Phase Error Scaling Law in Two-Wavelength Adaptive Optics

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***Abstract*— We derive a simple, physical, closed-form expression for the optical-path difference (OPD) of a two-wavelength adaptive-optics (AO) system. Starting from Hogge and Butts' classic OPD variance integral expression [*J. Opt. Soc. Am.* 72, 606 (1982)], we apply Mellin transform techniques to obtain series and asymptotic solutions to the integral. For realistic two-wavelength AO systems, the former converges slowly and has limited utility. The latter, on the other hand, is a simple formula in terms of the separation between the AO sensing (i.e., the beacon) and compensation (or observation) wavelengths. We validate this formula by comparing it to the OPD variances obtained from the aforementioned series and direct numerical evaluation of Hogge and Butts' integral. Our simple asymptotic expression is shown to be in excellent agreement with these exact solutions. The work presented in this letter will be useful in the design and characterization of two-wavelength AO systems.**

***Index Terms*— Adaptive optics, atmospheric turbulence.**

I. INTRODUCTION

ONE of the first analytical treatments of adaptive-optics (AO) correction concerned two-wavelength compensation, where atmospheric turbulence sensing—performed at one wavelength λ_{B}—is used to correct for wavefront aberrations at another λ_{T} [1], [2], [3], [4]. The motivation for this early work was the development of artificial AO beacons, either Rayleigh beacons or sodium guide stars [5], [6], [7], [8]. For the latter, a laser excites sodium atoms in the mesosphere that emit light at 589.2 nm. This light is used to sense and correct for atmospheric-turbulence-induced optical wavefront distortions. In astronomy, scientific observations are typically performed at longer wavelengths in the short-wave infrared. Quantifying the residual phase error or drop in image quality (Strehl ratio) therefore became crucial.

Arguably the most influential work in this regard was that of Hogge and Butts [3], who derived an integral expression for the optical-path-difference (OPD) variance assuming weak atmospheric turbulence (i.e., negligible irradiance fluctuations or scintillation). Primarily motivated by the astronomical application, they evaluated the integral numerically using the Hufnagel–Valley model [9] for the index of refraction structure constant C_n^2 and found that as long as $\lambda_{\mathrm{B}} < \lambda_{\mathrm{T}}$ and $|\lambda_{\mathrm{B}} - \lambda_{\mathrm{T}}|$ was less than a few microns, the OPD variance (and ultimately, the uncorrected phase error) was not overly punitive.

Others soon followed, including Holmes and Gudimetla [4], Winocur [10], and Wallner [11]. Holmes and Gudimetla quantified the effects of λ_{B}-λ_{T} separation on Strehl ratio. Winocur investigated the OPD variances for the first five Zernike modes/aberrations. Lastly, Wallner compared the diffractive OPD error (initially studied by Hogge and Butts) to refractive error caused by dispersion.

Based on the work of Hogge and Butts as well as these other scientists, diffractive error was generally found to be negligible for astronomical applications. Consequently, in the last few decades, two-wavelength AO research has shifted toward quantifying and correcting refractive or chromatic errors, which are more prevalent in astronomical observation [9], [12], [13], [14] as well as ground-to-satellite and satellite-to-ground optical communications [15].

However, this focus has begun to change with the development and fielding of terrestrial-based free-space optical communication, laser remote-sensing, and other directed energy systems [16]. The operational environment of these systems (horizontal propagation paths and small fields of view) mean that chromatic error is negligible and diffractive effects become significant. Furthermore, in many instances, these systems violate the conditions identified by Hogge and Butts, e.g., $\lambda_{\mathrm{B}} > \lambda_{\mathrm{T}}$. Consequently, being able to easily quantify the potential diffractive wavefront errors of a two-wavelength AO system is important.

Several recent studies have investigated this problem [17], [18], [19], but like the original works [3], [4], [10], the research was predominately computational, either numerical evaluation of integral relations (i.e., quadrature) or wave-optics simulations. Reference [17] does derive closed-form relations for two-wavelength phase error; yet, the expressions are relatively complicated. Indeed, deriving simple formulas to predict two-wavelength AO performance was not the purpose of their work.

