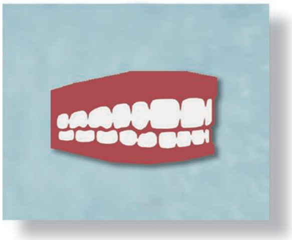

Microwave Bytes

Canned Electromagnetics

■ Steve C. Cripps

In scanning the subjects I have traversed in these columns, some patterns, or groupings, emerge. The "cultural conflicts" between digital and/or low-frequency analog electronics and microwaves are perhaps the most notable and frequent. But another would be an apparent fascination, some might say fixation, with "unconventional" electromagnetic (EM) structures. This admitted personal interest has usually come about for pragmatic reasons, although perhaps the attraction of the "unknown" has played a part as well. In more recent times, the adventurous aspect has rather had to succumb to the onward march of computer horsepower and EM simulation. Indeed, in defining what I mean by an unconventional structure I would probably suggest any structure that cannot be analyzed using the "laws" of electromagnetism. I suppose by that I mean Mr. Maxwell's "laws," but I happen to belong to the minority faction who feel his "ownership" of the subject ignores contributions of numerous others, most notably the reclusive English scientist Oliver Heaviside. But thereby hangs another story; before the advent of the digital computer, many useful structures could only be "solved" by experimental measurement, often using scaled model structures.

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One such article ("Squaring Up" [1]), appeared (gosh) as long ago as 2007 and addressed a subject which had continued to trouble me ever since I designed my first microstrip matching circuit; the standard Smith chart approach to designing transmission line matching networks assumes that the structure is essentially one-dimensional and, in particular, that its width dimension is very small compared to its length. Given that some of the elements I was coming up with did not obviously comply with this restriction, I always wondered how increasing width would affect the result, other than a simple reduction in nominal characteristic impedance. In particular, at higher frequencies, and using the then statutory 0.025-in alumina substrate, I was being forced into using microstrip structures that were almost squares. Clearly, in practice, these structures had "properties" that differed substantially from any attempt to pretend they were conventional transmission lines but could presumably still be useful were we to be able to predict their properties. This was, of course, because they did not lend themselves to the nice analytical solutions to the field equations encountered in one's academic studies, not to mention the admirable EM textbooks [I see my old copy of Ramo, Whinnery, and Van Duzer [2]

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sitting on my desk at this very moment ; -).] Readily available EM simulators, or, more specifically, readily available computing horsepower to run them, were half a lifetime away and even today are still mainly retrospective; they tell you what your circuit does, but there is no a priori design methodology to enable the use of such structures to one's advantage.

Another item under this heading, which certainly does still make an occasional appearance in the literature and on the conference circuit (remember that?!), is the tapered transmission line, which I duly addressed in a 2013 missive ("Cutting Tapers" [3]). This is an interesting case, inasmuch as there actually is an analytical solution for tapers having certain profiles (see RWV [2, p. 54]). But my conclusion on this one remains that their usefulness only really becomes apparent when they become inconveniently long, maybe at least a half wavelength and preferably more than that.

My latest "unusual" structure, about which I now focus, is rather different in that it is three-dimensional. As such, it takes me into the relatively well documented, and largely mathematically tractable, zone of microwave cavities. The structure in question is essentially a cylindrical metal box, where the spacing between the circular end plates is much smaller than the diameter of the cylinder; one thinks perhaps of a movie film canister, or, more to my current scale, the flat metal boxes in which one used to purchase a roll of adhesive Sellotape, shown in Figure 1 with my equally vintage copy of Ramo, Whinnery, and Van Duzer. Such boxes, circa the 1960s, have become collectors' items, apparently selling for US$20 on eBay; my own relic is currently used to store small screws. I will disclose the reasons for my interest in this structure later, and although in some cases they could be instantly characterized as "eclectic," they do not, however, include measuring the microwave properties of Sellotape ; -).

