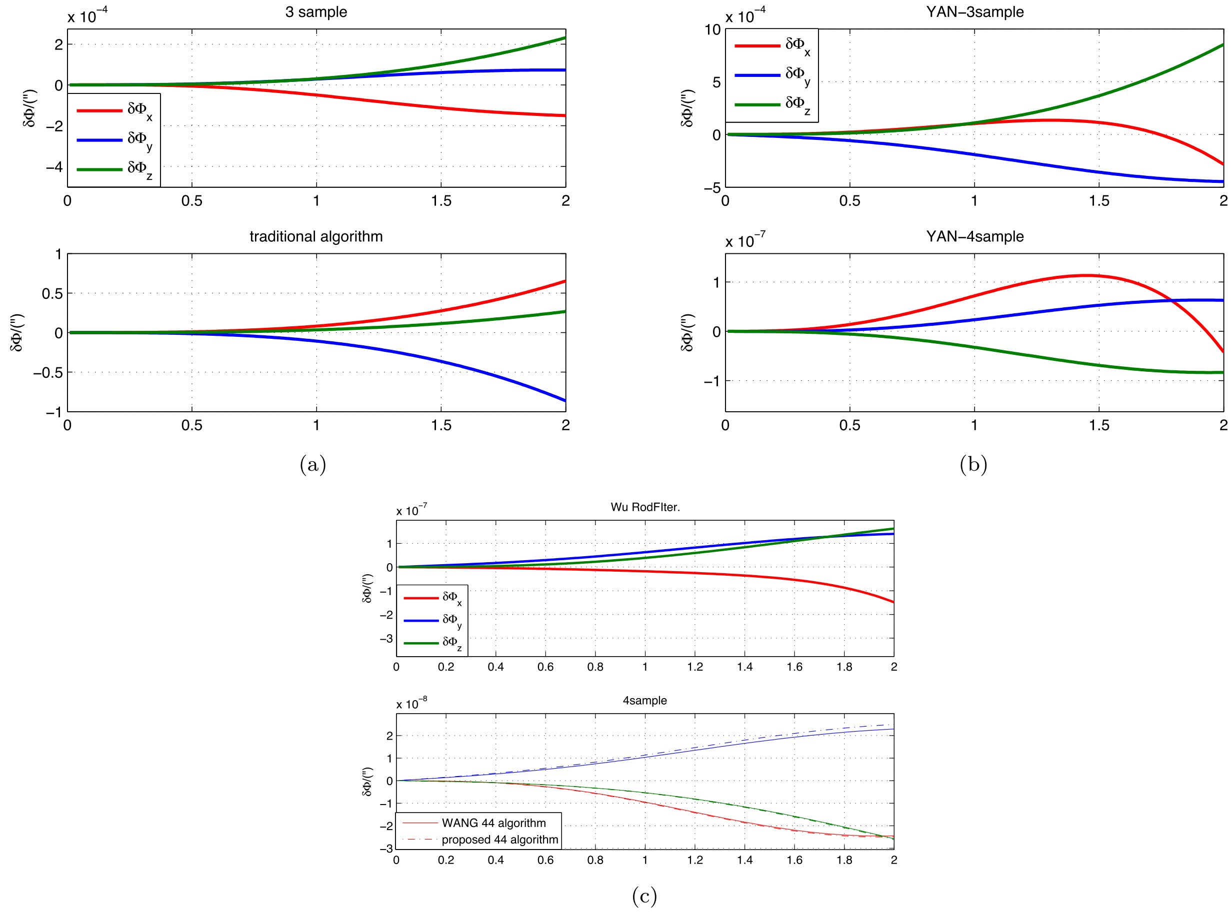

Fig. 4. Attitude error comparison in high dynamic motion environment. (a) Proposed 3 sample algorithm and traditional 3 sample algorithm (b) Yan 3 sample algorithm and Yan 4 sample algorithm (c) WuYuanxin's RodFIter, WANG 4 sample 4 order and 4 sample 4 order algorithm proposed in this paper.

YAN's three-sample algorithm; The algorithm accuracy of Wu's RodFIter is equivalent to the accuracy of Yan's four-sample algorithm; WANG 4 sample 4 order and the 4 sample 4 order attitude error compensation algorithm proposed in this paper with comparable accuracy, both are superior to YAN's four-sample algorithm. Because of that the attitude compensation coefficients obtained by this paper and WANG are almost the same, there is no significant difference in accuracy. In other words, the algorithm proposed in this paper uses different ideas, but it can also achieve excellent results.

V. Conclusion

In order to improve the performance of the navigation algorithm, a improved high-dynamic cone error compensation algorithm is proposed in this paper. To address the limitations of traditional algorithms in the case of large cone angles or in high dynamic maneuvering environments, the three order and four order cross products and the cross multiplication terms are reconsidered in the proposed algorithm avoiding errors caused by approximate. And then the three sub-sample attitude compensation algorithm is deduced in detail. What's more, we also give the derivation idea and the error compensation cofficient of the four sub-sample attitude compensation algorithm. The derivation method given in this paper does not need to know the coefficient compensation model in advance, resulting that the derivation of high-sample and high-order attitude compensation algorithms are much easier to implement. In addition pure coning and high dynamic environment simulations were come out to evaluate the performance of the proposed algorithm. The results show that the attitude error of the improved algorithm is less than the one of the traditional algorithm in typical conical motion case as well as high dynamic environment, proving its effectiveness and superiority.

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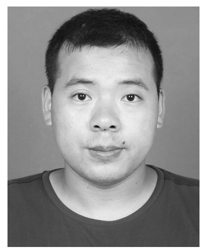

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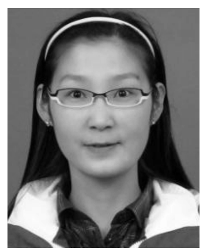

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