Our goal in this short letter is to obtain such a formula. Starting with Hogge and Butts' integral expression for the OPD variance and applying Mellin transform techniques, we obtain an exact series solution, expressed in terms of hypergeometric functions, and an asymptotic approximation

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for the OPD variance. The former slowly converges for realistic two-wavelength AO systems and is solely used to validate the latter, which is both simple and physical. We verify our asymptotic solution by comparing it to both the series OPD variance as well as direct numerical evaluation of Hogge and Butts' integral. We lastly conclude with a brief summary of our work.

II. Theory

The starting point for our analysis is the two-wavelength OPD variance given in Eq. (18) of [3], namely,

$$\left\langle \Delta\ell^2 \right\rangle = 1.303 \int_0^z C_n^2(\zeta) \int_0^\infty \kappa^{-8/3} \left[1 - \mathrm{jinc}^2\left(\frac{D}{2}\kappa\right)\right] \times \left[\cos\left(\frac{z-\zeta}{2k_\mathrm{B}}\kappa^2\right) - \cos\left(\frac{z-\zeta}{2k_\mathrm{T}}\kappa^2\right)\right]^2 \mathrm{d}\kappa\,\mathrm{d}\zeta, \quad (1)$$

where z is the propagation distance, D is the diameter of the circular receiving aperture, C_n^2 is the index of refraction structure constant, $\mathrm{jinc}(x) = 2J_1(x)/x$, and J_1 is a first-kind, first-order Bessel function. Lastly, k_B and k_T are the AO beacon and transmitter wavenumbers, respectively. Note that Eq. (1) is derived assuming Kolmogorov atmospheric turbulence [9], [20], [21] and that the beacon and transmitter fields are plane waves. The phase variance σ_ϕ^2 can be obtained simply by computing the product $\sigma_\phi^2 = k_\mathrm{T}^2 \left\langle \Delta\ell^2 \right\rangle$ [3].

For constant C_n^2 (generally applicable to horizontal propagation paths), the ζ integral can be evaluated in closed form and Eq. (1) becomes

$$\left\langle \Delta\ell^2 \right\rangle = 1.303\ C_n^2 z \int_0^\infty \kappa^{-8/3} \left[1 - \mathrm{jinc}^2\left(\frac{D}{2}\kappa\right)\right] \times \left[1 + \frac{1}{2}\mathrm{sinc}\left(\frac{z}{k_\mathrm{B}}\kappa^2\right) + \frac{1}{2}\mathrm{sinc}\left(\frac{z}{k_\mathrm{T}}\kappa^2\right) - \mathrm{sinc}\left(z\frac{k_\mathrm{B}-k_\mathrm{T}}{2k_\mathrm{B}k_\mathrm{T}}\kappa^2\right) - \mathrm{sinc}\left(z\frac{k_\mathrm{B}+k_\mathrm{T}}{2k_\mathrm{B}k_\mathrm{T}}\kappa^2\right)\right] \mathrm{d}\kappa, \quad (2)$$

where $\mathrm{sinc}(x) = \sin(x)/x$. Equation (2) can be written as the sum of five integrals, such that

$$\left\langle \Delta\ell^2 \right\rangle = 1.303\ C_n^2 z \int_0^\infty \frac{\mathrm{d}\kappa}{\kappa}\kappa^{-11/3} \left[1 - \mathrm{jinc}^2\left(\frac{D}{2}\kappa\right)\right] + 1.303\ C_n^2 z \sum_{i=1}^4 c_i \int_0^\infty \kappa^{-8/3}\,\mathrm{sinc}\left(\alpha_i\kappa^2\right) \times \left[1 - \mathrm{jinc}^2\left(\frac{D}{2}\kappa\right)\right] \mathrm{d}\kappa, \quad (3)$$

where $\boldsymbol{c} = \left[1/2\ 1/2\ -1\ -1\right]$, and

$$\boldsymbol{\alpha} = \left[\frac{z}{k_\mathrm{B}}\ \frac{z}{k_\mathrm{T}}\ \frac{z}{2}\left(\frac{1}{k_\mathrm{T}} - \frac{1}{k_\mathrm{B}}\right)\ \frac{z}{2}\left(\frac{1}{k_\mathrm{T}} + \frac{1}{k_\mathrm{B}}\right)\right]. \quad (4)$$

