Letter

Distributed Nash Equilibrium Seeking Over Random Graphs

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Dear Editor,

This letter is concerned with the distributed Nash equilibrium (NE) seeking in an N-player game over random graphs. We develop a distributed stochastic forward-backward (DSFB) algorithm based on local information exchange between agents. We prove that the DSFB algorithm can converge to an NE almost surely, and analyze the convergence rate of the proposed algorithm. Compared with the existing works on distributed NE seeking, the communication graph in this letter is supposed to be time-varying and stochastic, which makes the NE seeking algorithm more suitable for practical scenarios, but brings a great challenge in both the design and convergence analysis of the algorithm. Besides, by establishing a variational inequality on NE, we relax the co-coercivity or strong monotonicity assumption on the extended pseudo-gradient.

NE seeking is an essential topic in game theory and several classical NE seeking algorithms have developed. Recently, benefit from the technique of consensus, distributed NE seeking approaches have been proposed. Different from the classical NE seeking algorithms where all the players are available to the strategy information of all other players, in the setting of the distributed NE seeking algorithms, each player may only access to the strategy information of its neighbors. The distributed NE seeking problems has attracted much attention, due to its widely application on vehicle networks [1], distributed power control [2], wireless communication networks [3], [4] and other fields.

The early works on distributed NE seeking focused on the case with balanced communication graphs. In [5], the distributed NE seeking problem over a connected undirected communication graphs was considered, and the consensus protocol and gradient method were combined to design a continuous-time NE seeking algorithm. In [6], the authors considered the distributed NE seeking problem with bounded control input. In [7], the distributed NE seeking problem for aggregative games was further studied. In [8], a distributed method based on the forward-backward splitting method was presented to solve generalized NE in non-cooperative games. Recently, in [9]–[11], the distributed NE seeking problem over unbalanced communication graphs was investigated, while in [12], [13] the distributed NE seeking problem over switching communication graphs was studied. In [14], an asynchronous Gossip based algorithm was proposed and it was shown that if the the pseudo-gradient of the cost functions is bounded, the proposed algorithm converges to an NE in probability.

In summary, the communication graph is required to be deterministic in most existing works on distributed NE seeking. In fact, according to [15], to deal with the uncertainties of communication process in practice, one effective approach is to model the communication networks as random graphs. Though a number of works focused on distributed optimization over random graphs [15]–[17], yet there are rare existing results on distributed NE seeking over random graphs, which motivates this work. To solve the distributed NE seeking problem over random graphs, we design a DSFB algorithm which is able to converge to an NE in probability 1.

The main contributions of this letter are stated as follows. First, in contrast to the existing works on distributed NE seeking over fixed communication graphs, the communication graph considered in this letter is time-varying and stochastic, which is more general and more practical. Besides, our work provides a new approach for the distributed NE seeking problem with random packet-loss and gossip communication. Second, compared with the cocoercivity assumption in [18] and strong monotonicity assumption in [1], [5], a more relaxed assumption on the extended pseudo-gradient is used. Third, the proposed algorithm is able to deal with the games where a Nash equilibrium exists but the strategy set is not bounded, for instance, coercive games [19] and non-compact qualitative games [20]. While the boundness of pseudo-gradient and strategy set is required in [14].

Problem formulation: Consider a game with N players of which the set can be described as $\mathcal{V}=\{1,\ldots,N\}$. For player i, $x_i\in\mathcal{X}_i$ represents its strategy, and $\mathcal{X}_i\subset\mathbb{R}^{n_i}$ represents the private local strategy set of player i. Then, $\mathcal{X}=\prod_{i\in\mathcal{V}}\mathcal{X}_i\subset\mathbb{R}^{nN}$ represents the strategy set of all players, where $\prod$ denotes the Cartesian product and $n=\sum_{i=1}^{N}n_i$. For simplicity, denote $x_{-i}\in\mathcal{X}_{-i}:=\prod_{j\in\{1,\ldots,N\}\setminus\{i\}}\mathcal{X}_j$ as the strategies of all players except i-th player. Denote the stacked vector of all the players' stra tegies as $x=\mathrm{col}(x_i,i\in\mathcal{V})\in\mathbb{R}^n$. $J_i(x_i,x_{-i})$ represents the cost function of player i under the strategy (x_i,x_{-i}). In summary, an N-players game can be denoted as $G(\mathcal{V},\mathcal{X}_i,J_i)$. In context of a networked game, each player aims to minimize its own cost function, that is

$$\begin{aligned}\min_{x_i}\quad & J_i(x_i,x_{-i})\\ \text{s.t.}\quad & x_i\in\mathcal{X}_i,\quad \forall i\in\mathcal{V}.\end{aligned}\tag{1}$$

Next, we give the formal definition of the NE.