So the basic question is, if I connect a vector network analyzer (VNA) across opposite sides at the center of the box, as shown schematically in Figure 2, what frequency/impedance response do I see? Being what I would loosely term a "Pragmatic Microwave Engineer" (or "PME"), I would speculate that it will resemble some sort of short-circuited transmission line; the various fields, voltages, and currents will presumably take on an axially symmetric formation, and perhaps the whole structure can be represented as a "wheel" of tapered short-circuited shunt stubs connected in parallel at the center. As such, I am essentially assuming the field solution as "transverse electromagnetic," which keeps me in relatively familiar territory.

Well, it so happens I have some colleagues who know a great deal more about microwave cavities than I do, as they use them to do some admirable work on characterizing various materials, including biological specimens [4]. I hope they will not be offended if I characterize them as "Smart Microwave Physicists" ("SMPs"). So, catching a coffee break with one such colleague, and showing him my problem, he immediately asserted, "Aha, yes, the TM_{0N0} cavity resonance modes." So my simplistic model for the structure appeared to disintegrate in the face of such learned advice, and I embarked on what turned out to be a fairly lengthy odyssey through the literature, some quite old, and even going back to the dusty tomes of the MIT Microwave Radiation Lab series (notably, the admirable volume 8 [5]). The analysis can, indeed, be found in various places, but my journey was interesting. In describing this odyssey, I do not want to drift into reproducing what is essentially mathematical bookwork, and I will not dwell on every detail but will share some of the places where my mathematical engine stalled for a while.

Figure 3 shows the basic formulation: the radially symmetric structure has an equipotential "ring" at each radius *r*, yielding a distributed series impedance of *Z* and shunt admittance *Y* per unit length.

Some basic circuit analysis delivers the following differential equations describing the voltage, *V*(*r*), and current, *I*(*r*), as functions of the radius variable *r*, where the *Z* and *Y* values are now also themselves functions of *r*:

$$\frac{d^2V(z)}{dr^2} - \left(\frac{Z'}{Z}\right)\frac{dV(z)}{dr} - (ZY)V = 0,$$

$$\frac{d^2I(z)}{dr^2} - \left(\frac{Y'}{Y}\right)\frac{dI(z)}{dr} - (ZY)I = 0.$$

So, of course it would be nice if the coefficients in brackets were constants,

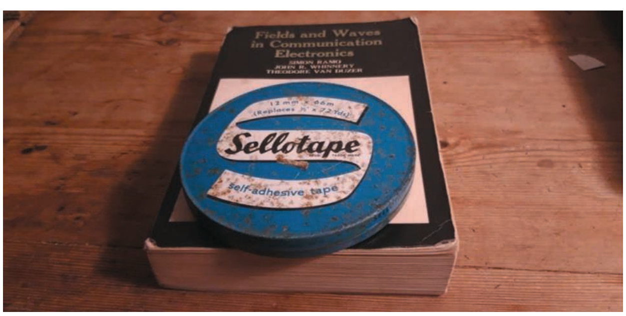

Figure 1. *A Sellotape can and my 1971 copy of Ramo, Whinnery, and Van Duzer.*

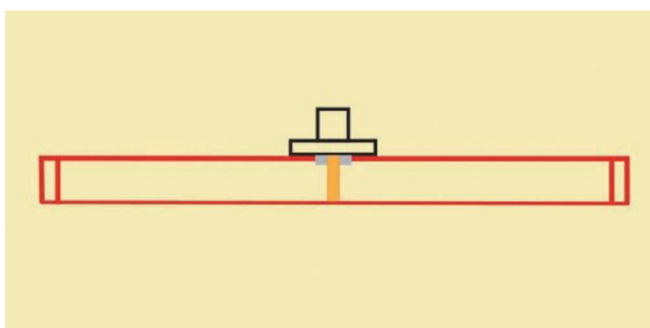

Figure 2. *Radial cavity measurement.*

so we could recognize both as wave equations and solvable using sinusoidal, or complex exponential, functions. This happens to be the case if we stipulate that the Z and Y functions of r are exponential, as for the tapered transmission line (see [2, pp. 53–56]).