While the c_i are simply the coefficients of the sinc functions, the α_i define the "two-wavelength Fresnel zone size," which determines when (or if) diffractive phase errors are significant [20]. We can estimate this parameter from the argument of the third sinc function as $l \sim \sqrt{z\left|\lambda_\mathrm{T} - \lambda_\mathrm{B}\right|}$. For small l, which implies short propagation distances z or small λ_T-λ_B separations, $\left\langle \Delta\ell^2 \right\rangle \approx 0$. This situation corresponds to the geometric optics region (or approximation) in classic turbulence theory [9], [20], [21]. On the other hand, for large l—implying large z and large $\left|\lambda_\mathrm{T} - \lambda_\mathrm{B}\right|$—diffractive phase errors are significant and $\left\langle \Delta\ell^2 \right\rangle > 0$. Indeed, under these conditions, the contributions to $\left\langle \Delta\ell^2 \right\rangle$ from the sinc $\left(\alpha_3\kappa^2\right)$ and sinc $\left(\alpha_4\kappa^2\right)$ terms are negligible, implying that the phases at λ_B and λ_T are uncorrelated. Consequently, $\left\langle \Delta\ell^2 \right\rangle$ simplifies to the sum of the individual OPD variances at λ_B and λ_T [3].

Returning to Eq. (3), the first integral, after substituting $x = \kappa D/2$, is equal to the Mellin transform of $J_1^2(x)$ evaluated at $s = -11/3$ [9]. The result is

$$1.303\ C_n^2 z \int_0^\infty \frac{\mathrm{d}\kappa}{\kappa}\kappa^{-11/3} \left[1 - \mathrm{jinc}^2\left(\frac{D}{2}\kappa\right)\right] = 0.4370\ C_n^2 z D^{5/3}. \quad (5)$$

A. Series Solution

We now focus on evaluating the second integral in Eq. (3). Using Mellin transform properties and the Mellin convolution theorem [9], [22], the second integral in Eq. (3), which we represent as I hereafter, is equal to

$$I = \int_0^\infty \kappa^{-8/3}\,\mathrm{sinc}\left(\alpha_i\kappa^2\right)\left[1 - \mathrm{jinc}^2\left(\frac{D}{2}\kappa\right)\right]\mathrm{d}\kappa = 2^{-23/6}\left(\frac{D}{2}\right)^{-2}\alpha_i^{11/6}\frac{1}{\mathrm{j}2\pi}\int_C \mathrm{d}s\left(\frac{D}{\sqrt{2\alpha_i}}\right)^{-s} \times \frac{\Gamma(-s/4 - 11/12)\,\Gamma(1+s/2)\,\Gamma(1/2-s/2)}{\Gamma(29/12+s/4)\,\Gamma(2-s/2)\,\Gamma(1-s/2)}, \quad (6)$$

where the integration contour C crosses the real s axis between $-4 < \mathrm{Re}(s) < -11/3$. The integrand has simple poles at s locations where the arguments of the numerator gamma functions are negative integers. The integral converges for all values of the parameter $D/\sqrt{2\alpha_i}$ when the integration contour is closed to the left. The result is an infinite series—formed by the sum of the pole residues at $s = -2m - 2$ for $m = 1, 2, \cdots, \infty$—that converges slowly for large values of $D/\sqrt{2\alpha_i}$. Recall that $\alpha_i \sim z/k$, and therefore, large values of the parameter are physically expected. Consequently, this series solution is not desired nor useful in achieving our goal of deriving a simple scaling law for $\left\langle \Delta\ell^2 \right\rangle$. Nevertheless, since we do use it for the results presented in Section III, we include it here for completeness:

$$I = 0.2016\ D^2\alpha_i^{-1/6}\,{}_4F_5\left(\left.\begin{matrix}\frac{1}{12}, \frac{5}{4}, \frac{7}{4}, -\frac{5}{12}\\ 2, \frac{5}{2}, \frac{3}{2}, 2, \frac{3}{2}\end{matrix}\right| \frac{-D^4}{64\alpha_i^2}\right) + 0.4715\alpha_i^{5/6}\left[{}_4F_5\left(\left.\begin{matrix}-\frac{5}{12}, \frac{3}{4}, \frac{5}{4}, -\frac{11}{12}\\ \frac{3}{2}, 2, 1, \frac{3}{2}, \frac{1}{2}\end{matrix}\right| \frac{-D^4}{64\alpha_i^2}\right) - 1\right], \quad (7)$$

where ${}_mF_n$ is a generalized hypergeometric function [23], [24]. Note that Eqs. (5) and (7) must be substituted back into Eq. (3) to ultimately find $\left\langle \Delta\ell^2 \right\rangle$.

B. Asymptotic Solution

On the other hand, an asymptotic solution to Eq. (6) can be found by closing the integration contour to the right, circumscribing the poles at $s = -2$, $s = 4m - 11/3$, and $s = 2m + 1$ for $m = 0, 1, 2, \cdots, \infty$. For large values of the parameter, this series converges quickly and therefore, we only need a few terms of the series to obtain accurate results. Focusing on the parameter's exponent reveals

$$I \sim D^{5/3}\left(\frac{D}{\sqrt{2\alpha_i}}\right)^{-s-11/3}, \tag{8}$$

which for the poles at $s = -11/3$, -2, $1/3$, and 1, yields powers equal to 0, $-5/3$, -4, and $-14/3$, respectively. Consequently, we only need to include the contributions from the first two poles as the others are negligible.

Applying Cauchy's integral theorem [25], [26], we obtain

$$I \approx 0.3354\ D^{5/3} - 0.4715\alpha_i^{5/6}. \tag{9}$$

Substituting this and Eq. (5) back into Eq. (3) and simplifying produces

$$\left\langle \Delta\ell^2 \right\rangle \approx 0.3071\ C_n^2 z\left(2\alpha_3^{5/6} + 2\alpha_4^{5/6} - \alpha_1^{5/6} - \alpha_2^{5/6}\right). \tag{10}$$

Reintroducing the α_i yields the final result:

$$\left\langle \Delta\ell^2 \right\rangle \approx 0.0664\ C_n^2 z^{11/6}\Big[2^{1/6}\left|\lambda_\mathrm{T} - \lambda_\mathrm{B}\right|^{5/6} + 2^{1/6}\left(\lambda_\mathrm{T} + \lambda_\mathrm{B}\right)^{5/6} - \lambda_\mathrm{B}^{5/6} - \lambda_\mathrm{T}^{5/6}\Big]. \tag{11}$$

The most interesting aspect of Eq. (11) is that it is independent of the aperture diameter D. Indeed, when multiplied by $1.303\ C_n^2 z$, the first term in Eq. (9) equals the result in Eq. (5). This term is subsequently canceled by the sums and differences in Eq. (3). If we include additional pole contributions, Eq. (11) does depend on D; however, the dependence is weak. We can quantify $\left\langle \Delta\ell^2 \right\rangle$'s dependence on D by evaluating Eq. (8) at $s = 1/3$; the result is $\left\langle \Delta\ell^2 \right\rangle \sim D^{-7/3}$. Consequently, the accuracy of Eq. (11) generally suffers for two-wavelength AO systems with small diameters. This is to be expected since small D results in a small value of the parameter, which violates our assumption that $D/\sqrt{2\alpha_i} \to \infty$.

Before proceeding, we note that we can obtain an even simpler $\left\langle \Delta\ell^2 \right\rangle$ by assuming that $2^{1/6}\left(\lambda_\mathrm{T} + \lambda_\mathrm{B}\right)^{5/6} \approx \lambda_\mathrm{B}^{5/6} + \lambda_\mathrm{T}^{5/6}$. In turn, Eq. (11) becomes

$$\left\langle \Delta\ell^2 \right\rangle \approx 0.0745\ C_n^2 z^{11/6}\left|\lambda_\mathrm{T} - \lambda_\mathrm{B}\right|^{5/6}. \tag{12}$$

This result implies that the OPD error in two-wavelength AO systems is approximately "wavelength shift-invariant," i.e., it depends only on the difference between the beacon and transmitter wavelengths and not on the wavelengths themselves.