Definition 1: Given a game $G(\mathcal{V},\mathcal{X}_i,J_i)$, a strategy profile $(x_i^*, x_{-i}^*)\in\mathcal{X}$ is said to be a NE if $J_i(x_i^*,x_{-i}^*)\le J_i(x_i,x_{-i}^*),\forall i\in\mathcal{V},\ x_i\in\mathcal{X}_i$.

To proceed further, the following assumptions are needed.

Assumption 1: For each player i,

1) The strategy set $\mathcal{X}_i\neq\emptyset$, and is compact and convex;

2) For any x_{-i}, J_i is continuously differentiable and convex in x_i, that is, for any $x,\ y\in\mathbb{R}^{n_i}$, and $x_{-i}\in\mathbb{R}^{n-n_i}$, $J_i(y,x_{-i})\ge J_i(x,x_{-i})+\langle y-x,\nabla_i J_i(x,x_{-i})\rangle$, where $\nabla_i J_i(x,x_{-i})=\frac{\partial J_i}{\partial x}(x,x_{-i})$.

Assumption 2: Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be the pseudo-gradient of the game (1) defined as$F(x)=[\nabla_{x_i}J_i(x_i,x_{-i})]$, $i\in\mathcal{V}$. The pseudo-gradient F is μ-coercive and θ_0-Lipschitz continuous, that is, there are two positive constants μ and θ_0 such that $\langle x-\tilde{x},F(x)-F(\tilde{x})\rangle\ge\mu\|x-\tilde{x}\|^2$, $\|F(x)-F(\tilde{x})\|\le\theta_0\|x-\tilde{x}\|,\quad\forall\ x,\ \tilde{x}\in\mathbb{R}^n$.

Remark 1: Assumption 1 is the convexity assumption and is used to guarantee the existence of an pure NE. While Assumption 2 is used to guarantee the uniqueness of the NE. Both assumptions are standard and widely used in the existing works on NE seeking problem, for instance [7], [10], [18]. As a matter of fact, a range of practical problem settings satisfy Assumptions 1 and 2, such as Nash-Cournot games, rate allocation problems and so on.

Due to external interference, communication links may be interrupted during the wireless transmission. It is assumed in this letter that

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players interact with each other through a random graph $\mathcal{G}_k = \{\mathcal{V},\mathcal{E}_k\}$, $k = 0,1,\ldots,$ containing no self-edges, where $\mathcal{V} = \{1,...,N\}$ is the set of players and $\mathcal{E}_k$ is the set of bidirectional edges (i, j) at time k. Specifically, $(j,i) \in \mathcal{E}_k$ and the edge (j,i) is associated with positive weight w^k_{ij} if player i can get information from player j; $(j,i) \notin \mathcal{E}_k$ and $w^k_{ij} = 0$, otherwise. Denote the weight matrix as $W_k = [w^k_{ij}]_{N\times N}$. The following assumption on the weight matrix of random communication graphs is required.

Assumption 3: The weight matrix W_k is required to satisfy that

1) $W_1, W_2,\ldots,$ are independent and identically distributed;

2) W_k is weight balance, that is, $W_k 1_N = 1_N$ and $1^T_N W_k = 1^T_N$;

3) The second largest eigenvalue of $E\{W^T_k W_k\}$ is less than 1, i.e., $\rho(E\{W^T_k W_k\} - 1_N 1^T_N/N) < 1$, where $\rho(\cdot)$ represents the spectral radius, and $E\{W^T_k W_k\}$ represents the expectation of the stochastic matrix $W^T_k W_k$.

Next, define the Laplacian matrix of $\mathcal{G}_k$ as $\mathcal{L}_k = I_N - \mathcal{W}_k$. Letting $\overline{\mathcal{L}} = E\{\mathcal{L}_k\}$, we obtain that the eigenvalue of $\overline{\mathcal{L}}$ can be sorted as $0 = \lambda_1(\overline{\mathcal{L}}) < \lambda_2(\overline{\mathcal{L}})\cdots \le \lambda_N(\overline{\mathcal{L}})$. Besides, by disk theorem, $\lambda_N(\mathcal{L}_k) \le 2$ almost surely. For an event A, if $P(A) = 1$, we say A holds almost surely. Define the expectation of the random graphs as $\overline{\mathcal{G}}(\mathcal{V},\overline{\mathcal{E}})$, where the edges set $\overline{\mathcal{E}}$ is associated by the weight matrix $\overline{W} = E\{W_k\}$.

