Letter

An Extended Convex Combination Approach for Quadratic $\mathcal{L}_2$ Performance Analysis of Switched Uncertain Linear Systems

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Dear Editor,

This letter is concerned with quadratic $\mathcal{L}_2$ performance (quadratic stability and $\mathcal{L}_2$ gain) for switched uncertain linear systems (SULS) with norm-bounded uncertainties. Assuming that no single uncertain subsystem achieves quadratic $\mathcal{L}_2$ performance γ but a convex combination of the subsystems can make it, we propose a state-dependent switching law such that the SULS achieves the desired quadratic $\mathcal{L}_2$ performance. The discussion is extended to the SULS with state feedback control, and a sufficient condition is proposed to design the state feedback gains and the switching law simultaneously.

There have been a few references studying quadratic stability/stabilization of switched certain linear systems. Reference [1], [2] show that if there exists a stable convex combination of subsystem matrices, then a state-dependent switching law can be proposed with a quadratic Lyapunov function to stabilize the switched system. Reference [3] investigates quadratic stability/stabilization of a class of switched nonlinear systems by using a nonlinear programming (Karush-Kuhn-Tucker condition) approach. Reference [4] extends the discussion and results in [1], [2] to achieve quadratic stability for switched linear systems with norm-bounded uncertainties, and a state-dependent switching law has been proposed for quadratic stabilization. Recently, [5] extends the discussion and results in [1], [2], [4] to output dependent switching law design for quadratic stabilization of switched linear systems with norm-bounded uncertainties. In both [4] and [5], a matrix inequality approach is used for the convex combination of subsystems to design the Lyapunov matrix in the switching law. Note that the motivation of dealing with switched uncertain systems is both theoretical and practical, when a single uncertain subsystem can not achieve certain desired performance.

In this letter, we polish the quadratic stability/stabilization result in [4], [5], and extend the discussion to quadratic $\mathcal{L}_2$ performance analysis and design for SULS. It is noted that such control problem has been studied in [6] for a class of switched non-linear systems with norm-bounded uncertainties, and the main approach is the average dwell time method. In that context, exponential stability of the switched system and a weighted $\mathcal{L}_2$ performance is obtained. Here, under the assumption that no single subsystem achieves certain quadratic $\mathcal{L}_2$ performance, we propose an extended convex combination based condition and a state-dependent switching law such that the switched system achieves the desired quadratic $\mathcal{L}_2$ performance. The extended convex combination is a major extension to the one proposed in [4], [5], which incorporates both uncertainties tolerance and disturbance attenuation. It is emphasized that since the switching law can not use either the uncertainty term or the disturbance input, we extend the existing switching law by incorporating the corresponding system matrices. Moreover, the established condition and the switching law are extended to the case of SULS with state feedback control in a natural and reasonable manner.

Concerning the comparison with more references in the literature, References [7] and [8] have considered robust $\mathcal{H}_\infty$ stabilization of switched linear uncertain systems, but the assumption on each subsystem and the technical approach is different. For example, the discussion in [7] is based on the design of multiple Lyapunov functions, and the subsystems in [8] are assumed to have certain quadratic $\mathcal{L}_2$ performance so as to deal with the affine terms. Reference [9] has considered the problem of disturbance tolerance/rejection for a family of linear systems (without uncertainties) subject to actuator saturation and disturbances, where state-dependent switching laws are proposed for bounded state stability and $\mathcal{L}_2$ gain analysis. Since the approach is based on multiple Lyapunov functions, the design condition is reduced to solving a set of complicated matrix inequalities, which is different from the convex combination approach in this letter.

Problem formulation: We consider the SULS described by

$$\begin{cases}\dot{x}(t) = (A_\sigma + D_\sigma F_\sigma(t) E_\sigma)\, x(t) + B_\sigma w(t) \\ z(t) = C_\sigma x(t)\end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^r$ is the disturbance input, and $z(t) \in \mathbb{R}^p$ is the controlled output. The index function $\sigma(t) : \mathbb{R} \to S_\mathcal{N} = \{1, \ldots, \mathcal{N}\}$ is the switching law (signal) defining which subsystem is activated at the time instant t, and $\mathcal{N}$ is the number of subsystems. The constant matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times r}$, $C_i \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{n\times l}$, $E_i \in \mathbb{R}^{k\times n}$ represent the dynamics of the i-th subsystem, and $F_i(t) \in \mathbb{R}^{l\times k}$ denotes the norm-bounded uncertainty satisfying $F_i(t)^T F_i(t) \le \zeta^2 I_k$, where ζ is a given positive scalar.