III. Numerical Results

In this section, we validate Eqs. (11) and (12) (and most importantly, their physical implications) by comparing $\left\langle \Delta\ell^2 \right\rangle$ to those obtained from the series solution in Eq. (7) and direct numerical evaluation of the integral in Eq. (2).

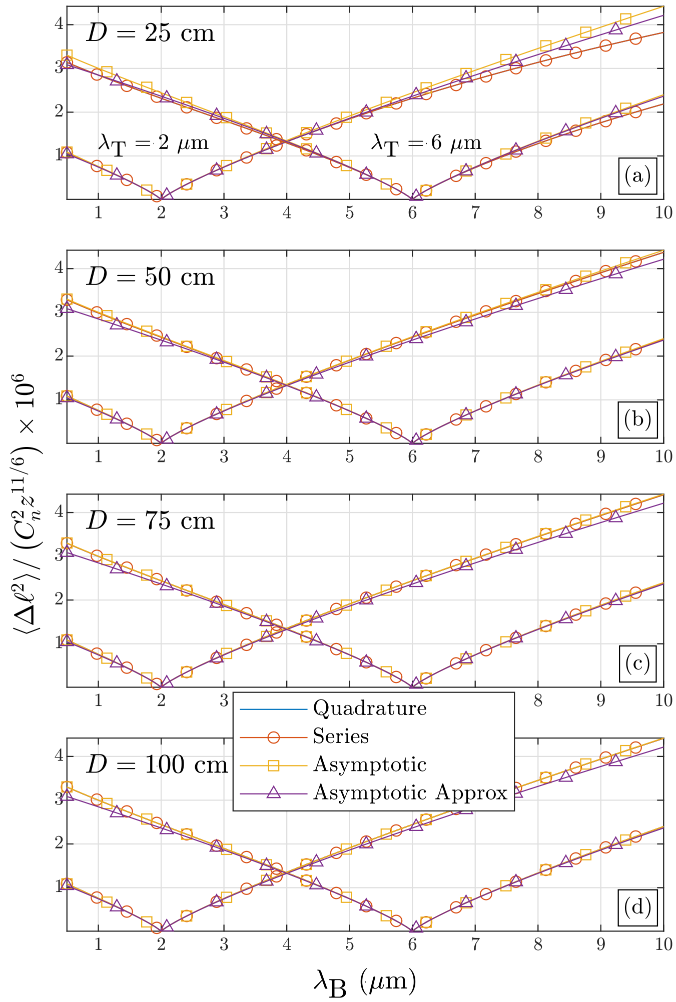

Fig. 1. Scaled OPD variance versus λ_B comparing direct numerical integration of Eq. (2), the series solution in Eq. (7), the asymptotic solution in Eq. (11), and the asymptotic solution approximation in Eq. (12). These $\left\langle \Delta\ell^2 \right\rangle$ are labeled "Quadrature," "Series," "Asymptotic," and "Asymptotic Approx," respectively and are distinguished by trace color and symbol. (a)–(d) show the $\left\langle \Delta\ell^2 \right\rangle$ for $\lambda_\mathrm{T} = 2\ \mu$m, $\lambda_\mathrm{T} = 6\ \mu$m, and $D = 25$ cm, 50 cm, 75 cm, and 100 cm, respectively.

