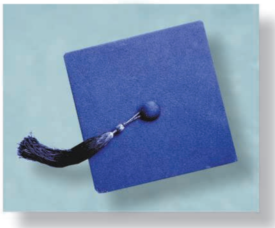

Educator's Corner

A Radio Engineer's Voyage to Double-Century-Old Plane Geometry

■ Takashi Ohira

Sail to the mysterious triangles and arcs of supreme elegance.

Academia may approach radio engineering through two possible educational ways. One is based on the analysis of formulas stemming from electromagnetics and circuit theory. Students are required to have mastered calculus and linear algebra beforehand. Although this way is mathematically rigorous, students sometimes become exhausted by the nabla-manifold vector and matrix equations. The other possible approach is based on plane geometry. The instructor draws an impedance locus on a blackboard to illustrate the behavior of circuit components, such as *LCR* and transmission lines. According to our academic experience, this friendly way is effective to stimulate first-year students to intuitively start from the introduction. The special geometry that can be exploited in radio engineering originated in the early 19th century, i.e., long before the Smith chart was invented. One may then ask, what in the world took place two centuries ago? To answer this question, we have in this article a time-traveling vessel. The final boarding gong is now beating, and we will soon hoist the sails for a mysterious adventure.

Takashi Ohira (ohira@tut.jp) is with Toyohashi University of Technology, Aichi, Japan. He is a Life Fellow of IEEE.

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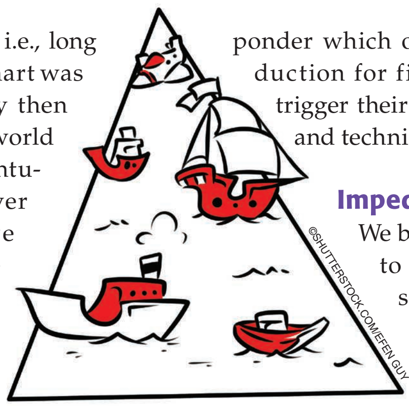

Maiden Voyage

Imagine that we make our maiden voyage across the ocean. Every mate on duty must be able to use an indispensable item: the navigation chart. This idea also applies to voyages in radio engineering. Through analogy to Mercator, Mollweide, and Lambert's conic projections in cartography, there are several different schemes we can use to draw a chart of electric impedance on a complex plane. Each scheme has pros and cons in accordance with the purpose of the chart, as comprehensively overviewed by Harold Wheeler in [1]. University lecturers may then ponder which one is the best introduction for first-year students to trigger their interest in RF theory and techniques.

Impedance Plane

We believe that the best way to introduce the representation of impedance is to use a Cartesian coordinate system, as presented in Figure 1. Complex impedance is decomposed into its real and imaginary parts as $Z = R + jX$. This is simply called the *impedance plane* or,

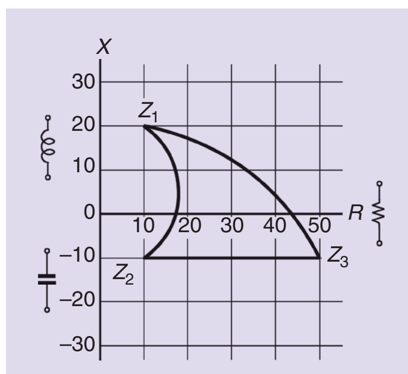

Figure 1. *The impedance half-plane with a mysterious triangle.*

to be more specific, the impedance *half-plane,* when the load is assumed to be passive, i.e., $R > 0$. The concept of such a half-plane was originally proposed by Henri Poincaré as a visible model of hyperbolic geometry [2], [3].

This concept is exploited for radio engineering by assuming that the vertical axis represents a reactance X, which physically implies inductors for positive X and capacitors for negative X. This R-X system utilizes a linear and orthogonal grid, which provides straight-line routes to represent the behavior of LCR elements connected in series to the load. Therefore, this simple approach is highly recommended for first-semester young sailors or possibly prospective captains, rather than trying to demonstrate other sophisticated circular charts abruptly. That is to say, simple is beautiful.