Definition 1 [5], [10]: The SULS (1) is said to be quadratically stable if there exist a quadratic function $V(x) = x^T P x$ with $P > 0$ and a positive scalar ϵ such that when $w(t) = 0$, $\dot{V}(x) < -\epsilon V(x)$ is satisfied for all nonzero $x(t)$ of the system (1) with any uncertainties $F_i(t)^T F_i(t) \le \zeta^2 I_k$.

The derivative of $V(x)$ may not exist at some switching points since the right-hand side of (1) is not continuous in our problem setting. For this purpose, we choose the Filippov solutions introduced in [11]. That is, when a switching occurs at time instant t_k, we define the derivative of $V(x)$ at t_k by

$$\dot{V}(x(t_k)) = \sup_{\eta\in[0,1]} \left\{\eta \dot{V}(x(t_k^-)) + (1-\eta)\dot{V}(x(t_k^+))\right\} \tag{2}$$

where $\dot{V}(x(t_k^-)) = \lim_{t\to t_k, t<t_k} \frac{d}{dt} V(x(t))$, and $\dot{V}(x(t_k^+)) = \lim_{t\to t_k, t>t_k} \frac{d}{dt} V(x(t))$.

Definition 2 [8], [12]: The SULS (1) is said to achieve quadratic $\mathcal{L}_2$ performance γ if it is quadratically stable with a quadratic function $V(x) = x^T P x$, and

$$\int_0^t z^T(\tau) z(\tau)\, d\tau < V(x(0)) + \gamma^2 \int_0^t w^T(\tau) w(\tau)\, d\tau \tag{3}$$

holds for any time $t > 0$ and any disturbance input $w(t)$ satisfying $\int_0^\infty w^T(\tau) w(\tau)\, d\tau < \infty$.

The control problem in this letter is for given positive scalar γ, assuming that each subsystem in (1) does not achieve quadratic $\mathcal{L}_2$ performance γ, we propose a state-dependent switching law such that the SULS (1) achieves quadratic $\mathcal{L}_2$ performance γ.

Remark 1: The assumption that each subsystem in (1) does not achieve quadratic $\mathcal{L}_2$ performance γ is motivated by one of the three basic problems in switched systems and control. In such situation, we can not activate only one subsystem, and it is also not possible to only use the (average) dwell time approach for desired performance

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of the entire switched system. Moreover, since both quadratic stabilization and $\mathcal{L}_2$ gain are considered, the cases involved in the above problem formulation are multi-faced. The subsystems in the SULS may not be quadratically stable, or the $\mathcal{L}_2$ gain of certain subsystem may be greater than γ even if it is quadratically stable.

Quadratic $\mathcal{L}_2$ performance analysis: We first state the following design condition, which plays a central rule throughout this letter.

Design condition: Find a set of scalars $\lambda_i \geq 0$ $(i = 1, \ldots, \mathcal{N})$ satisfying $\sum_{i=1}^{\mathcal{N}} \lambda_i = 1$ such that

$$A_\lambda = \sum_{i=1}^{\mathcal{N}} \lambda_i A_i \tag{4}$$

is Hurwitz, and furthermore

$$\left\| \begin{bmatrix} E_\lambda \\ C_\lambda \end{bmatrix} (sI_n - A_\lambda)^{-1} \begin{bmatrix} \zeta D_\lambda & \frac{1}{\gamma} B_\lambda \end{bmatrix} \right\|_\infty < 1 \tag{5}$$

where B_λ, C_λ, D_λ, E_λ are constant matrices satisfying

$$B_\lambda B_\lambda^T = \sum_{i=1}^{\mathcal{N}} \lambda_i B_i B_i^T, \; C_\lambda^T C_\lambda = \sum_{i=1}^{\mathcal{N}} \lambda_i C_i^T C_i,$$

$$D_\lambda D_\lambda^T = \sum_{i=1}^{\mathcal{N}} \lambda_i D_i D_i^T, \; E_\lambda^T E_\lambda = \sum_{i=1}^{\mathcal{N}} \lambda_i E_i^T E_i. \tag{6}$$