Remark 2: Assumption 3 is a mild assumption on random graphs and is widely used in the existing works on distributed optimization over random graphs [15]–[17], [21]. By item 1) of Assumption 3, the expectation of weight matrix $\overline{W}$ exists, and is also weight balance. The item 3) of Assumption 3 is equivalent to the fact that the expectation $\overline{\mathcal{G}}$ is connected, and is thus necessary. Although Assumption 3 requires the graphs to be weight balanced, yet it indeed relaxes requirement on the connectivity of graphs. Under Assumption 3, the graph $\mathcal{G}_k$ is allowed to be always disconnected at any time. While for the case with deterministic graphs, the graph needs to be always connected, e.g., [1], [12].

Next, we present the problem statement of this letter.

Problem 1 (Distributed NE seeking problem over random graphs): Given N players and a random communication graph $\mathcal{G}_k$, design a distributed NE seeking algorithm such that the strategy of all players $x = \mathrm{col}(x_1,\ldots,x_N)$ converges to an NE almost surely.

To solve Problem 1, the following technical lemma is required.

Lemma 1 (Robbins-Siegmund quasi-martingale [22]): Given a probability space $(\Omega,\mathcal{F},\mathcal{P})$, assume that the filtration $\{\mathcal{F}_k, k \in \mathbb{N}\}$ is a collection of sub-$\sigma-$fields satisfying $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_k \subseteq \cdots \subseteq \mathcal{F}$. The stochastic processes $V(k)$, $S(k)$, and $U(k)$ are $\mathcal{F}_k-$ measurable random variables for any $k \in \mathbb{N}$. The stochastic processes $V(k)$, $S(k)$ and $U(k)$ are assumed to be positive almost surely. If $E\{V(k+1)|\mathcal{F}_k\} \le (1+\gamma_k)V(k) - S(k) + U(k)$, $\sum^{\infty}_{i=0} U(i) < \infty$, hold almost surely, where γ_k, $k \in \mathbb{N}$ is a positive and decaying real sequence and satisfies $\sum^{\infty}_{k=0}\gamma_k < \infty$, then the limit $V(\infty)$ exists almost surely, and $\sum^{\infty}_{k=0} S(k) < \infty$ almost surely.

Main results: To solve the distributed NE seeking problem over random graphs, a DSFB algorithm is presented. In context of the distributed NE seeking problem, the information of all other players may not be available to each player i. Hence, in most of distributed NE seeking algorithm, for each player i, an estimator $x^i = (x^i_i, x^i_{-i})$ is implemented to estimate the strategies of all players. Specifically, x^i_i is the strategy of player i. For notation consistent, we use x^i_i instead of x_i. While x^i_{-i} is the estimation of the strategies of all other players by player i. In this case, the NE seeking problem (1) can be reformulated as in the following distributed manner:

$$\begin{aligned} \min_{\mathbf{x}^i_i\in\mathcal{X}_i} \quad & J_i(x^i_i, x^i_{-i}) \\ \text{s.t.} \quad & x^i = x^j, \quad \forall i,\ j \in \mathcal{V}. \end{aligned} \tag{2}$$

To solve the distributed NE seeking problem (2), the following DSFB algorithm (Algorithm 1) over random graphs is proposed.

The parameters of DSFB algorithm are designed as follows：

1) $c > 0$ is a constant and will be given in (6);

2) The step-size α_k is a decaying positive sequence, and satisfies $\cdots \le \alpha_1 \le \alpha_0$, $\sum^{\infty}_{k=0}\alpha_k = \infty$, $\sum^{\infty}_{k=0}(\alpha_k)^2 < \infty$.

To satisfy 2), the step-size α_k can be chosen as

$$\alpha_k = \frac{C}{(k+1)^p}, \quad k = 0,1,\ldots \tag{3}$$

where $1/2 < p < 1$, and $C > 0$.

Algorithm 1 DSFB With Random Graphs

1: **Initialization** $x^i_i(0) \in \mathcal{X}_i$, $x^i_{-i}(0) \in \mathbb{R}^{(n-n_i)}$
2: **for** $k = 1,2,...$ **do**
3: **for** each agent $i \in \mathcal{V}$ **do**
Iteration
4: $x^i_i(k+1) = P_{\mathcal{X}_i}\Big[x^i_i(k) - \alpha_k(\nabla_{x_i} J_i(x^i_i(k), x^i_{-i}(k)) + c\sum^N_{j=1} w_{ij}(k)(x^i_i(k) - x^j_i(k))\Big)\Big]$
5: $x^i_{-i}(k+1) = x^i_{-i}(k) - \alpha_k c\sum^N_{j=1} w_{ij}(k)(x^i_{-i}(k) - x^j_{-i}(k))$
6: **end for**
7: **end for**

Remark 3: The parameter c is required to be sufficiently large satisfying (6). In this case, the consensus term $c\sum_{j\in\mathcal{N}_i} w_{ij}(k)(x^i_i(k) - x^j_i(k))$ can guarantee that $x^i(k) - x^j(k) \to 0$. That is, each player can accurately estimate the strategies of all other players, which is significant for the convergence analysis of Algorithm 1. According to Theorem 1, the step-size α_k determines the convergence rate. For a faster converge speed of Algorithm 1, p is chosen smaller and thus is close to $1/2$.