Figure 2. *A variety of VSWR indicators in our vessel's radio shack.*

Curious Enigma

One stimulating supplement to this plane is a mysterious triangle drawn on the R-X grid, as shown in Figure 1. This is, apparently, distorted in comparison to the usual triangle. However, in a sense, the three routes shown all represent the shortest links between two vertices. This may sound like a tricky riddle. In spite of this, it is mathematically true when a special metric or scale intended for the impedance half-plane is employed. To address this curious enigma in an analytical way, let us set out on our voyage by first examining the meanings of "distance" and "length" on a chart. These concepts will be clarified in the following sections.

Geometric Distance

Let us consider two points, Z_1 and Z_2, located anywhere on the half-plane. For example, look again at the triangle's west side in Figure 1. The question here is, how should we define the distance, hereafter denoted as D, between Z_1 and Z_2? In any geometry, the distance must be a single, scalar, and real function of the relative position of two points. More strictly, it must exhibit four properties: identity, nonnegativity, commutativity, and triangle inequality [2], [3].

Even taking all these requirements into account, the definition of *distance* is not unique; rather, it has alternative formulations. A quick idea could be $D = |Z_1 - Z_2|$, where the twin vertical bars mean complex modulus. Although this definition is mathematically simple, it does not work properly in radio engineering, as the impedance has a dimension of ohms. Another idea for the definition could be $D = |Z_1 - Z_2|/Z_o$, where the denominator, Z_o, denotes some reference impedance, such as 50 Ω. In this definition, the physical dimension is nullified by the normalization. However, universe-strong versatility is expected from the general definition of *distance*. What we are looking for is a dimension-free, purely relative quantity that is invariant to any reference impedance.

Among the possible candidates for a valid definition of *distance*, we single out the natural logarithm of voltage standing-wave ratio (VSWR) ρ, which is simply formulated as

$$D = \ln \rho. \tag{1}$$

This is because the logarithmic VSWR strikes a deep chord with radio engineers, as richly suggested by Madhu Gupta in [3]. The aforementioned D is called the *Poincaré distance* in hyperbolic geometry, and (1) indicates our selection as the starting postulate of this exciting voyage.

We employ log VSWR as our starting postulate.

Reflectance Magnitude

To estimate the distance D between Z_1 and Z_2 using (1), we need to recall how ρ is derived from the two impedances. This can be formulated by the wave-reflectance approach as follows.

Radio engineers, from amateur to professional, often use ρ to tune a load, such as antennas or a wireless coupler for RF power transfer. Various measurement instruments are illustrated in Figure 2. They all indicate ρ to notify the radio operator of how much power is being reflected from the load. A basic and well-known relation is given by

$$\rho = \frac{1+\gamma}{1-\gamma}, \tag{2}$$

where γ denotes the reflectance magnitude. This equation tells us that the VSWR starts from unity at zero reflection, becomes three at half reflection, and goes to infinity at full reflection. Typical numerical examples of this behavior are given in Table 1.

Radio engineers, from amateur to professional, often use ρ to tune a load such as antennas or a wireless coupler for RF power transfer.

In addition to the numeric table, a more instructive route to the γ-to-ρ conversion is shown in Figure 3. Let us place a perpendicular pole at point (1, 0) up to height γ. We draw a straight line (elastic cord) from the pivot point (–1, 1) that passes through the pole's top. This cord is extended down to intercept the horizon at coordinate ρ, which gives us the exact VSWR. An inverse conversion is also available. Slide ρ from left to right along the horizon while keeping the pivot point fixed. The height γ at which the pole is truncated by the cord gives the reflectance magnitude.

This mechanical model enables us to estimate the VSWR even without any formula or electronic calculator. In fact, such a graphical approach cultivates engineering intuition in students' minds much more fully than just starting with the user's manual for fully automated simulation software.

Our next question is that of how γ comes from the two impedances. Stating the conclusion first, the aforementioned reflectance can be formulated as

$$\gamma = \left| \frac{Z_2 - Z_1}{Z_2 + Z_1^*} \right|, \tag{3}$$

where the superscript asterisk * denotes the complex conjugate.