The condition (4) and (5) requires a kind of "Hurwitz convex combination with $\mathcal{L}_2$ performance 1". The form is similar to the quadratic stability condition [10] of single system with matched norm-bounded uncertainty, but the matrices in (4) and (5) are constructed by the convex combination (6). Therefore, although we have assumed that each single subsystem does not achieve quadratic $\mathcal{L}_2$ performance γ, the design condition requires that a kind of convex combination of subsystems should achieve quadratic $\mathcal{L}_2$ performance γ. It is noted that the concept of Hurwitz convex combination A_λ has been proposed and used in the literature [1], [2], [8], [13], and the convex combination triple $(A_\lambda, D_\lambda, E_\lambda)$ has been proposed in [4], [5] to deal with stabilization of SULS with norm-bounded uncertainties (without disturbance input). The idea of the convex combination (4) and (5) with (6) presents a major extension to the above references. This is illustrated by the following observation: if one can find a Hurwitz convex combination A_λ, then the norm condition (5) is always true for small or zero ζ and released (large enough) $\mathcal{L}_2$ performance index γ. Moreover, when $B_i = 0$, $C_i = 0$, the condition (5) shrinks to $\|E_\lambda(sI_n - A_\lambda)^{-1} D_\lambda\|_\infty < \zeta^{-1}$, which is exactly the condition established in [4], [5].

Now, we give some discussion on how to check the above design condition. According to the bounded real lemma [10], the condition (5) together with A_λ being Hurwitz is equivalent to finding $P > 0$ and λ_i's such that

$$\begin{bmatrix} \begin{pmatrix} \mathrm{He}\{PA_\lambda\} \\ +E_\lambda^T E_\lambda + C_\lambda^T C_\lambda \end{pmatrix} & P\begin{pmatrix} \zeta D_\lambda & \frac{1}{\gamma} B_\lambda \end{pmatrix} \\ \begin{pmatrix} \zeta D_\lambda^T \\ \frac{1}{\gamma} B_\lambda^T \end{pmatrix} P & -I_{l+r} \end{bmatrix} < 0 \tag{7}$$

or equivalently,

$$\mathrm{He}\{PA_\lambda\} + E_\lambda^T E_\lambda + C_\lambda^T C_\lambda + P\left(\zeta^2 D_\lambda D_\lambda^T + \frac{1}{\gamma^2} B_\lambda B_\lambda^T\right) P < 0. \tag{8}$$

The condition (7) is a bilinear matrix inequality (BMI) w.r.t. $P > 0$ and λ_i's, and generally is not easy to solve globally. It is noted that one necessary condition for (7) is $\sum_{i=1}^{\mathcal{N}} \lambda_i \mathrm{He}\{PA_i\} < 0$ or $\mathrm{He}\{PA_\lambda\} < 0$, which is equivalent to A_λ being Hurwitz. This motivates that if we can manage to find the scalars λ_i such that A_λ is Hurwitz, we can use those scalars to solve the inequality (7) with respect to $P > 0$. However, it is commonly known that to find the set of stabilizing scalars λ_i is generally difficult. One comparatively efficient strategy to achieve such task is the so-called gridding method (or traversal search), which is based on the observation of $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. Here, we extend the gridding method in the following algorithm to solve (7) with respect to λ_i's and $P > 0$. Due to continuity with respect to the scalars λ_i, if the matrix inequality (7) is feasible, the algorithm will succeed when the division integer m is large enough.

Algorithm for solving (7):

Step 1: Set the division number m of the interval $[0, 1]$ as a moderate integer, for example, $m = 10$, and define $\mathcal{M} = \{0, 1/m, \ldots, (m-1)/m\}$.

