Letter

Symmetry and Nonnegativity-Constrained Matrix Factorization for Community Detection

Zhigang Liu, Guangxiao Yuan, and Xin Luo

Dear editor,

This letter presents a novel symmetry and nonnegativity-constrained matrix factorization (SNCMF)-based community detection model on undirected networks such as a social network. Community is a fundamental characteristic of a network, making community detection a vital yet thorny issue in network representation. Owing to its high interpretability and scalability, a symmetric nonnegative matrix factorization (SNMF) model is frequently adopted to address this issue. However, it adopts a unique latent factor (LF) matrix for representing an undirected network's symmetry, which leads to a reduced latent space that impairs its representation learning ability. Motivated by this discovery, the proposed SNCMF model innovatively adopts the following three-fold ideas: 1) Leveraging multiple LF matrices to represent a network, thereby enhancing its representation learning ability; 2) Introducing a symmetry regularization term that implies the equality constraint between multiple LF matrices to illustrate the network's symmetry; and 3) Incorporating graph regularization into the model to preserve the network's intrinsic geometry. Experimental results on several real-world networks indicate that the proposed SNCMF-based community detector outperforms the benchmark and state-of-the-art models in achieving highly-accurate community detection results.

Networks are ubiquitous in the age of the Internet. In general, multitudinous entities in a real system and their interactions form an undirected network, e.g., wireless sensor networks and social networks. Commonly, a community in a network can be considered as its sub-graph in which a group of network nodes are connected tightly with each other through direct or indirect connections. Communities are pervasive in a network, and play a significant role in revealing its mechanism of organization and operation. Based on accurately detected communities, various network analysis tasks are facilitated, such as graph classification and social recommendation.

Related work: To date, community detection has attracted great attention from researchers, leading to a pyramid of detection models [1]. Among them, a nonnegative matrix factorization (NMF) model [2] has proven to be highly scalable and interpretable. Hence, it has been frequently applied to community detection. Given a network, an NMF-based detector works by building a low-rank approximation to the target adjacency matrix relaying on a set of nonnegative LF matrices. The achieved LFs can be considered as either the soft-threshold to identify the community of a specific entity, or the input of a hierarchical community detector. For example, Leng *et al*. [3] present a graph-regularized L_p smooth NMF model for data representation, which considers the intrinsic geometric information of target data and generates a smooth and stable solution. By incorporating the prior information into the factorization process, Ma *et al*. [4] present a semi-supervised joint NMF model to improve the community detection performance in a multi-layer network.

The above-mentioned approaches, despite of their efficiency in community detection, fail to correctly represent a given undirected network's intrinsic symmetry. SNMF works by adopting a unique LF matrix to learn a low-rank approximation to a symmetric matrix, thereby correctly representing its symmetry, which is equivalent to the Laplacian-based spectral clustering and kernel *k*-means, and can well be utilized to perform community detection. Yang *et al*. [5] present a unified interpretation to SNMF-based community detectors. Ye *et al*. [6] propose a homophily preserving SNMF model that combines the link topology and node homophily of a network, thereby better describing community structures. Luo *et al*. [7] propose to linearly or non-linearly control the scaling-factor of a nonnegative multiplicative update scheme for a graph regularized SNMF model, resulting highly-accurate community detectors.

However, existing SNMF-based models commonly adopt a single LF matrix only for ensuring its rigorous symmetry, which is actually a so strong constraint that restricts its representation learning ability [8]. Thus, this work aims to design an SNCMF-based community detector that represents an undirected network's symmetry with multiple LF matrices, thereby achieving accurate results.

Problem statement: Given a network $G = (V, E)$ as V denotes a set of n nodes and E denotes a set of m edges, it can be described by an adjacency matrix $A = [a_{ij}]^{n\times n}$, where a_{ij} describes the relationship between nodes v_i and v_j. For an unweighted G, a_{ij} is one if $e_{ij}\in E$, and zero otherwise. If G is a weighted graph, then A is real-valued. In addition, A is symmetric if G is undirected, and asymmetric otherwise. Note that in our context the undirected network is taken into consideration. $\forall v_i \in V$, a community detector aims to identify its proper affiliation to its closely related nodes.