Some undergraduates may be puzzled as to why the asterisk stays solely in the denominator of (3) and not in the numerator. The answer is easy: this formula can be derived by just imposing the basic Ohm's law on the RF voltages and currents in a simple lumped-constant system. This can be done even without assuming transmission lines or wave propagation. A full description of this elegant theory is provided in [4].

We now have a powerful tool to measure the Poincaré distance, that is, the three-step sequence (3)–(2)–(1) displayed in Figure 4. This sequence is highly versatile because (1), (2), and (3) are valid not only for a Cartesian chart; they are also valid for any kind of chart, regardless of its shape, scale, or coordinate gradation. This wide validity can be directly derived from the formulas themselves. Moreover, reference impedance Z_o does not affect γ, ρ, or D at all. Obviously, these quantities are functions only of Z_1 and Z_2.

Triangle Mystery

Charles Berlitz warned travelers never to sail into the Bermuda Triangle, but we now gather momentum toward the triangle shown in Figure 1 with full curiosity in mind. Upon reaching vertex Z_1, our first assignment is to measure the distance between Z_1 and Z_1 itself. A zero distance may sound trivial, but it actually makes sense, as it resembles the 0-Ω calibration of circuit testers or impedance analyzers. By substituting $Z_1 = Z_2$ into (1), (2), and (3) in Figure 4, we can sequentially confirm that $\gamma = 0$, and thus $\rho = 1$, resulting in $D = 0$ as expected. The verification is successful. Our instruments are go!

Next we steer our boat to vertex Z_2, where our second assignment is to measure the distance between Z_1 and Z_2. Reading out their positions as $Z_1 = 10 + j20$ and $Z_2 = 10 - j10$ from the grid and inserting these complexes into (3), we obtain

$$\gamma_{12} = \left| \frac{(10 + j20) - (10 - j10)}{(10 + j20) + (10 + j10)} \right| = \frac{3}{\sqrt{13}}, \tag{4}$$

where the combined subscript $_{12}$ denotes the corresponding points Z_1 and Z_2. Substituting this result into (2) and (1), we obtain

$$D_{12} = \ln \frac{\sqrt{13} + 3}{\sqrt{13} - 3} \approx 2.4. \tag{5}$$

This is the distance between Z_1 and Z_2 observed in the Poincaré metric.

In (5), there is no unit appended to the figure. This is quite natural for dimension-free quantities. However, if units are required, the answer is neper (Np) for the quantity stemming from the natural logarithm. Another

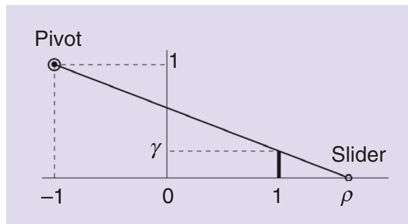

Figure 3. *An elastic-cord model to explain the γ-to-ρ conversion.*

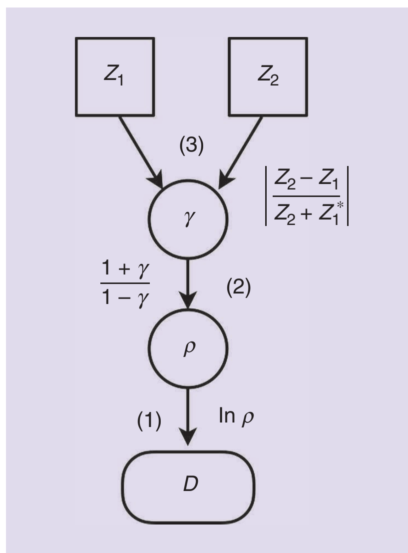

Figure 4. *A flowchart to measure the Poincaré distance between Z_1 and Z_2.*

TABLE 1. The reflectance-to-VSWR conversion lookup table.

γ	0	0.2	0.5	0.6	0.75	0.8	0.9	1
ρ	1	1.5	3	4	7	9	19	∞

option we have is to convert Np into decibel (dB), which is defined as 20 times the common logarithm and is thus familiar to radio engineers. It is quite impressive that 20 $\log_{10}$ VSWR lives in perfect harmony with M.C. Escher's art work, as pointed out by Gupta in [3]. For quick conversion, 1 Np $\approx$ 8.686 dB.