Step 2: Equation (1) choose λ_1 from $\mathcal{M}$ in ascending order; (2) fix λ_1 and choose λ_2 from $\mathcal{M}$ in ascending order under the constraint $\lambda_1 + \lambda_2 \leq 1$; (3) fix λ_1, λ_2 and choose λ_3 from $\mathcal{M}$ in ascending order under the constraint $\lambda_1 + \lambda_2 + \lambda_3 \leq 1$; … ($i$) fix $\lambda_1, \ldots, \lambda_{i-1}$ and choose λ_i from $\mathcal{M}$ in ascending order under the constraint $\sum_{j=1}^{i} \lambda_j \leq 1$, and so on, until $\lambda_{\mathcal{N}}$ is chosen.

Step 3: Solve (7) with the λ_i's chosen in Step 2. If (7) is feasible, record the solution and end the algorithm. If (7) is not feasible, go back to Step 2 for another set of λ_i's. Or, go back to Step 1 to increase the division integer m.

With the positive definite matrix P satisfying (7), we define our state dependent switching law as

$$\mathcal{SW}1: \quad \sigma(x) = \arg\min_{i \in S_{\mathcal{N}}} g_i(x) \tag{9}$$

$$g_i(x) = x^T\Big[\mathrm{He}\{PA_i\} + E_i^T E_i + C_i^T C_i + P\left(\zeta^2 D_i D_i^T + \frac{1}{\gamma^2} B_i B_i^T\right) P\Big] x. \tag{10}$$

Under $\mathcal{SW}1$, we obtain that for any x and any $i \in S_{\mathcal{N}}$,

$$\begin{aligned} & x^T\Big[\mathrm{He}\{PA_\sigma\} + E_\sigma^T E_\sigma + C_\sigma^T C_\sigma \\ &\quad + P\left(\zeta^2 D_\sigma D_\sigma^T + \frac{1}{\gamma^2} B_\sigma B_\sigma^T\right) P\Big] x \\ & \leq x^T\Big[\mathrm{He}\{PA_i\} + E_i^T E_i + C_i^T C_i \\ &\quad + P\left(\zeta^2 D_i D_i^T + \frac{1}{\gamma^2} B_i B_i^T\right) P\Big] x. \end{aligned} \tag{11}$$

Multiplying both sides of (11) by non-negative scalars λ_i and summing up the inequalities from $i = 1$ to $i = \mathcal{N}$ leads to

$$\begin{aligned} & x^T\Big[\mathrm{He}\{PA_\sigma\} + E_\sigma^T E_\sigma + C_\sigma^T C_\sigma \\ &\quad + P\left(\zeta^2 D_\sigma D_\sigma^T + \frac{1}{\gamma^2} B_\sigma B_\sigma^T\right) P\Big] x \\ & \leq x^T\Big[\mathrm{He}\{PA_\lambda\} + E_\lambda^T E_\lambda + C_\lambda^T C_\lambda \\ &\quad + P\left(\zeta^2 D_\lambda D_\lambda^T + \frac{1}{\gamma^2} B_\lambda B_\lambda^T\right) P\Big] x. \end{aligned} \tag{12}$$

We are ready to state and prove the main theorem in this letter.

Theorem 1: The SULS (1) under the state-dependent switching law $\mathcal{SW}1$ (9) achieves quadratic $\mathcal{L}_2$ performance γ.

Proof: We prove quadratic stability and $\mathcal{L}_2$ gain of the switched system by using the Lyapunov function candidate $V(x) = x^T P x$, where $P > 0$ is the matrix satisfying (7) (equivalent to (8)).

First, according to (8), one can always find a scalar $\epsilon > 0$ satisfying

$$\mathrm{He}\{PA_\lambda\} + E_\lambda^T E_\lambda + C_\lambda^T C_\lambda + P\left(\zeta^2 D_\lambda D_\lambda^T + \frac{1}{\gamma^2} B_\lambda B_\lambda^T\right) P + \epsilon P < 0. \tag{13}$$

Combining the above inequality with (12), we reach that under the switching law $\mathcal{SW}1$,

$$x^T\Big[\mathrm{He}\{PA_\sigma\} + E_\sigma^T E_\sigma + C_\sigma^T C_\sigma + P\left(\zeta^2 D_\sigma D_\sigma^T + \frac{1}{\gamma^2} B_\sigma B_\sigma^T\right) P\Big] x < -\epsilon x^T P x \tag{14}$$

holds for any non-zero x.