Let $\mathbf{x} = \mathrm{col}(x^1,\ldots,x^N) \in \mathbb{R}^{nN}$, and $x = \mathrm{col}(x^1_1,\ldots,x^N_N)$. Define the following two operators $\mathbb{A}_k := \mathcal{R}\boldsymbol{F}(\mathbf{x}) + cL_k\mathbf{x}$, $\mathbb{B} := \mathcal{R}N_{\mathcal{X}}(x)$ where $\mathcal{R} := \mathrm{diag}(\mathcal{R}_1,\ldots,\mathcal{R}_N)$ with $\mathcal{R}_i = \mathrm{col}(0_{n_1\times n_1},\ldots, 0_{n_{i-1}\times n_{i-1}}, I_{n_i\times n_i}, 0_{n_{i+1}\times n_{i+1}},\ldots,0_{n_N\times n_N})$, $L_k = \mathcal{L}_k \otimes I_N$, the extended pseudo-gradient $\boldsymbol{F}(\mathbf{x}) = \mathrm{col}(\nabla_1 J_1(x^1),\ldots, \nabla_N J_N(x^N))$ and $N_{\mathcal{X}}(x) = \prod^n_{i=1} N_{\mathcal{X}_i}(x_i)$, with $N_{\mathcal{X}_i}(x_i) = \{y \in \mathbb{R}^{n_i} | y^T(x - z) \le 0,\ \forall z \in \mathcal{X}_i\}$. It is worth pointing out that Algorithm 1 is a typical stochastic forward-backward algorithm corresponding to the operators $\mathbb{A}_k$ and $\mathbb{B}$.

Lemma 2: Under Assumptions 1 and 2, Algorithm 1 has a unique fixed point $\mathbf{x}^* = 1_N \otimes x^*$, where x^* is an NE of game (1).

Proof: By Karush-Kuhn-Tucker (KKT) condition, if x^* is an NE of game (1), then $\mathbf{0}_{n_i} \in \nabla_{x_i} J_i(x^*_i, x^*_{-i}) + N_{\mathcal{X}_i}(x^*_i)$, $i \in \mathcal{V}$. It implies that $\mathbf{x}^* = 1_N \otimes x^*$ is a fixed point of Algorithm 1. Assume that $\mathbf{x}^* = \mathrm{col}((x^1_1)^*, (x^1_{-1})^*,\ldots,(x^1_N)^*, (x^1_{-N})^*)$ is a fixed point of Algorithm 1. By Step 5 in Algorithm 1, $(x^i_i)^* = (x^j_{-i})^*$, which implies that $\mathbf{x}^* = \overline{x} \otimes I_N$ holds for some $\overline{x} \in \mathbb{R}^n$. By Step 4 in Algorithm 1, we can obtain that $\mathbf{0}_{n_i} \in \nabla_{x_i} J_i((x^i_i)^*, (x^i_{-i})^*) + N_{\mathcal{X}_i}((x^i_i)^*)$, $i \in \mathcal{V}$. By KKT condition, $\overline{x} = x^*$, and thus $\mathbf{x}^* = 1_N \otimes x^*$. That is, the fixed point of Algorithm 1 is unique. ■

Lemma 3: Let $\mathbf{x}(k) = \mathrm{col}(x^1(k),\ldots,x^N(k)) \in \mathbb{R}^{nN}$ and $G(\mathbf{x}(k)) = \mathrm{col}(G^1(x^1(k)),\ldots,G^N(x^N(k))) \in \mathbb{R}^{nN}$, where $G^i(x^i(k)) = \mathrm{col}(0_{n_1},\ldots, 0_{n_{i-1}}, \partial\mathcal{I}_{\mathcal{X}_i}(x^i_i(k)), 0_{n_{i+1}}\ldots, 0_{n_N}) \in \mathbb{R}^n$. Then, the updates in Algorithm 1 can be rewritten as

$$\mathbf{x}(k+1) = \mathbf{x}(k) - \alpha_k(\boldsymbol{F}(\mathbf{x}(k)) + c\mathcal{R}L_k\mathbf{x}(k) - G(\mathbf{x}(k+1)) \tag{4}$$

where $L_k := \mathcal{L}_k \otimes I_N$.