We finally sail to vertex Z_3 of the triangle, carrying a twofold mission: to measure both D_{23} and D_{31} at once. Repeating the same calculation process as in (4) and (5), we obtain

$$\gamma_{23} = \left| \frac{(50 - j10) - (10 - j10)}{(50 - j10) + (10 + j10)} \right| = \frac{2}{3}, \quad (6)$$

$$D_{23} = \ln \frac{3+2}{3-2} \approx 1.6, \quad (7)$$

$$\gamma_{31} = \left| \frac{(50 - j10) - (10 + j20)}{(50 - j10) + (10 - j20)} \right| = \frac{\sqrt{5}}{3}, \quad (8)$$

$$D_{31} = \ln \frac{3+\sqrt{5}}{3-\sqrt{5}} \approx 1.9. \quad (9)$$

Comparing the three distances, we can conclude that $D_{23} < D_{31} < D_{12}$. This inequality apparently disagrees with how the triangle looks in Figure 1 under the lamp of traditional geometry. That is why this plane seems so mysterious.

Which route is shorter, the curve or the straight line?

Piecewise Segmentation

When we make a long-haul voyage, the shortest or geodesic route is not always available. Sometimes, a tidal current or a jet stream dominates our course. This is also true in radio engineering. Imagine that we design an impedance transformer from Z_1 to Z_2. A wide variety of lumped and distributed-constant elements can be used. However, these elements seldom follow the shortest path to reach the target impedance on the plane [5]–[7]. This is why the length of curved routes must be considered apart from the direct distance described in the previous section.

According to differential geometry, a smooth curve can be measured by a line integral of infinitesimal segments along the curve. We begin with a small segment truncated at Z_1 and Z_2, as shown in Figure 5. The two points are denoted as

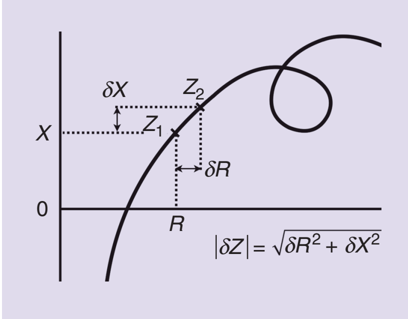

Figure 5. *A smooth curve and its segment on the impedance half-plane.*

$$Z_1 = R + jX, \quad (10)$$

$$Z_2 = R + jX + \delta Z, \quad (11)$$

where δZ implies a slight difference between the two points on the R-X plane.

Again, we follow the three-step sequence (3)–(2)–(1) depicted in Figure 4. Substituting (10) and (11) into (3), we obtain

$$\gamma = \left| \frac{Z_2 - Z_1}{Z_2 + Z_1^*} \right| = \frac{|\delta Z|}{|2R + \delta Z|}. \quad (12)$$

Then, substituting (12) into (2) and expanding the result into its first-order Maclaurin series with respect to $|\delta Z|$, we obtain

$$\rho = \frac{1+\gamma}{1-\gamma} = \frac{|2R + \delta Z| + |\delta Z|}{|2R + \delta Z| - |\delta Z|} = 1 + \frac{|\delta Z|}{R}, \quad (13)$$

in which the higher-order terms can be omitted because they are negligible. Finally, substituting (13) into (1) and applying the same expansion mentioned previously, we obtain

$$D = \ln \rho = \ln\left(1 + \frac{|\delta Z|}{R}\right) = \frac{|\delta Z|}{R} \quad (14)$$

for the distance between Z_1 and Z_2. See Figure 6 for graphical assistance to grasp what (14) means.