In this case, the shunt capacitive element increases linearly with r, and the series inductance decreases as $1/r$, so that

$$Z = \frac{j\omega L_o R}{r},$$
$$Y = \frac{j\omega C_o r}{R},$$

where R is the outer radius; ignoring, for now, the problem when $r = 0$, the equations for voltage and current become

$$\frac{d^2V(r)}{dr^2} + \left(\frac{1}{r}\right)\frac{dV(r)}{dr} + (\omega^2 L_0 C_0)\,V(r) = 0, \quad (1)$$

$$\frac{d^2I(r)}{dr^2} - \left(\frac{1}{r}\right)\frac{dI(r)}{dr} + (\omega^2 L_0 C_0)\,I(r) = 0. \quad (2)$$

Equation (1) is the most basic form of the Bessel equation and, as such, has solutions involving zero-order Bessel functions. The negative sign in (2) actually poses a considerably greater challenge for a direct analytical solution, but it turns out that we can bypass this problem by solving (1) and engaging a slick bit of EM theory. I have to admit I have always had a problem with Bessel functions. Part of the trouble is that no sooner has one got a head around the "basic" Bessel function ("J_0") with its almost-familiar damped cosinusoidal appearance, we are immediately blasted with Bessel functions of the second kind, modified Bessel functions, Hankel functions, and a few more variants (remember "Ber" and "Bei"?!), which, I must admit, were all a bit too much for me to absorb at the time. Part of my journey through the literature on this subject certainly involved my reacquaintance with most of the above but also getting a bit stalled on the universal use, in electrical engineering, of complex exponentials to represent cosinusoidal time variations. This has been a "bee in my bonnet," to use a characteristically arcane English proverb, for a very long time and has, inevitably, previously cropped up in the "Microwave Bytes" column ("Youthful Complexity" [6]).

Just to revisit that subject: the "Vs" and "Is" in the previous equations are not actually direct measurable instantaneous values of the voltage and current at the specified values of time and distance; they have complex values that contain this information but require some postprocessing to get to the literal voltage and current values. I suppose, for the most part, we are all familiar enough with using such complex values for anything we encounter that has a cosinusoidal form, but now and again we do have to exercise caution to keep our precious cipher intact.

For example, and just backtracking to the more familiar formulation for the case of a uniform transmission line, we are usually presented, up-front with little explanation, the following solution for the voltage along the line:

$$v(z,t) = (Ae^{-j\gamma z} + Be^{+j\gamma z})\,e^{j\omega t}, \quad (3)$$

thereby, *de facto*, adopting the same cipher for the distance, or z-variation, as for time t. As such, we are told, it enables the action of the line to be visualized as a forward and reverse traveling wave; indeed, as if to really ram the point down our throats, this is usually written as

$$v(z,t) = (V_+ e^{-j\gamma z} + V_- e^{+j\gamma z})e^{j\omega t},$$

where the constants V_+ and V_- quantify the forward and reflected waves. Furthermore, we are expected to accept without question the corresponding expression for the current,

$$i(z,t) = (1/z_0)(V_+ e^{-j\gamma z} - V_- e^{+j\gamma z})e^{j\omega t},$$

something which I always thought looked intuitively reasonable but was never presented with the precise reasoning behind it. In any event, although this formulation may give some physical insight into how transmission lines behave, any actual measurement of the voltage, for example, using a slotted line, shows that the voltage amplitude variation with distance z displays a "standing wave" pattern which can, equally intuitively and correctly, be written as

$$v(z,t) = (A\cos\gamma z + B\sin\gamma z)e^{j\omega t}. \quad (4)$$

I say "correctly" with more confidence now than when this issue first caught my attention on my aforementioned odyssey; was I absolutely sure this was correct? I could not actually recall seeing it in any textbook, which would inevitably jump straight into the complex exponential formulation for the "z" part of the solution. It is also considerably less clear, to me, how the current variation $i(z,t)$ can be expressed in this format. One's doubts can always be resolved by expanding (3) and (4) and taking the real part; the key point being that, in general, the constants A and B have complex values, which makes this task a bit more cumbersome than it first appears.