Figure 1 shows these results. The figure is composed of four plots, wherein we show the "Quadrature" [numerical evaluation of Eq. (2)], "Series" [see Eq. (7)], "Asymptotic" [see Eq. (11)], and "Asymptotic Approx" [see Eq. (12)] scaled OPD variances versus beacon wavelength $\lambda_\mathrm{B}\ (\mu\mathrm{m}) \in [0.5, 10]$. The $\left\langle \Delta\ell^2 \right\rangle$ traces for each are delineated by color and symbol. In (a)–(d), $\lambda_\mathrm{T} = 2$ and $6\ \mu$m, and $D = 25$ cm, 50 cm, 75 cm, and 100 cm, respectively. Table I reports the percent errors for the Series, Asymptotic, and Asymptotic Approx results relative to the Quadrature $\left\langle \Delta\ell^2 \right\rangle$ averaged over λ_B.

Overall, the results shown in Fig. 1 and Table I validate our asymptotic analysis. Comparing the $\lambda_\mathrm{T} = 2\ \mu$m values of $\left\langle \Delta\ell^2 \right\rangle$ in (a)–(d), we see the weak dependence of $\left\langle \Delta\ell^2 \right\rangle$ on D. We observe the same weak dependence on D for the $\lambda_\mathrm{T} = 6\ \mu$m $\left\langle \Delta\ell^2 \right\rangle$. In the $D = 25$ cm plot, there is some error in Eqs. (11) and (12), especially at large λ_B-λ_T separations. Examining Table I, we see that the error in Eq. (11) decreases

TABLE I
SERIES, ASYMPTOTIC, AND ASYMPTOTIC APPROXIMATION PERCENT ERRORS

	D (cm)	$\lambda_{\mathrm{T}} = 2\ \mu$m	$\lambda_{\mathrm{T}} = 6\ \mu$m
Series	25	0.2305×10^{-5}	0.3547×10^{-5}
	50	0.2273×10^{-5}	0.3503×10^{-5}
	75	0.2269×10^{-5}	0.3496×10^{-5}
	100	0.2264×10^{-5}	0.3483×10^{-5}
Asymptotic	25	5.3496	4.1168
	50	0.4508	0.3605
	75	0.1305	0.1062
	100	0.0547	0.0446
Asymptotic Approx	25	2.8457	2.5311
	50	2.2818	1.2394
	75	2.5903	1.4884
	100	2.6633	1.5487

as D increases (as predicted), while the error in Eq. (12) remains relatively constant and less than 3%. Furthermore, comparing the $\lambda_{\mathrm{T}} = 2\ \mu$m $\langle \Delta\ell^2 \rangle$ with the $\lambda_{\mathrm{T}} = 6\ \mu$m $\langle \Delta\ell^2 \rangle$ in each of the Fig. 1 subplots, we observe the approximate wavelength shift-invariance of $\langle \Delta\ell^2 \rangle$, as the V-shaped traces simply shift to the right.

Before concluding, it is important to note that the two-wavelength OPD variance expressions derived in this letter are applicable in weak atmospheric turbulence (i.e., negligible scintillation). In turbulence conditions where scintillation is present, the problem is not analytically tractable. Reference [19] numerically studied how scintillation affected two-wavelength, phase-conjugate AO performance. Through a series of wave-optics simulations, the authors found that the Hogge and Butt's $\langle \Delta\ell^2 \rangle$ (and therefore, ours as well) underestimated the true phase error. This difference was due to the weak correlation of the rotational phases [8], [27], [28] at λ_{T} and λ_{B} as well as not correcting the optical field's amplitude (phase-conjugate versus field-conjugate AO).

IV. CONCLUSION

In this letter, we derived a simple formula to predict the OPD variance of a two-wavelength AO system. Starting from Hogge and Butts' OPD variance integral and applying Mellin transform techniques, we obtained both an exact, slowly converging series solution and an accurate, asymptotic approximation for the aforementioned variance. The latter implied that the two-wavelength wavefront error was weakly dependent on the aperture diameter and furthermore, was wavelength shift-invariant, i.e., only depended on the difference between the beacon and transmitter wavelengths and not on the wavelengths themselves. To our knowledge, these two findings have not been reported before. Lastly, we validated our asymptotic approximation by comparing it to both the series solution and direct numerical evaluation of Hogge and Butts' integral. The results were found to be in excellent agreement. The work presented in this letter will be useful in the design of optical systems that use two-wavelength AO.

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