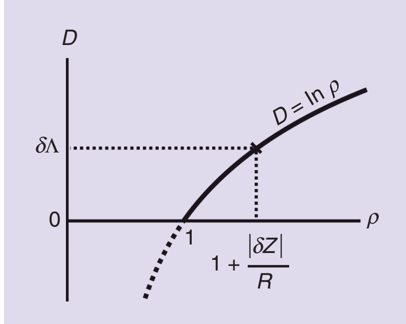

Figure 6. *A small segment length stems from the natural logarithm of VSWR. We find that $\delta\Lambda = |\delta Z|/R$ because the curve has a 45° slope in the vicinity of unity.*

The segment length is a small increment of VSWR from unity.

Assuming that the curve is smooth and the segment sufficiently small, the aforementioned distance can be equivalently regarded as the segment length. The impedance difference is decomposed into its real and imaginary parts as

$$\delta Z = \delta R + j\delta X. \quad (15)$$

Looking at Figures 5 and 6 together, we can rewrite (14) as

$$\delta\Lambda = \frac{|\delta Z|}{R} = \frac{1}{R}\sqrt{\delta R^2 + \delta X^2}. \quad (16)$$

This is the Poincaré distance between two adjacent impedances. From this result, we conclude that the small segment length is given by the absolute difference in impedance normalized to its original resistance. Note that the original reactance X does not contribute to the length at all. Finally, above $\delta\Lambda$ is accumulated as

$$\Lambda = \int_C d\Lambda \quad (17)$$

to measure the entire curve length Λ, where the prefix δ is replaced by the infinitesimal operator d. Sequence (15)–(16)–(17) is summarized as a flowchart in Figure 7.

Curve Length

Now let us look back at the mysterious triangle presented in Figure 1. We sail from vertex Z_1 to Z_2 once again. This time, however, the route is segmented into small pieces that are integrated in length. For simplicity, we assume that this route is a circular arc lying along the circle that passes through Z_1 and Z_2 and is centered on the x-axis. Reading out its location from the grid, the circle is formulated as

$$R^2 + (X-5)^2 = 325. \tag{18}$$

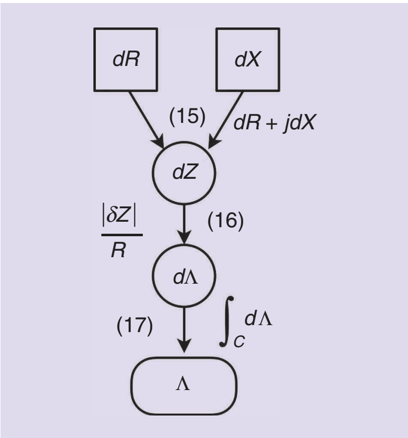

Figure 7. *A flowchart to measure the Poincaré length of a smooth curve.*

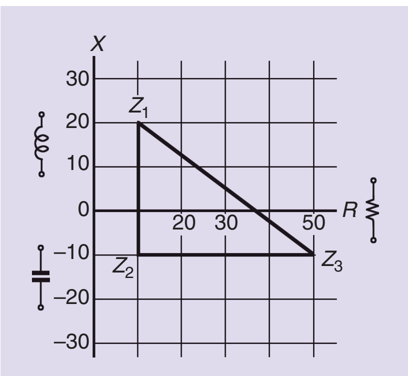

Figure 8. *A pseudotriangle on the impedance half-plane.*

TABLE 2. The triangle dimensions in the Poincaré metric.

	Z_1–Z_2	Z_2–Z_3	Z_3–Z_1
Distance D	2.4	1.6	1.9
Length Λ in Figure 1	2.4	1.6	1.9
Length Λ in Figure 8	3	1.6	2

By differentiation, the curve's slope is calculated as

$$\frac{dR}{dX} = -\frac{X-5}{R}. \tag{19}$$

This slope enables us to eliminate δR from (16), resulting in

$$\begin{aligned} d\Lambda &= \frac{1}{R}\sqrt{dR^2 + dX^2} = \frac{5\sqrt{13}}{R^2}dX \\ &= \frac{1}{2}\left| \frac{1}{X-5+5\sqrt{13}} \right. \\ &\qquad \left. - \frac{1}{X-5-5\sqrt{13}} \right| dX, \end{aligned} \tag{20}$$

where δ reduces to the infinitesimal operator d. Integrating this $d\Lambda$ from Z_2 to Z_1, the arc is measured as

$$\begin{aligned} \Lambda_{12} &= \int_{Z_2}^{Z_1} d\Lambda \\ &= \left[\frac{1}{2}\ln\left| \frac{X-5+5\sqrt{13}}{X-5-5\sqrt{13}} \right| \right]_{-10}^{20} \\ &= \ln\frac{\sqrt{13}+3}{\sqrt{13}-3} \approx 2.4. \end{aligned} \tag{21}$$