So the solution to the Bessel equation for the radial voltage on my Sellotape can, (1), is stated by at least two of my revered references [x, y], as

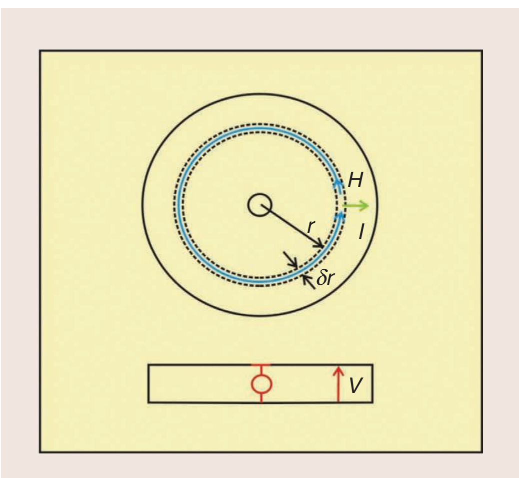

Figure 3. *Analysis of radial cavity excited at midpoint.*

$$v(z,t) = (AJ_0(\gamma r) + BY_0(\gamma r))e^{j\omega t}, \quad (5)$$

where J_0 is the zero-order Bessel function of the first kind, Y_0 the zero-order Bessel function of the second kind ("N_0" in some books), and γ the propagation constant ω/c. But hang on, what's this "Second Kind" thing, one may well ask, if not out loud (?!); why do we need this extra complication? An intuitive answer can be surmised by comparing (4) and (5). Bessel functions are not entirely independent, separate entities from the familiar trig functions; they have an insidious relationship with them. In fact, as the J_0 and Y_0 arguments get larger, the amplitudes stabilize, and the zero-crossings start to line up with the corresponding cosine and sine functions (respectively) having the same arguments. Trying, still, to avoid letting this article slide into a math class, I will avoid the temptation to plot the curves for the various functions, which can be found in many books. But it does occur to me that while we recognize other functions as having some amount of kinship with basic trig functions, such as the hyperbolic functions "sinh" and "cosh," we refrain from recognizing such in the Bessel family; "cosbe" and "sinbe" maybe don't quite roll off the tongue, although perhaps it could also be said that the kinship unfolds somewhat as higher Bessel function orders are considered.

Returning briefly to my concerns about the validity of (4) and/or (5), and given the universal fixation of textbooks on using the complex exponential form in (3), I should note that some books, as if to allay such concerns, actually transmogrify the basic form of (5) into a "quasi-exponential" form using so-called Hankel functions. This involves defining Hankel, or "modified Bessel," functions as

$$H^1{}_0(x) = J_0(x) + jY_0(x),$$
$$H^2{}_0(x) = J_0(x) - jY_0(x)$$

and expressing these in exponential form, so (5) can be recast in the form

$$v(z,t) = G_0(\gamma z)\left(Ae^{j\theta(\gamma z)} + Be^{-j\theta(\gamma z)}\right),$$

where $G_0(\gamma z)$ and $\theta(\gamma z)$ are, in turn, functions of $J_0(\gamma z)$ and $Y_0(\gamma z)$. Got that? Don't worry, I will stick with my preferred "standing wave" formulation in (5) (!).

So, returning to my problem, (5) gives me a solid starting point to solve the Sellotape can structure. I can determine a relationship between the constants A and B by defining the voltage to be zero at the outer radius, but I then run into a problem solving (2) for the current, which, of course, is needed in order to obtain the desired impedance variation with radius and, in particular, at the central measurement point. This is another key point on my odyssey where SMP comes to my rescue. If we go back to the field equations, there exist some fundamental relationships between the E and H fields that enable us to bypass the sticky problem displayed in (2). In particular, and with some apologies, I take it we all remember

$$\nabla X \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

which fortunately [and admittedly showing the virtue of the complex exponential cipher ;-)] simplifies to the more palatable

$$\frac{\partial E}{\partial r} = -\mu j\omega H,$$

and, better still, at radius r,

$$E = \frac{V}{h}, \quad I = 2\pi r H,$$

whereby we complete our circumnavigation of (2), obtaining

$$i(r,t) = j\frac{2\pi r}{\eta h}(AJ_1(\gamma r) + BY_1(\gamma r)) \quad (6)$$

utilizing the Bessel function property that $J_0'(x) = -J_1(x)$ etc. (see [2, p. 213]).