Proof: Let $g(x^i_i(k), x^i_{-i}(k)) = x_i(k) - \alpha_{ki}(\nabla_{x_i} J_i(x_i(k), \mathbf{x}^i_{-i}(k)) + c\sum_{j\in\mathcal{N}_i} w_{ij}(k)(x_i(k) - \mathbf{x}^j_i(k)$. Then,

$$\begin{aligned} x^i_i(k+1) &= \arg\min_{x^i_i}\left\{\frac{1}{2}\|x - g(x^i_i(k), x^i_{-i}(k))\|^2 + \partial\mathcal{I}_{\mathcal{X}_i}(x)\right\} \\ &\in g(x^i_i(k), x^i_{-i}(k)) - \partial\mathcal{I}_{\mathcal{X}_i}(x^i_{-i}(k+1)) \end{aligned} \tag{5}$$

which implies that (4) holds. ■

The following assumption is required to establish a monotonicity property for the augmented space $\mathbb{R}^{Nn}$ in Lemma 4, and is significant in the convergence analysis of DSFB algorithm.

Assumption 4: The extended pseudo-gradient $\boldsymbol{F}$ is θ-Lipschitz continuous, that is, $\|\boldsymbol{F}(x)-\boldsymbol{F}(y)\| \le \theta\|x-y\|,\ \forall\ x, y$.

It is worth pointing out that Assumption 4 holds when the gradient ∇J_i is Lipschitz continuous and is weaker than co-coercivity in [18] and strong monotonicity in [1], [5].

The following lemma is required for the convergence analysis of DSFB Algorithm.

Lemma 4: Under Assumptions 2–4, the expectation matrix $\overline{\mathcal{L}} = E\{\mathcal{L}_k\}$ satisfies $\lambda_2(\overline{\mathcal{L}}) > 0$. Choose c as any constant satisfying

$$c > c_0 = \frac{\left(\frac{(\theta+\theta_0)^2}{4\mu}+\theta\right)}{\lambda_2(\overline{\mathcal{L}})} \tag{6}$$

where μ is any positive constant.Then, the following variational inequality holds:

$$\langle \mathcal{R}\boldsymbol{F}(\mathbf{x}) + c\widetilde{L}\mathbf{x} + G(\mathbf{x}), \mathbf{x}-\mathbf{x}^*\rangle > 0, \quad \forall\ \mathbf{x} \ne \mathbf{x}^*. \tag{7}$$

Proof: According to the third item In Assumption 3, $\lambda_2(\overline{\mathcal{L}}) > 0$. According to [23, Lemma 2], when c satisfies (7), we have the matrix $\Psi = \begin{bmatrix} \mu/n & -(\theta+\theta_0)/2\sqrt{n} \\ -(\theta+\theta_0)/2\sqrt{n} & c\lambda_2(\mathcal{L})-\theta \end{bmatrix}$ is positive defined. Then, for any $\mathbf{x}$ and $\tilde{\mathbf{x}} \in Ker(\mathcal{L})$,

$$(\mathbf{x}-\tilde{\mathbf{x}})^T(\mathcal{R}\boldsymbol{F}(\mathbf{x}) - \mathcal{R}\boldsymbol{F}(\tilde{\mathbf{x}}) + cL_k(\mathbf{x}-\tilde{\mathbf{x}})) \ge \bar{\mu}\|\mathbf{x}-\tilde{\mathbf{x}}\|^2 \tag{8}$$

where $\bar{\mu} := \lambda_{\min}(\Psi) > 0$.

According to the definition of $G(\mathbf{x})$, we have

$$\begin{aligned}&\langle G(\mathbf{x}) - G(\mathbf{x}^*), \mathbf{x}-\mathbf{x}^*\rangle \\ &\quad = \sum_{i=1}^{N} \langle \partial\mathcal{I}_{\mathcal{X}_i}(x_i^i) - \partial\mathcal{I}_{\mathcal{X}_i}((x_i^i)^*), x_i^i - (x_i^i)^*\rangle. \end{aligned} \tag{9}$$

Then, we obtain

$$\begin{aligned}&\langle G(\mathbf{x}) - G(\mathbf{x}^*), \mathbf{x}-\mathbf{x}^*\rangle \\ &\quad = \langle \mathcal{N}_{\mathcal{X}_i}(x_i^i) - \mathcal{N}_{\mathcal{X}_i}((x_i^i)^*), x_i^i - (x_i^i)^*\rangle \ge 0. \end{aligned} \tag{10}$$

Since $\mathcal{R}\boldsymbol{F}(\mathbf{x}^*) + c\widetilde{L}\mathbf{x}^* + G(\mathbf{x}^*) = 0$, then we have

$$\begin{aligned}&\langle \mathcal{R}\boldsymbol{F}(\mathbf{x}) + c\widetilde{L}\mathbf{x} + G(\mathbf{x}), \mathbf{x}-\mathbf{x}^*\rangle \\ &\quad = \langle \mathcal{R}\boldsymbol{F}(\mathbf{x}) - \mathcal{R}\boldsymbol{F}(\mathbf{x}^*) + c\widetilde{L}\mathbf{x} - c\widetilde{L}\mathbf{x}^*, \mathbf{x}-\mathbf{x}^*\rangle \\ &\qquad + \langle G(\mathbf{x}) - G(\mathbf{x}^*), \mathbf{x}-\mathbf{x}^*\rangle \ge \bar{\mu}\|\mathbf{x}-\mathbf{x}^*\|^2 > 0. \end{aligned} \tag{11}$$

■

Next, we present the main result of this letter.