Although the calculation has been a little bit tough this time, the resultant length is identical to the distance observed in (5). This identity means that the assumed arc is the shortest route that links the two vertices. This is because, in general, the length can never be less than the distance. In a similar way, we can also confirm that $D_{23} = \Lambda_{23}$ and $D_{31} = \Lambda_{31}$. Thus, we conclude that this shape should be called a *triangle* even though it looks distorted.

Pseudotriangle

We now move on to another mysterious triangle, as seen in Figure 8. We find that it resembles the triangle in Figure 1; the only difference is that its three sides are all straight lines rather than curves. Looking at Figures 1 and 8 in parallel, let us pose the following question: Which side is longer, the curve or the straight line?

The straight line from Z_1 to Z_2 is read out as $R = 10$ and $-10 < X < 20$. Therefore, $dR = 0$, and, thus, (16) simply reduces to

$$d\Lambda = \frac{1}{10}|dX|. \tag{22}$$

Integrating from Z_2 to Z_1, the line measures

$$\begin{aligned} \Lambda_{12} &= \int_{Z_2}^{Z_1} d\Lambda \\ &= \frac{1}{10}\int_{-10}^{20} dX = \frac{1}{10}[X]_{-10}^{20} = 3. \end{aligned} \tag{23}$$

This result finds a roughly 25% increment from (5) or (21). We can therefore answer the quiz: the straight line is longer than the curve. This answer is diametrically opposed to common sense, considering usual geometry.

Let us see the next line shown in Figure 8. The straight line from Z_2 to Z_3 is read out as $10 < R < 50$ and $X = -10$. Therefore, $dX = 0$, and, thus, (16) reduces to

$$d\Lambda = \frac{1}{R}|dR|. \tag{24}$$

Integrating from Z_2 to Z_3, the line measures

$$\begin{aligned} \Lambda_{23} &= \int_{Z_2}^{Z_3} d\Lambda \\ &= \int_{10}^{50} \frac{1}{R}dR = [\ln R]_{10}^{50} = \ln 5 \approx 1.6. \end{aligned} \tag{25}$$

This is identical to the result from (7), which is completely natural because this route is a straight baseline common to both triangles.

The final line shown in Figure 8 runs from Z_1 to Z_3 at a slant, which is formulated as

$$X = -\frac{3}{4}R + \frac{55}{2} \tag{26}$$

for $10 < R < 50$. Therefore, $dX = -3/4\,dR$, and, thus, (16) reduces to

$$d\Lambda = \frac{5}{4R}|dR|. \tag{27}$$

Integrating from Z_1 to Z_3, the line measures

$$\begin{aligned} \Lambda_{13} &= \int_{Z_1}^{Z_3} d\Lambda \\ &= \int_{10}^{50} \frac{5}{4R}dR = \frac{5}{4}[\ln R]_{10}^{50} \\ &= \frac{5}{4}\ln 5 \approx 2.0. \end{aligned} \tag{28}$$

By comparing this result with (9), we can say again that the straight line is longer than the curve. See Table 2 for an overview of the two triangles in dimension.

In summary, the shape in Figure 8 looks like a triangle, but its sides do not always link the corresponding vertices with the minimum length. In this sense, even having three angles, it should be called a *pseudotriangle*.

The route looks straight but is curved in a sense.

Zero and Infinity

Although the adventure on the impedance half-plane is not yet exhausted, we leave its further exploration to brave future challengers. Let us now steer our vessel from the triangular zone toward a circular island with full rudder.

