# Understanding Normalized Mean Squared Error in Power Amplifier Linearization

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***Abstract*— This letter provides a detailed analysis of the normalized mean squared error (NMSE) of an ideal orthogonal frequency-division multiplexing (OFDM) system, subject to soft clipping of the power amplifier (PA) output. Based on the central limit theorem, the OFDM baseband input is modeled as white Gaussian noise, and based on the Bussgang theorem, the NMSE is studied in some detail. Asymptotes describing the NMSE for the considered system are derived, when the system is working in the linear operation mode and when working under compression, respectively. Detailed yet simple expressions between the NMSE, average output power, saturation voltage of the PA, and the noise level of the additive channel or thermal noise are derived, which are believed to provide insight into the system's behavior.**

***Index Terms*— Linearization, nonlinear systems, normalized mean square error (NMSE), power amplifier (PA).**

## I. Introduction

Studies of orthogonal frequency-division multiplexing (OFDM) systems and their performance under power amplifier (PA) nonlinearities is a well-established research area with several established papers [1]–[3] based on Bussgang theory [4]. Recently, there has been a renewed interest in the topic, including studies of back-off optimization [5], [6] and studies of the normalized mean squared error (NMSE) and its lower bound [7].

The recent work [7] presented an analysis of the NMSE as a function of the average output power $P$ of a nonideal transmitter driven by OFDM inputs. In this letter, the NMSE-versus-$P$ performance of an ideally predistorted PA was derived, leading to a bound on the performance of all practical systems. The results of [7] are quite complex, but can be numerically evaluated for different use cases. However, numerical evaluations alone provide limited understanding of the system's behavior. Understanding, the behavior of the NMSE is the goal of this letter. With the work [7] as a starting point, additional results are derived to gain insight and rule-of-thumbs on the achievable performance. Thanks to the intuitive interpretations of the derived results, they are judged valuable for the practitioners and researchers in the area.

In Section II, the considered model is introduced and the main results of [7] are reviewed. The main contributions are provided in Section III. In Section IV, a numerical example is presented and some further observations are provided. Section V concludes this letter.

Manuscript received May 31, 2018; revised August 5, 2018; accepted September 3, 2018. Date of publication October 17, 2018; date of current version November 6, 2018.

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Digital Object Identifier 10.1109/LMWC.2018.2869299

## II. System NMSE and Average Output Power

We consider an ideal DPD-PA chain which has a static and linear behavior up to a saturation voltage [7]. With $A$ denoting the saturation voltage, the ideal DPD-PA chain is given by

$$y = \begin{cases} u + w, & |u| \le A \\ Ae^{j\angle u} + w, & |u| > A. \end{cases} \tag{1}$$

In (1), the OFDM-modulated input $u$ is by the central limit theorem modeled as a complex-valued white Gaussian signal with variance $\sigma_u^2$, $w$ is a complex-valued circularly symmetric additive white Gaussian noise with variance $\sigma_w^2$, and $y$ denotes the baseband modeled noise corrupted PA signal. The variance $\sigma_y^2$ of the output $y$ is given by [7]

$$\sigma_y^2 = \sigma_u^2(1 - e^{-\gamma^2}) + \sigma_w^2 \tag{2}$$

where the *input back-off level* $\gamma^2$ was introduced as $\gamma^2 = A^2/\sigma_u^2$. The model (1) is adopted from the work in [7], where the model and the results derived from it were experimentally verified.

The NMSE is the focus in this letter, which is defined as

$$\text{NMSE} \triangleq \frac{\text{Var}(y - u)}{\sigma_u^2} \tag{3}$$

where $\text{Var}(z)$ denotes the variance of the stochastic variable $z$. Based on (1), an expression for the NMSE was derived in [7], that is

$$\text{NMSE} = e^{-\gamma^2} - 2\gamma\sqrt{\pi}\,Q(\sqrt{2\gamma^2}) + \frac{\gamma^2}{\beta^2} \tag{4}$$

where $Q(\cdot)$ denotes the tail probability of the standard normal distribution. Furthermore, the *clipping-to-input ratio* $\beta^2$ was introduced as $\beta^2 = A^2/\sigma_w^2$. The core of the work in [7] was a study of the NMSE as a function of the average output power $P = \sigma_y^2/2R$, where $R$ is the load impedance and $\sigma_y^2$ is given in (2). Thus,

$$P = \frac{A^2(1 - e^{-\gamma^2})}{2R\gamma^2} + \frac{\sigma_w^2}{2R} \tag{5}$$

where, for the forthcoming analysis, $\sigma_u^2$ was expressed as $A^2/\gamma^2$. The results (4) and (5) were the main results of [7].

