Enigmas, etc.

Poincaré Length

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A simple, lumped-constant ladder is loaded with resistor R, as shown in Figure 1(a). Reactors L and C are adjusted so that input impedance Z_{in} becomes a real number. This is projected as a geometrical trajectory in Figure 1(b). It starts from R, then runs along arcs L and C, and finally reaches Z_{in} back on the horizon. Find the overall length of this locus observed from Poincaré's (Figure 2) metric [1]. Which of the following is correct?

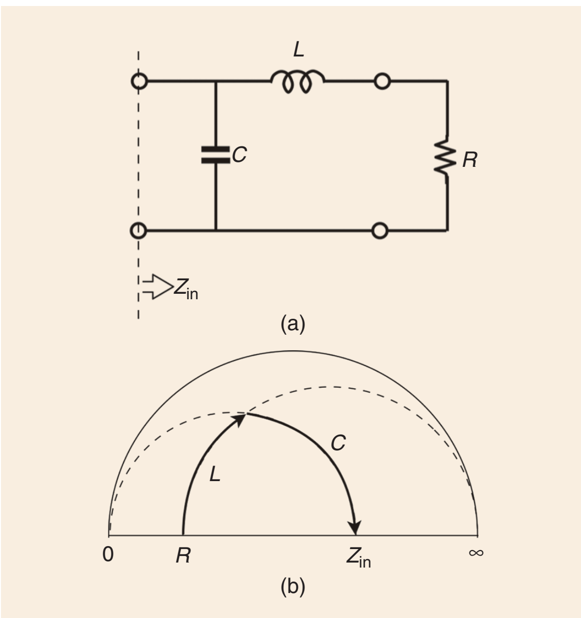

Figure 1. *An inductor–capacitor ladder for real-to-real impedance matching. (a) A circuit scheme and (b) a locus on a Smith chart.*

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(a) $2\omega CR$ (b) $\frac{2\omega L}{R}$ (c) $\omega^2 LC$ (d) $\omega CR + \frac{\omega L}{R}$.

One clue is that the length of a smooth curve can be calculated by integrating infinitesimal line segments along it. An infinitesimal line segment on the chart is regarded as the hypotenuse of a right triangle (see Figure 3) which measures

$$d\Lambda = \frac{|dZ|}{R} = \frac{1}{R}\sqrt{dR^2 + dX^2}, \quad (1)$$

Figure 2. *A portrait of French mathematician, theoretical physicist, and engineer Jules Henri Poincaré. (Source: Marimo Matsumoto; used with permission.)*

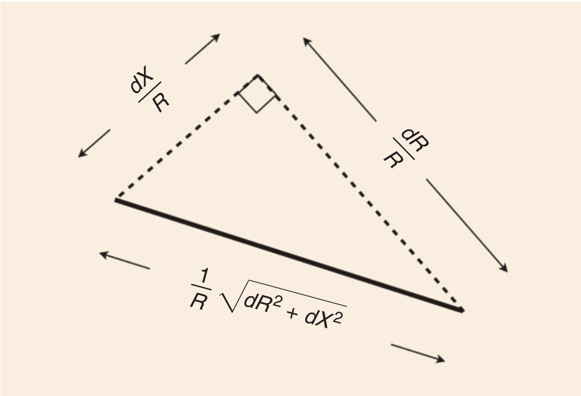

Figure 3. *A close look at an infinitesimal right triangle to measure a given segment (solid side). Each broken side lies on a constant-R or -X contour of the chart. Dimensions are commonly normalized by local scale R. See the final term of (6).*

from hyperbolic geometry [2]. This is exactly what we deduced for a small increment of the standing-wave ratio (SWR) in the previous "Enigmas, etc." challenge [3]. The Greek symbol Λ hereafter stands for Poincaré length. Keeping (1) in mind, let us start answering the problem. Enjoy this challenge until the correct answer appears in next month's column.

Solution to the February 2020 "Enigmas, etc." Challenge

As described in the February 2020 installment of "Enigmas, etc.," SWR ρ relates to the source and load impedances as

$$\frac{\rho-1}{\rho+1}=\left|\frac{Z_2-Z_1^*}{Z_2+Z_1}\right|. \qquad (2)$$

The impedances were specified as

$$Z_1 = R + jX, \qquad (3)$$
$$Z_2 = R - jX + \delta Z. \qquad (4)$$

Substituting them into (2), we achieve

$$\frac{\rho-1}{\rho+1}=\left|\frac{\delta Z}{2R+\delta Z}\right|. \qquad (5)$$

Solving it for ρ, and remembering $|\delta Z| \ll R$, we reach

$$\rho = 1 + \frac{|\delta Z|}{R}. \qquad (6)$$

Therefore, the correct answer is a.

For a physical interpretation, let us focus on the final term of (6), which tells us how much the SWR increases from unity due to load deviation δZ. We should pay particular attention to the denominator; it is not the entire impedance but rather its real part alone. In other words, the original reactance X does not affect ρ at all. This might be different from an intuitive prediction; however, the aforementioned increment of SWR gives us a crucial bridge between RF engineering and hyperbolic geometry [1]. We will cross that bridge in the next challenge.

References

[1] M. S. Gupta, "Escher's art, Smith chart, and hyperbolic geometry," *IEEE Microw. Mag.*, vol. 7, no. 5, pp. 66–76, Oct. 2006. doi: 10.1109/MW-M.2006.247916.

[2] K. Yamada and T. Ohira, "Graphical representation of the power transfer efficiency of lumped-element circuits based on hyperbolic geometry," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 64, no. 5, pp. 485–489, May 2017. doi: 10.1109/TCSII.2016.2580626.

[3] T. Ohira, "Standing wave ratio," *IEEE Microw. Mag.*, vol. 21, no. 2, pp. 108–109, Feb. 2020. doi: 10.1109/MMM.2019.2947392.