The question here is, on what kind of plane can we meet both zero (short) and infinity (open) (as they cannot simultaneously live in a Cartesian town)? This is because infinity is too far on the aforementioned impedance plane, whereas zero is too far on the admittance plane. To overcome this antipodal conflict, instead of the Cartesian scale, a nonlinear scale for the coordinates is required.

One solution to this puzzle is to employ the reflectance magnitude γ as a radial scale. It is already known that γ ranges from zero to unity, never diverging even for an open, short, or any positive real impedance. The plane, in general, must be a 2D world. Therefore, in addition to the magnitude, the phase θ of the reflection must be taken into account. Thus, we involve γ and θ in the polar-coordinate complex reflectance

$$\Gamma = \gamma e^{j\theta} = \frac{\rho - 1}{\rho + 1} e^{j\theta}, \tag{29}$$

where ρ denotes the VSWR once seen in (2). Explicitly, $\gamma = |\Gamma|$, and $\theta = \angle\Gamma$. The final right-hand side in (29) is called the *VSWR expression of complex reflectance*, which will be exploited in the final section.

It is also known that the reflectance can be calculated from

$$\Gamma = \frac{R + jX - 50}{R + jX + 50} \tag{30}$$

in a 50-Ω system. For example, $\Gamma = -1$ when $R = X = 0$. To see how Γ behaves in response to impedance, we sweep R and X from zero to infinity and plot Γ on a complex plane. The result is presented in Figure 9. We find that zero and infinity can successfully live together in one world. This elegant projection is called a *Poincaré disk* in hyperbolic geometry [2], [3], [8] and is also known as a *Smith chart* in radio engineering [9], [10].

The shape in Figure 8 looks like a triangle, but its sides do not always link the corresponding vertices with the minimum length.

How can zero and infinity live in one world?

Power Ratio

The disk shown in Figure 9 was indeed an elegant discovery and is even now working for us on the visual display of electromagnetic field simulators and vector network analyzers and on the blackboard in microwave engineering classrooms. However, at least in geometry, this disk is not the only solution for zero and infinity to coexist. Looking further forward, let us continue our voyage to seek another circular island.

If we accept the concept of the power standing-wave ratio (PSWR), denoted as ρ^2, a favorable wind will