As revealed in [5], [9], an NMF-based community detector usually involves four key process stages: 1) constructing the adjacency matrix of a given network, 2) designing an NMF-based model, 3) solving the built model, and 4) detecting community based on the achieved model. Its workflow is depicted in Fig. 1. In detail, it assumes that a given network has K communities as K be prior information, and then learns a rank-K approximation to A as $\hat{A} = XY^T$ (s.t. $X^{n\times K}$, $Y^{n\times K} \geq 0$). Note that X denotes the community feature matrix, and Y is taken as the community indicator for identifying the node-community indicator, i.e., $\forall j\in\{1{\sim}n\}$, $k \in\{1{\sim}K\}$, y_{jk} denotes the probability that node v_j belongs to community C_k. Hence, such a process of detecting community is formulated as

$$\forall v_j \in V: \ \ \forall v_j \in C_k, \text{ if } y_{jk} = \max\left\{y_{jl} | l \in \{1 \sim K\}\right\}. \quad (1)$$

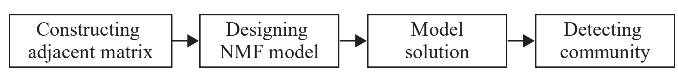

Fig. 1. Workflow for an NMF-based community detector.

To achieve X and Y, an NMF model adopts a loss function to express the difference between $\hat{A}$ and A. Such a learning objective is defined based on the Euclidean distance, i.e.,

$$O_{NMF} = \min_{X,Y\geq 0} \left\|XY^T - A\right\|_F^2 \quad (2)$$

where $\|\cdot\|_F$ is the Frobenius norm. X and Y are constrained to be nonnegative for describing the nonnegative probabilities standing for each node's community tendency. Note that O_{NMF} is non-convex in both X and Y, making their global optima intractable. However, X and Y's stationary point can be achieved via an alternative and iterative learning algorithm, i.e., nonnegative multiplicative update (NMU) scheme [2]

$$\begin{aligned} x_{ik} &\leftarrow x_{ik}\left((AY)_{ik}\Big/\left(XY^{\mathrm{T}}Y\right)_{ik}\right) \\ y_{jk} &\leftarrow y_{jk}\left(\left(A^TX\right)_{jk}\Big/\left(YX^{\mathrm{T}}X\right)_{jk}\right). \end{aligned} \quad (3)$$

With (3), an NMF-based community detector is established. Note that it acquires the network representation without considering the symmetry which is the intrinsic property for an undirected network.

An SNMF-based community detector adopts a unique LF matrix $X^{n\times K} = [x_{jk}]$ to represent the adjacency matrix A only. Its learning objective is given as

Zhigang Liu and Guangxiao Yuan contributed equally to this work. Corresponding author: Xin Luo.

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Z. G. Liu is with the School of Computer Science and Technology, Chongqing University of Posts and Telecommunications, Chongqing 400065, and also with the Chongqing Institute of Green and Intelligent Technology, Chinese Academy of Sciences, Chongqing 400714, China (e-mail: liuzhigangx @gmail.com).

G. X. Yuan is with the Faculty of Liberal Arts and Social Sciences, The Education University of Hong Kong, Hong Kong 999077, China (e-mail: 13397397ygx@gmail.com).

X. Luo is with the School of Computer Science and Technology, Dongguan University of Technology, Dongguan 523808, and also with the College of Computer and Information Science, Southwest University, Chongqing 400715, China (e-mail: luoxin21@gmail.com).

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Algorithm 1 SNCMF-Based Community Detector

Input: Network G, community count K, Parameters α and λ

Operation	**Cost**
Initialize: Adjacency matrix A; Degree matrix D with zeroes; LF matrices X and Y nonnegatively; Community set $C = \{C_1, C_2, \ldots, C_k\}$; Iteration count $t = 0$ and max-iteration-count T; α and λ with nonnegative constants	T_1
for $i = 1\sim n$	
for $l = 1\sim n$	
$D_{ii} = D_{ii} + A_{il}$	
end for	
end for	
while not converge **and** $t \le T$ **do**	T_2
Update X according to learning rule (12a)	
Update Y according to learning rule (12b)	
Update U according to learning rule (12c)	
$t = t + 1$	
end while	
for all $v_i \in V$	
Assign community affiliation according to (1) with U as a soft indicator	T_3
end for	

Output: Community set $C = \{C_1, C_2, \ldots, C_k\}$

$$O_{SNMF} = \min_{X\ge 0} \left\|XX^T - A\right\|_F^2 \tag{4}$$

where X can be found via an NMU algorithm, i.e.,

$$x_{ik} \leftarrow x_{ik}\left((AX)_{ik} \Big/ \left(XX^TX\right)_{ik}\right). \tag{5}$$

For stable convergence, the following learning rule with a linearly adjusted multiplicative term is commonly adopted:

$$x_{ik} \leftarrow x_{ik}\left(1/2 + (AX)_{ik} \Big/ \left(2XX^TX\right)_{ik}\right). \tag{6}$$

An SNMF-based model introduces strong symmetry into the model for precisely representing the symmetry of an undirected network. However, the unique LF matrix also leads to shrinkage of its LF space, which reduces its representation learning ability to A, as well as its performance for community detection.