Theorem 1: Under Assumptions 1–4, with the DSFB Algorithm, each $x^i(k)$, $i \in \mathcal{V}$, converges to x^* with the convergence rate $O(1/k^{\alpha})$, where x^* is an NE of game (1) and $0 < \alpha < 1/2$.

Proof: According to KKT condition, we have $\mathbf{0}_{nN} \in \mathcal{R}\boldsymbol{F}(\mathbf{x}^*) + c\tilde{L}\mathbf{x}^* + G(\mathbf{x}^*)$. Then, it follows from (4) that:

$$\begin{aligned}&\mathbf{x}(k+1) - \mathbf{x}^* - \alpha_k G\mathbf{x}(k+1)) + \alpha_k G(\mathbf{x}^*) \\ &\quad = \mathbf{x}(k) - \mathbf{x}^* - \alpha_k(\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + cL_k\mathbf{x}(k) - \mathcal{R}\boldsymbol{F}(\mathbf{x}^*) - c\tilde{L}\mathbf{x}^*) \end{aligned} \tag{12}$$

which implies that

$$\begin{aligned}\left\|\mathbf{x}(k+1)-\mathbf{x}^*\right\|^2 &\le \alpha_k^2\|\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + cL_k\mathbf{x}(k) \\ &\quad - (\mathcal{R}\boldsymbol{F}(\mathbf{x}^*) + c\tilde{L}\mathbf{x}^*)\|^2 + \|\mathbf{x}(k)-\mathbf{x}^*\|^2 \\ &\quad - \alpha_k^2\|G(\mathbf{x}(k+1)) - G(\mathbf{x}^*)\|^2 - 2\alpha_k\langle\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + cL_k\mathbf{x}(k) - \mathcal{R}\boldsymbol{F}(\mathbf{x}^*) \\ &\quad + c\tilde{L}\mathbf{x}^*, \mathbf{x}(k)-\mathbf{x}^*\rangle - 2\alpha_k\langle G(\mathbf{x}(k+1)) - G(\mathbf{x}^*), \mathbf{x}(k+1)-\mathbf{x}^*\rangle. \end{aligned} \tag{13}$$

Then, the first term of the right-hand side of (13) satisfies

$$\begin{aligned}&\|\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + cL_k\mathbf{x}(k) - (\mathcal{R}\boldsymbol{F}(\mathbf{x}^*) + c\tilde{L}\mathbf{x}^*)\|^2 \\ &\quad \le 2\|\mathcal{R}\|^2\|\boldsymbol{F}(\mathbf{x}(k)) - \boldsymbol{F}(\mathbf{x}^*)\|^2 \\ &\qquad + 4c^2\|L_k\|^2\|\mathbf{x}(k)-\mathbf{x}^*\|^2 + 4c^2\|L_k - \tilde{L}\|^2\|\mathbf{x}^*\|^2 \\ &\quad = (2{\theta_0}^2 + 16c^2)\|\mathbf{x}(k)-\mathbf{x}^*\|^2 + 32c^2\|\mathbf{x}^*\|^2 \end{aligned} \tag{14}$$

where the first inequality holds since $(a+b+c)^2 \le 2a^2 + 4b^2 + 4c^2$, $\forall a, b, c \in \mathbb{R}^{\mathbb{N}}$, and the second inequality holds since both of $\|L_k\|$, $\|\tilde{L}\|$ are smaller than 2. It follows from (10) that the last term of the right-hand side of (13) is larger than 0.