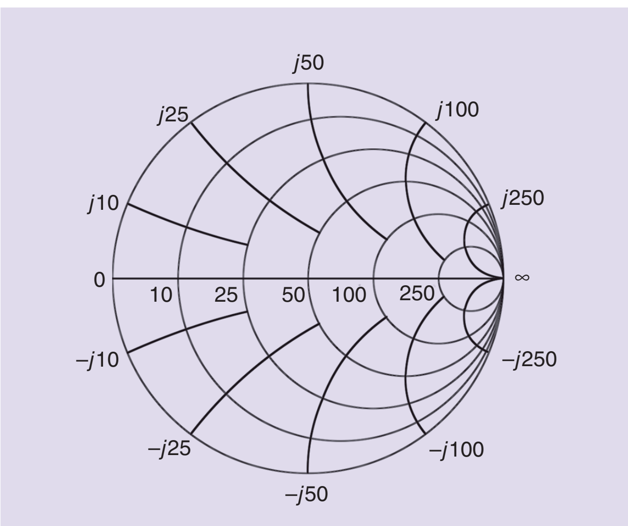

Figure 9. *A Poincaré disk with an impedance grid.*

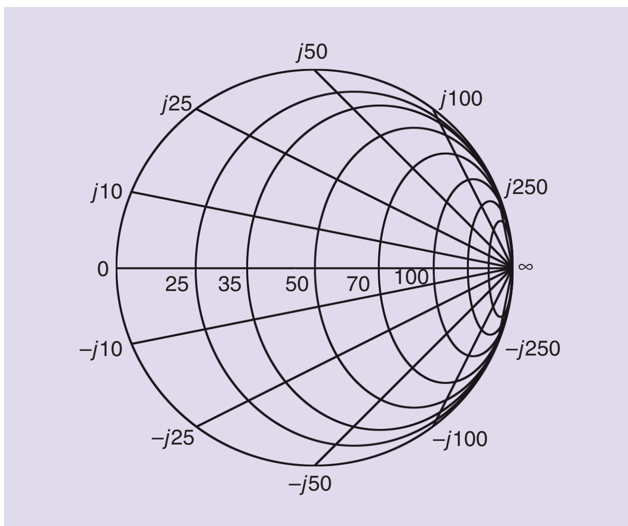

Figure 10. *A Beltrami–Klein disk with an impedance grid.*

blow to hoist our sails. This notation, ρ^2, is based on the common circuit theorem: voltage ratio squared equals power ratio. Now, even though it may sound somewhat abrupt, it is quite convenient to thrust this PSWR into the complex reflectance. By just replacing ρ with ρ^2 on the final right-hand side of (29), we define a new complex:

$$\Omega = \frac{\rho^2 - 1}{\rho^2 + 1} e^{j\theta}. \qquad (31)$$

This simple replacement makes a difference only in magnitude; the phase is kept unchanged from the original Γ. Because the right-hand side in (31) ranges from zero to unity in magnitude at any phase, we notice that Ω draws a circular disk on the complex plane as well as Γ does. In other words, this disk meets the prime requirement of zero and infinity as well as the Smith chart does.

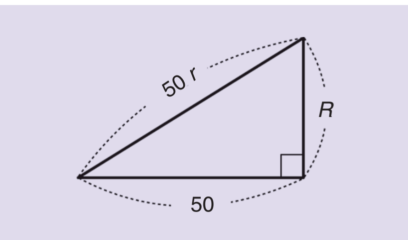

Figure 11. *A right triangle can be used to remember the axial ratio's elegant law.*

To thoroughly comprehend how Ω behaves on the plane, it is helpful to formulate Ω in terms of Γ. By substituting (2) and (29) into (31), we can eliminate ρ and θ at the same time, resulting in

$$\Omega = \frac{2\Gamma}{1 + \Gamma\Gamma^*}. \qquad (32)$$

Upon encountering this formula, might students feel something come to their attention? Yes, this is congruent with the double-angle rule of hyperbolic tangent, except for the conjugation on the denominator's final term. In any case, using this formula, we can sail directly from the familiar Γ island to the new Ω island.

With the help of (32), we can convert (30) into

$$\Omega = \frac{R^2 + (X + j50)^2}{R^2 + X^2 + 50^2}, \qquad (33)$$

which projects the R-X grid within the disk, as seen in Figure 10. This chart is called the *Beltrami–Klein disk* in hyperbolic geometry [2], [11], [12]. On this disk, we notice ellipses in the north–south symmetry with respect to the Equator, commonly contacting the East Pole (or West Pole in Australia-orientated maps). These ellipses are called *horocycles* in hyperbolic geometry. They physically mean constant-R contours, along which X ranges from negative to positive infinity. Focusing on the ellipse's axial ratio r, we find

$$50^2 + R^2 = 50^2 r^2, \qquad (34)$$

where R is the resistance read out at the ellipse's west-side interception across the Equator. This formula is so elegant that students can quickly grasp it. Those who are better at geometry than at algebra can memorize this law via the equivalent right triangle depicted in Figure 11 using the Pythagorean proposition.

We also notice straight lines all converging on the East Pole. These lines are called *geodesics* in hyperbolic geometry. They physically mean constant-X contours, along which R ranges from zero to infinity. It is a remarkable feature of this disk that all the constant-X contours lie in straight chords, in contrast with the circular arcs on the Smith chart shown in Figure 9.

Voltage ratio squared equals power ratio.

Back to the Future

Now it is almost time to bid goodbye to 19th-century geometry and fuel our vessel to sail back to the future. We have three triangles and two circular disks to keep in our treasure box. On finalizing this fruitful adventure, we sincerely express our full respect to Eugenio Beltrami and Henri Poincaré by displaying their portraits in Figure 12. Imagine if we could invite these two great mathematicians to our International Microwave Symposium and give them the chance to know our technical term SWR: they might have presented in a special memorial session how to formulate their circular disks, as in their speech balloons.

Make geometry great again.

Figure 12. *Henri Poincaré (left) (1854–1912) and Eugenio Beltrami (1835–1900).*

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