Letter

Distributed Optimization Algorithm for Multi-Robot Formation with Virtual Reference Center

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Dear editor,

In this letter, we use a distributed optimization approach to solve a class of multi-robot formation problem with virtual reference center. We investigate the design and analysis of the constrained consensus algorithm to solve the optimization problem with a sum of objective functions with some local constraints. In the multi-robot system with virtual reference center, each robot has messages on its own constraints and objective function, as well as the message about the formation that interacts with the virtual reference center. At the same time, all the robots collaborate to find the minimum value of the function defined by the formation. To find the optimal formation, we propose an algorithm with fixed step size with better performance. In addition, we use a combination of the Hungarian assignment algorithm and the proposed formation algorithm to get the optimal matching formation of the multi-robot system.

Distributed formation control of multi-robot systems has emerged as an active research area over the past decades. This problem finds applications in different fields, such as path planning, goal searching, formation and rendezvous [1], [2]. Generally, the distributed formation optimization is described with a connected network of many robots. In the literature, distributed formation control of multi-robot systems has been extensively studied. For the problem of formation forming and changing, [3]–[5] use the coordination errors between robots to propose a distributed formation control strategy without assuming each robot knowing the complete state of the leader. Reference [6] proposes a control framework in a nonlinear multi-agent system to deal with the problem of distributed fault-tolerant containment control (FTCC). For the assignment strategy, [1] proposes a method to assign the best goal to each robot and calculate the collision-free path for each robot to its goal destination iteratively. The convex optimization strategy for a large-scale robot team is considered with both algorithm scalability and real-time performance in [7] and [8]. Besides, [9] investigates the dynamic task assignment for multi-robot system and propose two task assignment strategies.

The main contribution of this letter is the proposition of the distributed optimization with gradient projection (GP) algorithm to minimize the formation distance and the Hungarian algorithm to minimize the assignment cost function. The formation is controlled using the relative position between the robots and the virtual reference center. The simulations are presented to verify the effectiveness of the proposed algorithm with good convergence performance for obtaining the global optimal solution, especially for the case of large-scale formations.

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Problem description and formation algorithm: The optimization problem for the distributed multi-robot formation with m robots is described as

$$\min f(q,\chi) = \frac{1}{2}\sum_{i=1}^{m}\|p^i - q^i + \sum_{j=1}^{m}\chi_{ij}\Delta^{ij}\|^2 \tag{1}$$

$$\text{s.t.}\begin{cases}\sum_{i=1}^{m}\chi_{ij}=1,\ \sum_{j=1}^{m}\chi_{ij}=1,\ i,j=1,2,\ldots,m\\ q^i\in\Omega^i,\ i=1,2,\ldots,m\end{cases} \tag{2}$$

$$\chi_{ij}=\begin{cases}1, & \text{robot } i \text{ is assigned to goal } j\\ 0, & \text{robot } i \text{ is not assigned to goal } j\end{cases} \tag{3}$$

where $\|\cdot\|$ is the Euclidean norm, p^i is the initial coordinate of robot i, q^i is the coordinate of destination i. $\Delta^{ij}=\Delta^i-\Delta^j$ is the desired relative coordinate between robots i and j when they reach the final destinations, $\Delta^i\in\mathbb{R}^n$ represents the desired relative position for robot i to the virtual reference center, and Ω^i is a nonempty and closed convex set in $\mathbb{R}^n$ which represents the restricted area for robot i.

For the objective function in (1), as it is a quadratic function, it is easy to get that $f(q,\chi)$ is convex and the gradient ∇f is Lipschitz continuous with respect to q.

Assume that the desired relative position for robot i to the virtual reference center is known, so the desired formation for the multi-robot system is predescribed by $\Delta^i\in\mathbb{R}^n$. Then, we define $\Delta=(\Delta^1,\Delta^2,\ldots,\Delta^m)^T\in\mathbb{R}^{mn}$, $\tilde{q}=(q^1,q^2,\ldots,q^m)^T\in\mathbb{R}^{mn}$ with $q^i\in\mathbb{R}^n$ to denote an estimated solution of (1) by robot i. Let $L_m\in\mathbb{R}^{m\times m}$ be the Laplacian matrix of graph $\mathcal{G}$ and $L=L_m\otimes I_n\in\mathbb{R}^{mn\times mn}$, where $\otimes$ is the Kronecker product and I_n is n-dimensional identity matrix. Besides, a function $g(u)$ is called a projection operator from $\mathbb{R}^n$ to $S\subseteq\mathbb{R}^n$ if $g(u)=\arg_{v\in S}\min v-u$, where S is a closed convex set.