This pretty much gets me to my destination; by dividing (4) and (5) I get the impedance at any chosen radius r, and the constants can be mitigated by imposing the short circuit condition at the outer radius.

So I would now appear to be ready to show what you have all been waiting for: a comparison between measurement and theory. In fact, to perform the measurement, I decided not to vandalize my priceless antique Sellotape can but manufacture a more RF-friendly specimen, shown in Figure 4.

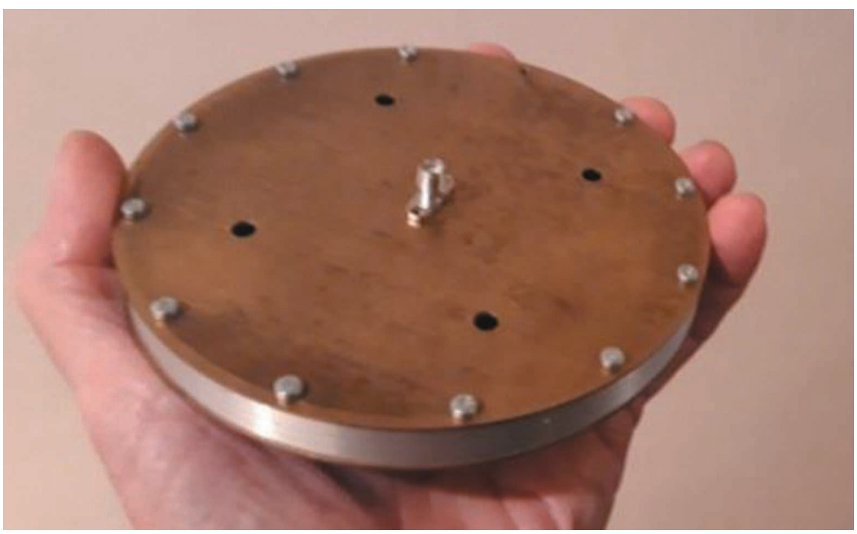

Figure 4. *The cavity used for measurements (the pre-Brasso, holes are machining anchor points).*

This model is a 60-mm radius and 10 mm deep, and, as such, the SMA launcher in the middle introduces a minor problem setting the phase reference on the VNA. However, this turns out to be quite a useful "tweak" to make the measurements tie up with the equations (have we all been there before?), as shown in Figure 5. In fact, the phase offset corresponds quite closely to the length of the SMA "stalk" that connects to the lower plate. So theory and measurement are in impeccable agreement, and the standout feature is that the "quarter-wave" resonance occurs at a significantly higher

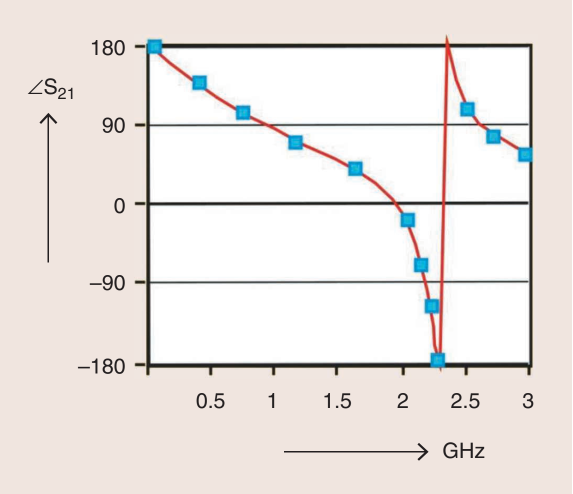

Figure 5. *The theoretical (red) and measured (blue points) S_{21} phase of the radial cavity shown in Figure 4.*