## III. Main Results

The performance analysis presents asymptotes for the NMSE valid for four different regions of operations, spanning from a small signal operation to full saturation, as defined in Table I. The two extreme cases (including $\gamma \to \infty$ and $\gamma \to 0$, as special cases) are merely of academic interest, but still provide some valuable insight into the problem under study.

TABLE I
REGIONS OF OPERATION AND THE CORRESPONDING INPUT BACK OFF $\gamma^2$

| Region of operation | Input back-off $\gamma^2$ | $\text{SNR} = \sigma_u^2/\sigma_w^2$ |
|---|---|---|
| Small signal operation | $\gamma^2 \gg 1$ | Low |
| Linear operation | $\gamma^2 \geq \ln \beta^2/2$ | High |
| Compression | $\gamma^2 < \ln \beta^2/2$ | High |
| Saturation | $\gamma^2 \ll 1$ | High |

### A. Small Signal and Linear PA Operation

We observe that the sum of the first two terms in (4) tends to zero as $\gamma^2$ increases, and the third term dominates the expression, that is

$$\text{NMSE} \simeq \frac{\gamma^2}{\beta^2} \tag{6}$$

where $\simeq$ denotes an equality where only the dominant terms have been retained. With the same argument, the average output power (5) is approximated as

$$P \simeq \frac{A^2}{2R\gamma^2} + \frac{\sigma_w^2}{2R} \simeq \frac{A^2}{2R\gamma^2} \tag{7}$$

where the second equality follows from the high-SNR assumption. It is relevant to comment on the fact that the final form of (7) simply equals $\sigma_u^2/2R$, but for the forthcoming discussions expressing the power as function of $\gamma^2$ is convenient. In particular, as $\gamma^2 \to \infty$ and under a load impedance of $R = 50$ [Ω], the average output power (7) is given by

$$P_{\text{dBm}}^{\text{min}} = 10\log_{10}\left(\sigma_w^2\right) + 10 \text{ [dBm]} \tag{8}$$

where $P_{\text{dBm}}^{\text{min}}$ is a lower bound on the average output power. Accordingly, the output power is limited by the power of the additive noise, whereas according to (6) the NMSE increases without bound as $\gamma^2$ increases. Combining (6) and (7), one has

$$\text{NMSE} \simeq \frac{\sigma_w^2}{2R} \cdot \frac{1}{P}. \tag{9}$$

With the NMSE approximated by (9) under a load impedance of $R = 50$ [Ω], the linear asymptote below describes the region of linear operation

$$\text{NMSE}_{\text{dB}} = -\text{P}_{\text{dBm}} + \text{P}_{\text{dBm}}^{\text{min}} \text{ [dBm]} \tag{10}$$

where $\text{NMSE}_{\text{dB}}$ is the NMSE in dB, $P_{\text{dBm}}$ the average output power $P$ in dBm, and $P_{\text{dBm}}^{\text{min}}$ the minimal output power according to (8).

As observed by (7), as the value of $\gamma^2$ increases, the average output power decreases, and so the NMSE increases. In addition, the high-SNR assumption SNR $\gg 1$ is violated as the additive noise becomes more pronounced relative to the power of the input. Thus, it is reasonable to define the transition from the small signal region to the region of linear operation as the point where $\text{NMSE}_{\text{dB}} = 0$ [dB] or by (6) for $\gamma^2 = \beta^2$. By (7), the transition point corresponds to an average power $P \simeq \sigma_w^2/R$, which by aid of (8) provides the approximation

$$P_{\text{dBm}}^{\text{lin}} = 3 + P_{\text{dBm}}^{\text{min}} \text{ [dBm]} \tag{11}$$

as the average output power when the PA enters from the small signal region into the linear region of operation.

### B. Operation in Compression and Saturation

*1) Minimum NMSE at $P_{dBm}^{bar}$:* For $\gamma > 1$, the NMSE in (3) is approximated by [6]

$$\text{NMSE} \simeq \frac{e^{-\gamma^2}}{2\gamma^2} + \frac{\gamma^2}{\beta^2}. \tag{12}$$