Next, the derivative of $V(x)$ along the trajectories of the SULS (1) for the activated subsystem is computed and evaluated by

$$\begin{aligned}\dot{V}(x) &= \frac{d}{dt}x^T P x = \text{He}\{x^T P\dot{x}\}\\ &= \text{He}\{x^T P(A_\sigma x + D_\sigma F E_\sigma x + B_\sigma w)\}\\ &= x^T\left(\text{He}\{PA_\sigma\} + \text{He}\{PD_\sigma F E_\sigma\}\right)x\\ &\quad + x^T P B_\sigma w + w^T B_\sigma^T P x\\ &\le x^T\Big(\text{He}\{PA_\sigma\} + \zeta^2 P D_\sigma D_\sigma^T P + E_\sigma^T E_\sigma\Big)x\\ &\quad -\left(\frac{1}{\gamma}x^T P B_\sigma - \gamma w^T\right)\left(\frac{1}{\gamma}x^T P B_\sigma - \gamma w^T\right)^T\\ &\quad + \frac{1}{\gamma^2}x^T P B_\sigma B_\sigma^T P x + x^T C_\sigma^T C_\sigma x - z^T z + \gamma^2 w^T w\\ &\le x^T\Big(\text{He}\{PA_\sigma\} + E_\sigma^T E_\sigma + C_\sigma^T C_\sigma\\ &\quad + P(\zeta^2 D_\sigma D_\sigma^T + \frac{1}{\gamma^2}B_\sigma B_\sigma^T)P\Big)x - z^T z + \gamma^2 w^T w\\ &< -\epsilon V(x) - z^T z + \gamma^2 w^T w \end{aligned} \tag{15}$$

for any non-zero x, where the final inequality is obtained by (14). Thus, (15) is true for any time t except the switching time instants t_k. Moreover, since (15) holds at any $t < t_k$ and any $t > t_k$, by combining the Filipov definition (2) with (15) for the continuous function $V(x)$, we immediately obtain (15) also holds at the switching time instant t_k. Therefore, (15) holds at any time instant.

When $w(t) = 0$, we obtain $\dot{V}(x) < -\epsilon V(x)$ from (15), and thus the SULS (1) is quadratically stable with the quadratic Lyapunov function $V(x) = x^T P x$.

Moreover, (15) leads to $\dot{V}(x(\tau)) < -z^T(\tau)z(\tau) + \gamma^2 w^T(\tau)w(\tau)$ for any $\tau \ge 0$. Integrating both sides of this inequality from $\tau = 0$ to $\tau = t$, with the fact of $V(x(t)) \ge 0$ for any t, we obtain the inequality (3), which implies that the SULS (1) achieves quadratic $\mathcal{L}_2$ performance γ. ■

Remark 2: The design condition (5) is reduced to the matrix inequality (7), which is equivalent to

$$\begin{bmatrix} \text{He}\{PA_\lambda\} & PD_\lambda & PB_\lambda & C_\lambda^T & E_\lambda^T\\ D_\lambda^T P & -\zeta^{-2}I_l & 0 & 0 & 0\\ B_\lambda^T P & 0 & -\gamma^2 I_r & 0 & 0\\ C_\lambda & 0 & 0 & -I_p & 0\\ E_\lambda & 0 & 0 & 0 & -I_k \end{bmatrix} < 0. \tag{16}$$

Multiplying the first row and column of (16) by $Q = P^{-1}$ and rearranging the order of rows/columns, we obtain

$$\begin{bmatrix} \text{He}\{A_\lambda Q\} & QC_\lambda^T & QE_\lambda^T & B_\lambda & D_\lambda\\ C_\lambda Q & -I_p & 0 & 0 & 0\\ E_\lambda Q & 0 & -I_k & 0 & 0\\ B_\lambda^T & 0 & 0 & -\gamma^2 I_r & 0\\ D_\lambda^T & 0 & 0 & 0 & -\zeta^{-2}I_l \end{bmatrix} < 0 \tag{17}$$

which, together with $Q > 0$, $\lambda_i \ge 0$, $\sum_{i=1}^N \lambda_i = 1$, is also equivalent to the design condition (5).