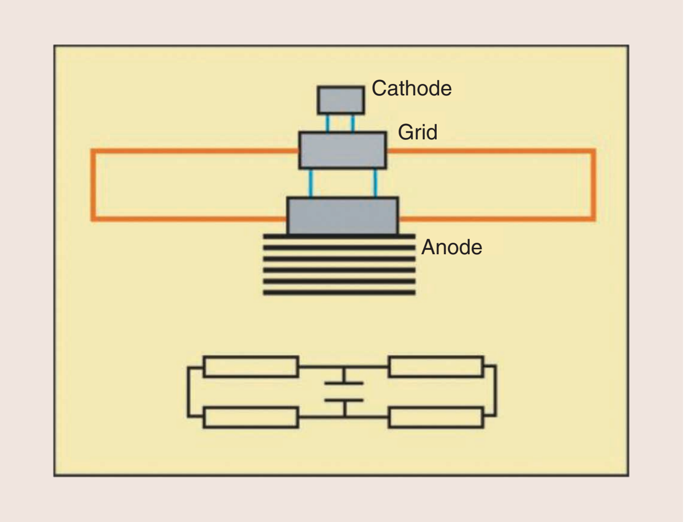

Figure 6. *The use of a radial cavity in a microwave triode circuit.*

frequency (very close to 2 GHz) than the physical quarter-wavelength frequency of 1.25 GHz. Despite the Bessel functions, the impedance (reactance) trajectory looks distinctly similar to the tangent function that would correspond to a uniform short-circuited stub having a fairly low Z_0.

There is one gremlin in the story, which concerns how both the equations, and, in principle, the measurement, become "singular" at the very point where I perform the measurement $r = 0$. In defining the formulation, as, for example, in Figure 3, I have carefully stipulated that the structure has a hole at this point. In fact, in practice, I do need to drill a hole at the center in order to allow the inner connection of the SMA launcher to pass through the upper lid, as shown in Figure 2, so, in this sense, I can sidestep the singularity issue without a guilty conscience, although it seems that the analysis holds up no matter how small the radius value gets, so long as it is not set to zero. My VNA, for sure, does not seem to have a problem, and the measurement seems fairly insensitive to the exact dimensions and placement of the launcher.

So, what now? Why my interest in this structure? At this point in time, I can certainly cite three application areas of interest: one quite historical, another an ongoing research area (and, as such, limited in what I can disclose), and a third that has actually emerged during this odyssey.

The historical application concerns microwave vacuum tubes, specifically, the "disk-sealed triodes" that were used for power amplification at frequencies up to 4 GHz and, as such, survived into the first generation of mobile phone base stations. I do intend to discuss this very interesting piece of microwave technological history in a forthcoming column (so be warned!), but Figure 6 gives a quick indication. These tubes had symmetrical construction with axial metal rings forming the cathode, grid, and anode connections. They were not physically small structures, and in order to connect a microwave circuit around them, a correspondingly "extensive" structure was required. Radial cavities fitted the bill very well, and Figure 6 shows how one such could connect across the grid and anode rings to form a shunt resonance with the internal capacitance (the common grid connection was universal). Coupling to the outside world, especially at the 50-Ω level, was an additional challenge, but more on that at a later date.

Radial cavities are also used in microwave heating, most notably in plasma physics but also a range of industrial and medical application areas. However, having finally managed to derive the impedance-versus-radius equations, I became interested in what happens when the circumferential short is replaced by a load. The load, of course, has to be evenly distributed around the circumference, and I immediately think about a number of ports evenly distributed around the periphery of the circular box. And, hey presto, I appear to have reinvented the radial power combiner(!). Well, maybe so, but with some significant deviations in terms of a possible increase in the number of combined ports, the bandwidth, and the harmonic performance, which should replicate the broad bandwidth of the tapered transmission line. More work in progress.

In conclusion I should identify my SMP, who is my colleague Adrian Porch, who helped to restart my engine at numerous points on this EM odyssey.

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