In (12), the first term is monotonically decreasing for increasing $\gamma^2$, whereas the latter term is monotonically increasing, with a minima for a particular $\gamma^2 = \bar{\gamma}^2$. In [6], it was shown that

$$\bar{\gamma}^2 \simeq \ln\left(\frac{\beta^2}{2}\right) - \ln\left(\ln\left(\frac{\beta^2}{2}\right)\right) \tag{13}$$

where $\ln(\cdot)$ is the natural logarithm. Since $A$ is a fixed constant, the solution $\bar{\gamma}^2$ determines the average signal power $\sigma_u^2$, where the PA operation switches from the linear region to weak compression. The corresponding NMSE is given by (12) for $\gamma^2 = \bar{\gamma}^2$. Under an assumption of $\beta^2 \gg 1$ and accordingly $\bar{\gamma}^2 \approx \ln(\beta^2/2)$, the minimum NMSE obtained at $\gamma^2 = \bar{\gamma}^2$ is well approximated by

$$\text{NMSE}_{\text{dB}}^{\text{bar}} = 10\log_{10}(\bar{\gamma}^2) - 10\log_{10}(\beta^2) \text{ [dB]} \tag{14}$$

where $\text{NMSE}_{\text{dB}}^{\text{bar}}$ denotes the minimum NMSE in dB. The power at the transition point $\bar{\gamma}^2$ follows from (5) and is after suitable approximations equal to (7) for $\gamma^2 = \bar{\gamma}^2$. The average output power $P_{\text{dBm}}^{\text{bar}}$ determining the transition from the linear to nonlinear region of operation of the amplifier is given by

$$P_{\text{dBm}}^{\text{bar}} = 10\log_{10}(A^2) - 10\log_{10}(\bar{\gamma}^2) + 10 \text{ [dBm]}. \tag{15}$$

The average power at the transition point $P_{\text{dBm}}^{\text{bar}}$ is a function of the PA output saturation voltage $A$ and the clipping-to-noise ratio via $\bar{\gamma}^2$ in (13).

*2) 1-dB Compression and Saturation:* The 1-dB compression point can be calculated from (5). Under the high-SNR assumption, the equation describing the 1-dB compression of the average output power reads $1 - \exp(-\gamma^2) = 10^{-1/10}$ so that $\gamma_{\text{P1dB}}^2 \approx 1.6$. Accordingly, inserting the numerical value $\gamma_{\text{P1dB}}^2$ into (5), the output average power (5) at the 1-dB compression point follows.

An operation in saturation ($\gamma \to 0$) limits the NMSE in (4) to $\text{NMSE}_{\text{dB}}^{\text{sat}} = 0$ dB. In addition, from (5)

$$\lim_{\gamma \to 0} P = \frac{A^2}{2R} \lim_{\gamma \to 0}\left(\frac{1 - e^{-\gamma^2}}{\gamma^2}\right) + \frac{\sigma_w^2}{2R} = \frac{A^2 + \sigma_w^2}{2R} \tag{16}$$

where L'Hôpital's rule was used in the second equality. Under the high-SNR assumption, the maximum output power (16) is limited by

$$P_{\text{dBm}}^{\text{sat}} = 10\log_{10}(A^2) + 10 \text{ [dBm]}. \tag{17}$$

With (17), 1-dB compression of the average output power reads

$$P_{\text{dBm}}^{\text{P1dB}} = P_{\text{dBm}}^{\text{sat}} - 3 \text{ [dBm]}. \tag{18}$$

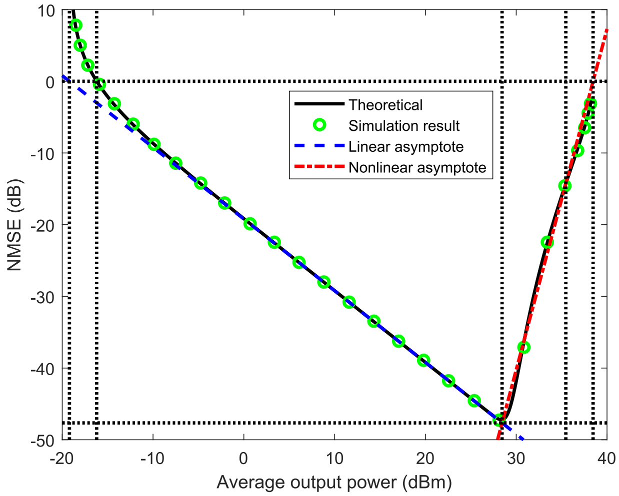


Fig. 1. Theoretical NMSE versus average output power (black solid line) and simulation results (green 'o'). Asymptotes for the linear region of operation (10) (blue dashed line) and the region of compression (19) with Δ according to (20) (red dashed-dotted line). The power levels $P_{\mathrm{dBm}}^{\mathrm{min}} = -19.2$ [dBm] (8), $P_{\mathrm{dBm}}^{\mathrm{lin}} = -16.2$ [dBm] (11), $P_{\mathrm{dBm}}^{\mathrm{bar}} = 28.4$ [dBm] (15), $P_{\mathrm{dBm}}^{\mathrm{P1dB}} = 35.5$ [dBm] (18), and $P_{\mathrm{dBm}}^{\mathrm{sat}} = 38.5$ [dBm] (17) are all shown by the vertical black dotted lines. $\mathrm{NMSE}_{\mathrm{dB}}^{\mathrm{bar}} = -47.7$ according to (14) is shown by the horizontal dashed line.