Remark 3: Since (16) is linear with respect to γ^2 and ζ^{-2}, if we wish to find smaller $\mathcal{L}_2$ performance index γ, then we proceed to solve the optimization problem

$$\begin{aligned} &\min \quad \gamma^2\\ &\text{s.t.} \quad (16),\ P > 0,\ \lambda_i \ge 0,\ \sum_{i=1}^N \lambda_i = 1. \end{aligned} \tag{18}$$

Or, if we need to tolerate larger uncertainty, we may deal with the optimization problem

$$\begin{aligned} &\min \quad \zeta^{-2}\\ &\text{s.t.} \quad (16),\ P > 0,\ \lambda_i \ge 0,\ \sum_{i=1}^N \lambda_i = 1. \end{aligned} \tag{19}$$

Furthermore, a weighted cost function $\min\ a\gamma^2 + b\zeta^{-2}$ with $a \ge 0$ and $b \ge 0$ can also be designed if we desire to incorporate both the $\mathcal{L}_2$ performance index and the uncertainty tolerance in a balanced manner.

Numerical simulation: Consider the SULS (1) where

$$A_1 = \begin{bmatrix} -16.08 & 25.44\\ 25.44 & -30.92 \end{bmatrix},\ A_2 = \begin{bmatrix} -36.96 & -30.72\\ -30.72 & -19.04 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1\\ 0 \end{bmatrix},\quad B_2 = \begin{bmatrix} -1\\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.5 & 1.5 \end{bmatrix},\quad C_2 = \begin{bmatrix} -1.0 & 0.5 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 1 & 1\\ 3 & 1 \end{bmatrix},\quad D_2 = \begin{bmatrix} 1 & 4\\ 1 & 2 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0.5 & 0\\ 0.5 & 0.5 \end{bmatrix},\quad E_2 = \begin{bmatrix} 0.5 & -0.5\\ 0.5 & 1.0 \end{bmatrix}$$

and

$$F_1(t) = \begin{bmatrix} 0.8 & -0.5\\ 0.5 & 0.8 \end{bmatrix}\begin{bmatrix} \cos t & -\sin t\\ \sin t & \cos t \end{bmatrix}$$

$$F_2(t) = \begin{bmatrix} 0.3 & -0.7\\ 0.7 & 0.5 \end{bmatrix}\begin{bmatrix} \cos t & -\sin t\\ \sin t & \cos t \end{bmatrix}.$$

Then, $F_i^T(t)F_i(t) \le \zeta^2 I_2$ holds with $\zeta = 1$.

Since A_1 and A_2 are not Hurwitz, each subsystem can not achieve any quadratic performance level. By using the algorithm proposed in the previous section, we find that the matrix inequality (7) is feasible with $\lambda_1 = 2/3$, $\lambda_2 = 1/3$, $\gamma = 0.2$ and

$$P = \begin{bmatrix} 0.4076 & -0.0117\\ -0.0117 & 0.4164 \end{bmatrix}.$$

To confirm the design condition (4) and (5), we have

$$A_\lambda = \frac{2}{3}A_1 + \frac{1}{3}A_2 = \begin{bmatrix} -23.04 & 6.72\\ 6.72 & -26.96 \end{bmatrix}$$

and use the Cholesky decomposition method in MATLAB to obtain for (6),

$$B_\lambda = \begin{bmatrix} 1.0000 & 0\\ -0.3333 & 0.4714 \end{bmatrix},\ C_\lambda = \begin{bmatrix} 0.7071 & 0\\ 0.4714 & 1.1667 \end{bmatrix}$$

$$D_\lambda = \begin{bmatrix} 2.6458 & 0\\ 2.1418 & 1.9355 \end{bmatrix},\ E_\lambda = \begin{bmatrix} 0.7071 & 0.3536\\ 0 & 0.6770 \end{bmatrix}.$$

Then, it is confirmed the norm condition (5) holds with $\gamma = 0.2$.

Using the switching law (9) for the SULS with the initial state $x(0) = [1\ \ -1]^T$ and the disturbance input $w(t) = 2e^{-6t}\cos 5t$, we obtain Fig. 1, where the state trajectories of the SULS converge to zero. And, (3) holds for any $t > 0$, which implies the desired $\mathcal{L}_2$ performance has been achieved.