Based on the above analysis, we can obtain

$$\begin{aligned}&\left\|\mathbf{x}(k+1)-\mathbf{x}^*\right\|^2 \\ &\quad \le (1 + (\alpha_k)^2(2{\theta_0}^2 + 16c^2))\|\mathbf{x}(k)-\mathbf{x}^*\|^2 + 32(\alpha_k)^2c^2\|\mathbf{x}^*\|^2 \\ &\qquad - 2\alpha_k\langle\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + cL_k\mathbf{x}(k) - \mathcal{R}\boldsymbol{F}(\mathbf{x}^*) + c\tilde{L}\mathbf{x}^*, \mathbf{x}(k)-\mathbf{x}^*\rangle \\ &\qquad - 2\alpha_k\langle G(\mathbf{x}(k+1)) - G(\mathbf{x}^*), \mathbf{x}(k+1)-\mathbf{x}^*\rangle. \end{aligned} \tag{15}$$

Let $V(k) = \|\mathbf{x}(k)-\mathbf{x}^*\|^2 + 2\alpha_k\langle G(\mathbf{x}(k)) - G(\mathbf{x}^*), \mathbf{x}(k)-\mathbf{x}^*\rangle$, and we have

$$\begin{aligned}&V(k+1)\|\mathbf{x}(k+1)-\mathbf{x}^*\|^2 \\ &\quad + 2\alpha_k\langle G(\mathbf{x}(k+1)) - G(\mathbf{x}^*), \mathbf{x}(k+1)-\mathbf{x}^*\rangle \\ &\quad \le (1 + \alpha_k^2(2{\theta_0}^2 + 16c^2))V(k) + 32\alpha_k^2c^2\|\mathbf{x}^*\|^2 \\ &\qquad - 2\alpha_k\langle\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + cL_k\mathbf{x}(k) + G(\mathbf{x}(k)), \mathbf{x}(k)-\mathbf{x}^*\rangle \\ &\qquad - 2\alpha_k\langle\mathcal{R}\boldsymbol{F}(\mathbf{x}^*) + cL_k\mathbf{x}^* + G(\mathbf{x}^*), \mathbf{x}(k)-\mathbf{x}^*\rangle. \end{aligned} \tag{16}$$

Next, we calculate the following conditional expectation:

$$\begin{aligned}&E\{V(k+1)|\mathcal{F}_k\} \\ &\quad \le (1 + \alpha_k^2(2{\theta_0}^2 + 16c^2))E\{V(k)\} + 32\alpha_k^2c^2\|\mathbf{x}^*\|^2 \\ &\qquad - 2\alpha_k\langle\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + c\tilde{L}\mathbf{x}(k) + G(\mathbf{x}(k)), \mathbf{x}(k)-\mathbf{x}^*\rangle. \end{aligned} \tag{17}$$

Since $\sum_{k=0}^{\infty}\alpha_k^2 < \infty$, we obtain $\sum_{k=0}^{\infty}(\alpha_k)^2(2\theta_0^2 + 16c^2) < \infty$, $\sum_{k=0}^{\infty}32(\alpha_k)^2c^2\|\mathbf{x}^*\|^2 < \infty$.

Then, it follows from Lemma 1 that $\sum_{k=0}^{\infty}\alpha_k\langle\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + c\tilde{L}\mathbf{x}(k) + G(\mathbf{x}(k)), \mathbf{x}(k)-\mathbf{x}^*\rangle < \infty$, $\lim_{k\to\infty}V(k) = V(k) < \infty$ almost surely.

According to Lemma 4, if $\mathbf{x}(k) \ne \mathbf{x}^*$, then $\langle\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + c\tilde{L}\mathbf{x}(k) + G(\mathbf{x}(k)), \mathbf{x}(k)-\mathbf{x}^*\rangle > 0$. Since $\sum_{k=0}^{\infty}\alpha_k = \infty$, we have $\lim_{k\to\infty}\langle\mathcal{R}\boldsymbol{F}(\mathbf{x}(k)) + c\tilde{L}\mathbf{x}(k) + G(\mathbf{x}(k)), \mathbf{x}(k)-\mathbf{x}^*\rangle = 0$.

It follows from (11) that $\lim_{k\to\infty}\mathbf{x}(k) = \mathbf{x}^*$. Choosing $1/2 < p < 1-\alpha$ in (3), we have $\alpha_k = C/(k+1)^{1-\alpha-\epsilon}$, where $0 < \epsilon < 1-\alpha-p$. It follows from () that $\sum_{k=0}^{\infty}C\bar{\mu}/(k+1)^{1-\alpha-\epsilon}\|\mathbf{x}(k)-\mathbf{x}^*\|^2 < \infty$. Hence, $\|\mathbf{x}(k)-\mathbf{x}^*\|^2 \le C_0/(k+1)^{\alpha}$, holds for some positive constant C_0. ■

Remark 4: As shown in the proof of [23, Lemma 2], the boundness of pseudo-gradient and strategy set is not used for establishing [23, Lemma 2] which is employed to prove Lemma 4. This implies that the compactness assumption on the strategy set in Assumption 1 can be replaced with some relaxed assumptions which guarantee the existence of a Nash equilibrium, for instance, the coercive game assumption [19]. Thus, Algorithm 1 can still be applied to the games under those relaxed assumptions, such as coercive games [19] and non-compact qualitative games [20]. While the boundness of pseudo-gradient and strategy set is necessary in convergence analysis of [14], which makes the algorithm in [14] not applicable to these games.