Proposed community detector: A unified SNCMF model for highly-accurate community detection is presented. Its main idea is to introduce equality-constraint-based symmetry regularization along with the graph regularization into its objective, thereby achieving highly-accurate representation to the target network.

To make an NMF model own the symmetry of a target undirected network, we put the symmetry constraint terms, i.e., $||X{-}U||_F$ and $||Y{-}U||_F$, into its learning objective as

$$\varepsilon = \min_{X,Y,U\ge 0}\left(\left\|XY^T - A\right\|_F^2 + \alpha\left(\|X-U\|_F^2 + \|Y-U\|_F^2\right)\right) \tag{7}$$

where A denotes the adjacency matrix, X and Y denote LF matrices, U is a bridge variable used to transfer the information between X and Y, and $\alpha > 0$ is a coefficient to balance the symmetry constraint and loss error. With (7), X and Y are required to be equal for indirectly representing the symmetry of A. On the other hand, this constraint is loose and can be adjusted with α: as α increases, X and Y become closer to make SNCMF better illustrate A's symmetry, and vice versa. However, LF space is not shrunk with two LF matrices, i.e., X and Y, as well as one intermediate matrix U. Therefore, (7) ensures SNCMF's high representation learning ability to A.

To ensure the local invariance of a target network, we incorporate the graph regularization with the bridge variable U into (7) as

$$\varepsilon = \min_{X,Y,U\ge 0}\left(\left\|XY^T - A\right\|_F^2 + \alpha\left(\|X-U\|_F^2 + \|Y-U\|_F^2\right) + \lambda \mathrm{tr}\left(U^TLU\right)\right) \tag{8}$$

where λ adjusts the effect of graph regularization, L is the Laplacian matrix and $L = D - S$. Note that the diagonal matrix D's element is computed as $D_{ii} = \sum_l S_{il}$, and the similarity matrix S measures the closeness among node pairs. It should be pointed that A and S are numerically equal in our context.

Let $\Phi = [\varphi_{ik}]$, $\Gamma = [\gamma_{jk}]$ and $\mathrm{K} = [\kappa_{sk}]$ be the Lagrangian multipliers for nonnegative constraints of $X = [x_{ik}] \ge 0$, $Y = [y_{jk}] \ge 0$ and $U = [u_{sk}] \ge 0$, and the corresponding Lagrangian function is obtained

$$\begin{aligned}\mathcal{L} = {} & \mathrm{tr}\left(XY^TYX^T - 2AYX^T + AA^T\right) + \lambda\mathrm{tr}\left(U^TLU\right)\\ & + \alpha\mathrm{tr}\left(XX^T - 2XU^T\right) + \alpha\mathrm{tr}\left(YY^T - 2YU^T\right) + 2\alpha\mathrm{tr}\left(UU^T\right)\\ & + \mathrm{tr}\left(\Phi X^T\right) + \mathrm{tr}\left(\Gamma Y^T\right) + \mathrm{tr}\left(\mathrm{K}U^T\right)\end{aligned} \tag{9}$$

where the commonly accepted property that $||A||_F = \mathrm{tr}(AA^T)$ is adopted as $\mathrm{tr}(\cdot)$ calculates the trace of a matrix. Hence, the partial derivatives of $\mathcal{L}$ with respect to X, Y and U are achieved as

$$\begin{aligned}\partial\mathcal{L}/\partial X &= XY^TY - AY + \alpha X - \alpha U + \Phi\\ \partial\mathcal{L}/\partial Y &= YX^TX - A^TX + \alpha Y - \alpha U + \Gamma\\ \partial\mathcal{L}/\partial U &= -\alpha X - \alpha Y + 2\alpha U + \lambda\mathcal{L}U + K.\end{aligned} \tag{10}$$

By setting $\partial\mathcal{L}/\partial X = 0$, $\partial\mathcal{L}/\partial Y = 0$ and $\partial\mathcal{L}/\partial U = 0$, a local minimum of (8) is achieved. Thus, by combining (10) and the KKT conditions: $\varphi_{ik}x_{ik} = 0$, $\gamma_{jk}y_{jk} = 0$ and $\kappa_{sk}u_{sk} = 0$, $\forall i, j, s \in \{1\sim n\}$, $k \in \{1\sim K\}$, we thus achieve the following inferences related to x_{ik}, y_{jk} and u_{sk}:

$$-\left(XY^TY - AY + \alpha X - \alpha U\right)_{ik} x_{ik} = 0 \tag{11a}$$

$$-\left(YX^TX - A^TX + \alpha Y - \alpha U\right)_{jk} y_{jk} = 0 \tag{11b}$$

$$-(-\alpha X - \alpha Y + 2\alpha U + \lambda LU)_{sk} u_{sk} = 0. \tag{11c}$$

Then, we achieve the learning rules for x_{ik}, y_{jk} and u_{sk} for SNCMF

$$x_{ik} \leftarrow x_{ik}\left((AY + \alpha U)_{ik} \big/ \left(XY^TY + \alpha X\right)_{ik}\right) \tag{12a}$$

$$y_{jk} \leftarrow y_{jk}\left(\left(A^TX + \alpha U\right)_{jk} \big/ \left(YX^TX + \alpha Y\right)_{jk}\right) \tag{12b}$$

$$u_{sk} \leftarrow u_{sk}\left((\alpha(X+Y) + \lambda AU)_{sk} \big/ (2\alpha U + \lambda DU)_{sk}\right). \tag{12c}$$

Hereto, an SNCMF-based community detector is obtained. Note that we take U as the indicator matrix for final community division, since it actually shares the information from both X and Y.

With (12), we thus design an algorithm for the proposed SNCMF-based community detector in Algorithm 1. According to the details shown in Algorithm 1, the time cost of SNCMF can be summarized as: $T_{\mathrm{SNCMF}} = T_1 + T_2 + T_3 = \Theta(n^2) + \Theta(n^2TK) + \Theta(nK) \approx \Theta(n^2TK)$, where the lower-order-terms are dropped since $K \ll n$. In practice, since K and T are small and positive constants, the computational cost of SNCMF is approximately quadratic with the node count, which is equal to most existing NMF-based community detectors [2]−[9]. Note that the time cost of the model can be greatly reduced via the help of GPU.

Experiments: For evaluating the performance of SNCMF, six publicly-available networks from real applications are used in our experiments as shown in Table 1. Normalized Mutual Information (NMI) for measuring the similarity between the resulted community assignment and the ground-truth community information is taken as the indicator to evaluate the detection accuracy of involved models [5], [7], [8]. Note that the NMI is in the scale of (0, 1), as large NMI stands for high performance of a detection model.

Table 1. Details of Adopted Networks

No.	Networks	Nodes	Edges	K	Description
D1	Youtube [10]	11 144	36 186	40	Youtube online
D2	Friendster [10]	11 023	280 755	13	Friendster online
D3	LiveJournal [10]	7181	253 820	30	LiveJournal online
D4	Orkut [10]	11 751	270 667	5	Orkut online
D5	Amazon [10]	5304	16 701	85	Amazon product
D6	DBLP [10]	12 547	35 250	4	DBLP collaboration

We compare our method with nine benchmark and state-of-the-art models: NMF [2], GNMF [11], SNMF [5], GSNMF [5], NSED [12], SymNMF [13], DeepWalk [14], LINE [15], and HPNMF [6].

To obtain objective experimental results, hyper-parameters are set as follows. For LINE and DeepWalk, we perform the experiments by the default settings in their official toolkits. For graph regularized models, i.e., GNMF, GSNMF and SNCMF, the graph regularization coefficient λ is uniformly set at 100 according to [11]. For HPNMF, both of its parameters, i.e., λ and γ, are set at one, according to [6]. In addition, for SNCMF, we set its symmetry regularization parameter, i.e., α at 2^{-8} uniformly on all datasets.

In addition, the final experiment results are achieved by repeating each separate experiment 20 times with various initial hypotheses. Average NMI values of ten involved methods on six social networks are summarized in Table 2. Moreover, the corresponding statistical test results, i.e., Win/Loss counts, Friedman ranks, and p-value of

Table 2. Community Detection Performance (NMI%±STD%) of Tested Models on Each Network (✪ Indicates That SNCMF Win the Compared Models)