Extension to feedback controller design: We now extend the discussion to the SULS with control input

$$\begin{cases} \dot{x}(t) = (A_\sigma + D_\sigma F_\sigma(t)E_\sigma)x(t) + B_\sigma w(t) + H_\sigma u(t)\\ z(t) = C_\sigma x(t) \end{cases} \tag{20}$$

where $x(t)$, $w(t)$, $z(t)$, $y(t)$ and the corresponding matrices are the same as in (1), while $u(t) \in \mathbb{R}^m$ is the control input to be designed and $H_i \in \mathbb{R}^{n\times m}$ is a constant matrix.

The control problem is to design a switching state feedback $u(t) = K_\sigma x(t)$ such that the switched closed-loop system achieves quadratic $\mathcal{L}_2$ performance γ. Note that the state feedback gains K_i and the switching law σ should be designed simultaneously in this formulation. The control problem is practical and not trivial when the design condition in Theorem 1 is not feasible (and thus, the SULS (1) without control input can not achieve the desired quadratic $\mathcal{L}_2$ performance γ).

Now, the closed-loop system composed of (20) and the switching state feedback is

$$\begin{cases} \dot{x}(t) = (A_\sigma + H_\sigma K_\sigma + D_\sigma F_\sigma(t)E_\sigma)x(t) + B_\sigma w(t)\\ z(t) = G_\sigma x(t). \end{cases} \tag{21}$$

Replacing A_λ in (17) with $A_\lambda + \sum_{i=1}^N \lambda_i H_i K_i$ and then letting

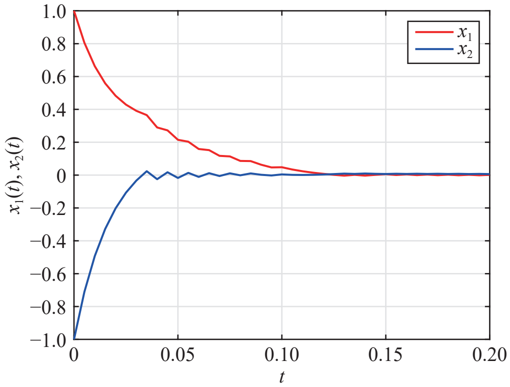

Fig. 1. State trajectories in the numerical simulation.

$K_iQ = M_i$, we obtain the following design condition:

$$\begin{bmatrix} \Omega_{11} & QC_\lambda^T & QE_\lambda^T & B_\lambda & D_\lambda \\ C_\lambda Q & -I_p & 0 & 0 & 0 \\ E_\lambda Q & 0 & -I_k & 0 & 0 \\ B_\lambda^T & 0 & 0 & -\gamma^2 I_r & 0 \\ D_\lambda^T & 0 & 0 & 0 & -\zeta^{-2} I_l \end{bmatrix} < 0 \quad (22)$$

where $\Omega_{11} = \mathrm{He}\left\{A_\lambda Q + \sum_{i=1}^N \lambda_i H_i M_i\right\}$.

Theorem 2: There is a switching state feedback for the SULS (20) such that the switched closed-loop system (21) achieves quadratic $\mathcal{L}_2$ performance γ, if there are matrix $Q > 0$, matrix M_i and non-negative scalars λ_i satisfying the matrix inequality (22).

When (22) is feasible, the feedback gain matrices are computed by $K_i = M_i Q^{-1}$, and the switching law is given by (9) and (10) with $P = Q^{-1}$.

Conclusion: We have dealt with the quadratic $\mathcal{L}_2$ performance analysis problem for switched uncertain linear systems. Under the assumption that no single uncertain subsystem achieves quadratic $\mathcal{L}_2$ performance γ but a convex combination of the subsystems can make it, we have proposed a state-dependent switching law such that the SULS achieves the desired quadratic $\mathcal{L}_2$ performance. We have also extended the discussion to the design of switching state feedback controller, together with its application to the quadratic stabilization of a boost converter. Our future work will consider the applicability and extension of such convex combination approach to practical disturbance attenuation [14] and [15], switched affine systems [16], [17], switched dynamical output feedback [18], and event-triggered control [19] and adaptive tracking control [20] for SULS.

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