Remark 5: The convergence analysis in this paper is different from that in the existing works on fixed communication graphs [5], [8], [13]. On the one hand, the convergence analysis in these existing works is established based on Lyapunov theory, and requires the second smallest eigenvalue of the Laplacian matrix of the graph to be positive, which implies that the communication graph has to be connected all the time. This condition does not hold under Assumption 3, and thus the approaches on convergence analysis in [5], [8], [13] cannot be applicable any longer. On the other hand, the update system in Algorithm 1 is stochastic, which brings a great challenge for convergence analysis of Algorithm 1. To handle this issue, we establish the stochastic convergence analysis by the aid of the Robbins-Siegmund quasi-martingale.

Remark 6: This paper provides a new approach for distributed NE seeking problem with packet-loss or gossip communication. The proposed approach can deal with the uncertainties of communication process effected by external random disturbances, and the constraints of communication capacity of each player.

Numerical example: Consider a game with five players and a random graph $\mathcal{G}_k$ shown in Fig. 1 which satisfies $P(\mathcal{G}_k = \mathcal{G}^i) = p_m$, $m = 1,2,3$ with $p_1 = 0.5$, $p_2 = p_3 = 0.25$. The weight in $\mathcal{G}^m$ is chosen as: $w_{12}^1 = w_{14}^1 = w_{54}^1 = 1/2$, $w_{21}^1 = w_{54}^1 = 1$, $w_{ij}^1 = 0$, otherwise; $w_{42}^2 = w_{43}^1 = 1/2$, $w_{34}^2 = w_{24}^2 = 1$, $w_{ij}^2 = 0$, otherwise; $w_{23}^3 = w_{25}^3 = 1/2$, $w_{32}^3 = w_{52}^3 = 1$, $w_{ij}^3 = 0$, otherwise. Denote the strategy of the player i as $x_i = \mathrm{col}(x_{i1}, x_{i2})$ and $x = \mathrm{col}(x_1, \ldots, x_5)$. The cost functions of the

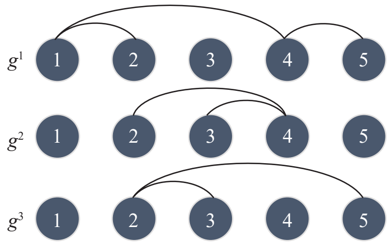

Fig. 1. The random graphs with five players.

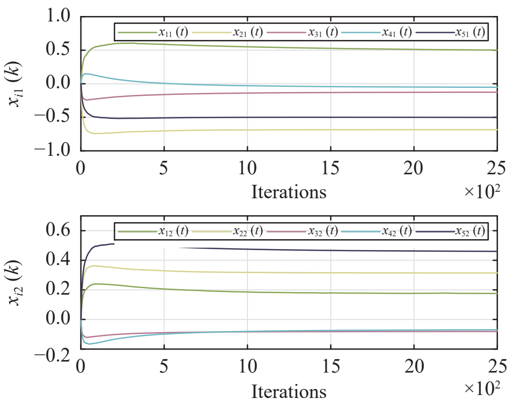

Fig. 2. Strategies of all players.

five players are $J_1(x)=x_{11}^2-2x_{11}+x_{12}^2-x_{12}+\|x_1-x_4\|^2$, $J_2(x)=2x_{21}^2+4x_{21}+2x_{22}^2-2x_{22}+\|x_2-x_4\|^2$, $J_3(x)=3x_{31}^2+2x_{31}+3x_{32}^2+x_{32}+\|x_3-x_1\|^2$, $J_4(x)=2x_{41}^2-x_{41}+2x_{42}^2+x_{42}+\|x_4-x_2\|^2$, $J_5(x)=x_{51}^2+2x_{51}+x_{52}^2-2x_{52}+\|x_5-x_4\|^2$.

The strategy set is set to $\mathcal{X}_i=[-1,1]^2$. Select $c=1$ and $\alpha_k=0.05/(k+1)^{0.6}$. The strategy simulation of all players is shown in Fig. 2, which shows that Algorithm 1 achieves the NE.

Conclusion: In this letter, we investigate the distributed NE seeking problem in an N-player game over random graphs. We develop a DSFB algorithm based on local information exchange between players. We prove that the DSFB algorithm converges to an NE almost surely. Compared with the existing works on distributed NE seeking, the communication graph in this letter is time-varying and stochastic. Future work includes extension to non-weight balance communication graphs, and algorithms with a fixed step-size to ensure linear convergence.

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