No.	NMF	GNMF	SNMF	GSNMF	NSED	SymNMF	DeepWalk	LINE	HPNMF	SNCMF
D1	17.26±2.58✪	38.99±2.47✪	16.95±2.10✪	49.45±2.18✪	18.44±1.44✪	18.59±1.85✪	18.43±0.6✪	17.84±1.39✪	49.10±2.38✪	**50.21±0.38**
D2	68.08±6.66✪	45.17±2.89✪	67.64±5.53✪	62.64±6.95✪	76.14±5.13✪	78.40±4.49✪	80.37±0.98✪	70.31±1.95✪	80.55±3.55✪	**81.85±4.03**
D3	23.47±3.87✪	49.89±2.41✪	33.78±4.49✪	**73.17±3.82**	19.79±5.19✪	42.22±2.81✪	48.38±2.71✪	42.33±1.59✪	66.82±0.39✪	69.45±3.59
D4	29.53±2.13✪	51.56±2.62✪	29.39±6.36✪	59.18±4.02✪	31.25±4.93✪	33.75±7.20✪	57.37±1.48✪	42.31±2.36✪	48.13±8.46✪	**66.76±7.29**
D5	42.21±1.46✪	67.54±1.23✪	45.58±1.72✪	62.29±4.25✪	38.17±1.64✪	47.17±2.07✪	51.98±1.49✪	48.23±1.65✪	61.21±3.93✪	**73.57±1.95**
D6	7.18±3.22✪	**19.18±4.86**	6.60±2.81✪	9.25±1.69✪	5.67±3.03✪	9.18±2.75✪	11.16±0.76✪	8.92±0.44✪	11.54±1.67✪	16.51±3.31
Win/Loss	6/0	5/1	6/0	5/1	6/0	6/0	6/0	6/0	6/0	—
Ranks*	8.4	4.2	8.6	4.0	8.6	6.2	4.0	6.2	3.4	**1.4**
p-value**	**0.0313**	0.0625	**0.0313**	0.0625	**0.0313**	**0.0313**	**0.0313**	**0.0313**	**0.0313**	—

*A lower Friedman rank value indicates a higher community detection accuracy. **The accepted hypotheses with a significance level of 0.05 are highlighted.

Wilcoxon test, are also provided in Table 2. From the results, we have the following findings:

1) Symmetry regularization enables an NMF-based community detector to achieve higher accuracy. As shown in Table 2, we see that SNCMF outperforms GNMF on five testing cases out of six in total significantly with a confidence level at 90%.

2) Symmetry and graph regularizations work compatibly to make SNCMF outperform its peers in terms of community detection accuracy. From the results, SNCMF achieves the best scores on four networks except D3 and D6. Besides, according to the Friedman statistical results, we observe that it achieves the lowest average Friedman rank among all tested methods, which means it obtains the best performance among its peers. In detail, SNCMF has significantly higher detection accuracy than NMF, SNMF, NSED, SymNMF, DeepWalk, LINE, and HPNMF with a confidence level at 95%, as well as GNMF and GSNMF with a confidence level at 90%, respectively.

Further, to check whether or not SNCMF can properly describe the symmetry of an undirected network, we plot the data distribution in the low-rank approximation $\hat{A}$ to D1 respectively generated by NMF, SNMF, and SNCMF in Fig. 2. From it, we can observe that: 1) NMF in Fig. 2(a) can not describe the symmetry of a target network at all; 2) SNMF in Fig. 2(d) precisely describes the symmetry of the target network; and 3) SNCMF can capture a portion of the target network' s symmetry, and its representation learning ability to the symmetry of the target network can be flexibly adjusted by α, as depicted in

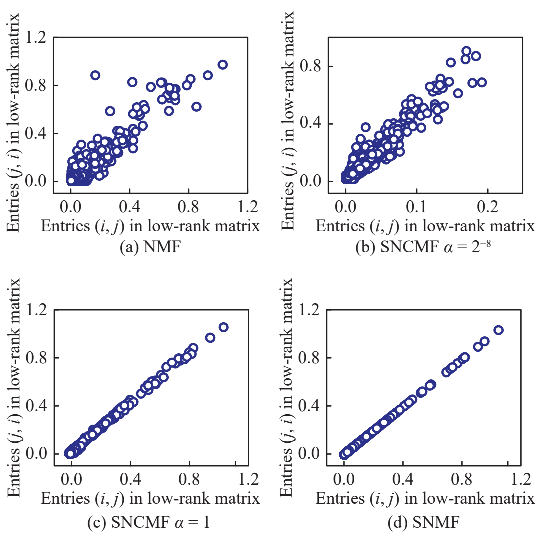

Fig. 2. Data distribution in $\hat{A}$ on D1.

Figs. 2(b) and 2(c).

Conclusions: This work presents a novel SNCMF model that adopts both symmetry and graph regularizations to accurately represent an undirected network. Empirical studies on six real-world social networks demonstrate that an SNCMF-based community detector achieves higher community detection accuracy than the benchmark and state-of-the-art methods. Thus, we conclude that the representation to the symmetry of a target undirected network and the LF space should be well balanced as in an SNCMF model for achieving highly accurate community detection results.

We plan to handle the problem of hyper-parameter adaptation via evolutionary computation techniques such as the particle swarm optimization (PSO) algorithm [16] in our future work.

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