*3) Asymptote in Compression:* Finally, the asymptote when the PA operates in compression is studied. Here, the slope Δ of a linear asymptote is determined by the transition and saturation points, or $\mathrm{NMSE}_{\mathrm{dB}} = \Delta \mathrm{P}_{\mathrm{dBm}} - \Delta \mathrm{P}_{\mathrm{dBm}}^{\mathrm{sat}}$ using $\mathrm{NMSE}_{\mathrm{dB}}^{\mathrm{sat}} = 0$ [dB]. Accordingly,

$$\mathrm{NMSE}_{\mathrm{dB}} = \Delta\big(\mathrm{P}_{\mathrm{dBm}} - \mathrm{P}_{\mathrm{dBm}}^{\mathrm{sat}}\big) \text{ [dBm]}. \tag{19}$$

The slope Δ follows from straightforward calculations using the coordinates for the minimum NMSE, that is $\mathrm{NMSE}_{\mathrm{dB}}^{\mathrm{bar}}$ in (14) and $P_{\mathrm{dBm}}^{\mathrm{bar}}$ in (15). A straightforward calculation reveals

$$\Delta = \frac{\log_{10}(\beta^2)}{\log_{10}(\bar{\gamma}^2)} - 1 \tag{20}$$

which is a function of $\beta^2$ only according to (13). For $\beta^2 \gg 1$, Δ linearly increases with increasing $\beta^2$ expressed in dB approximately as $\Delta \approx \beta_{\mathrm{dB}}^2/20 + 2$, where $\beta_{\mathrm{dB}}^2 = 10\log_{10}(\beta^2)$ [dB].

## IV. Discussion and Numerical Results

Although (4) and (5) are straightforward to numerically evaluate for different use cases, the result is typically not easily interpreted by visual inspection only, and thus, the derived results provide valuable insight and useful rule-of-thumbs. In Fig. 1, the NMSE (4) is numerically evaluated as a function of the average output power $P$ (5). The numerical values are adopted from [7], that is $A = 26.5$ [V], $\sigma_w^2 = 0.0012$ [V$^2$], and $R = 50$ [Ω]. From Fig. 1, one can note a satisfactory level of precision of the derived results. Despite the quite complex NMSE in (4), the behavior is well approximated by the derived linear asymptotes, (10) and (19), respectively. The value given by (14) provides an accurate approximation of the minimum of the NMSE, which is a function of $\beta^2$ only.

A deeper insight is provided by adding simulation results, based on 100 independent runs out of which each run is based on 100 baseband samples. There is an excellent agreement between the theoretical and simulation results, as evidenced by Fig. 1. Thus, the theoretical predictions are judged to determine the performance of practical systems in finite samples.

From Fig. 1, it is observed that during linear operation, the NMSE decays by 1 dB per dB increase in the average output power, a result that is also evident from (10). The offset is entirely determined by the additive noise level $\sigma_w^2$ according to (10). During compression, the slope of the asymptote is given by (20), that is an increase of $\Delta = 4.8$ dB per dB increase in output power, for the use case $\beta^2 = 26.5^2/0.0012$, that is $\beta_{\mathrm{dB}}^2 = 57$ [dB].

## V. Conclusion

The purpose of this letter is to enlighten the NMSE versus $P$ behavior of an idealized OFDM system, where the operating point of the PA varies. It is observed that all results are independent of the DPD input power $\sigma_u^2$, that is the behavior of the DPD-PA chain is entirely determined by the saturation voltage $A$ and the power of the additive noise $\sigma_w^2$, either directly or via the clipping-to-noise-ratio $\beta^2 = A^2/\sigma_w^2$.

The NMSE-$P$ relation is well described by two linear asymptotes, describing the mode of linear operation and compression, respectively. The former asymptote has a constant 1 [dB] slope per dB increase in average output power, and the latter a positive slope determined by $\beta^2$, where typically Δ is in the region [3, 6]. The crossing of the two asymptotes determines the mode of operation where the NMSE is minimized, that is in the transition from PA working in linear operation to a weak compression. Explicitly, the input back-off level minimizing the NMSE is determined by the average output power $P_{\mathrm{dBm}}^{\mathrm{bar}}$ in (15).

Due to their simplicity and ability to accurately predict the performance of the considered OFDM system, the derived results are believed to be useful for practitioners and researchers in wireless communications.

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