Lemma 1: Problem (1) with respect to q subject to constraints (2) and (3) is equivalent to the following optimization problem

$$\min\tilde{f}(\tilde{q})=\sum_{i=1}^{m}f^i\left(q^i\right)=\frac{1}{2}\sum_{i=1}^{m}\|p^i-q^i+\sum_{j=1}^{m}\chi_{ij}\Delta^{ij}\|^2$$

$$\text{s.t. } L(\tilde{q}-\Delta)=0,\ \tilde{q}\in\Omega \tag{4}$$

where $\Omega=\prod_{i=1}^{m}\Omega^i$ is the Cartesian product.

Proof: Since all the robots share the same coordinate of the virtual reference center, we have $q^1-\Delta^1=q^2-\Delta^2=\cdots=q^m-\Delta^m$, which follows the equality constraint $L(\tilde{q}-\Delta)=0$. ■

Lemma 2 [10]: Assume the graph $\mathcal{G}$ of the multi-robot system is undirected and connected, α is a positive constant. Then the optimal solution to (4) can be represented by $\tilde{q}^*\in\Omega\subseteq\mathbb{R}^{mn}$ if and only if there exists $\tilde{y}^*\in\mathbb{R}^{mn}$ such that

$$\begin{cases}\tilde{q}^*-g\left[\tilde{q}^*-\alpha\left(\nabla\tilde{f}(\tilde{q}^*)+L\tilde{y}^*\right)\right]=0\\ L(\tilde{q}^*-\Delta)=0.\end{cases} \tag{5}$$

We use the subscript to simplify the formulas of the iterations, $\tilde{q}_k=\tilde{q}(k)$, $\tilde{h}_k=\tilde{h}(k)$. From (5), the proposed algorithm to solve the equalities is described as

$$\begin{cases}\tilde{q}_{k+1}=g\left[\tilde{q}_k-\alpha\left(\nabla\tilde{f}(\tilde{q}_k)+\tilde{h}_k+L(\tilde{q}_k-\Delta)\right)\right]\\ \tilde{h}_{k+1}=\tilde{h}_k+L(\tilde{q}_{k+1}-\Delta)\end{cases} \tag{6}$$

where $\tilde{h}=\mathrm{vec}\left(h^1,h^2,\ldots,h^m\right)$ is the vectorization of matrix.

Robot i generates its new estimate at time $k+1$ according to the following formula:

$$\begin{cases} q_{k+1}^i = g^i\left[q_k^i - \alpha\left(\nabla f^i\left(q_k^i\right) + h_k^i + \sum_{j=1,j\neq i}^{m} a_{ij}\left(q_k^i - q_k^j - \Delta^{ij}\right)\right)\right] \\ h_{k+1}^i = h_k^i + \sum_{j=1,j\neq i}^{m} a_{ij}\left(q_{k+1}^i - q_{k+1}^j - \Delta^{ij}\right). \end{cases} \tag{7}$$

Let $\tilde{h}_k = L\tilde{y}_k$, then the algorithm (6) becomes

$$\begin{cases} \tilde{q}_{k+1} = g\left[\tilde{q}_k - \alpha\left(\nabla\tilde{f}(\tilde{q}_k) + L(\tilde{y}_k + \tilde{q}_k - \Delta)\right)\right] \\ \tilde{y}_{k+1} = \tilde{y}_k + \tilde{q}_{k+1} - \Delta. \end{cases} \tag{8}$$

Lemma 3 [10]: Assume that the initial conditions satisfy $\tilde{h}_0 = L\tilde{y}_0$ for any $\tilde{y}_0 \in \mathbb{R}^{mn}$, then algorithm (6) and algorithm (8) are equivalent.

Assume that $\tilde{q}^* \in \mathbb{R}^{mn}$ is an optimal solution for problem (4). According to Lemma 2, we can get that there exits $\tilde{y}^* \in \mathbb{R}^{mn}$ to satisfy the equations in (5).

Lemma 4: Following the above notations, we construct the following two functions:

$$M_1(\tilde{q}_k) = \|\tilde{q}_k - \tilde{q}^*\|^2,\ \ M_2(\tilde{y}_k) = (\tilde{y}_k - \tilde{y}^*)^T L(\tilde{y}_k - \tilde{y}^*)$$

then, we have

$$\begin{aligned} M_1(\tilde{q}_{k+1}) - M_1(\tilde{q}_k) \le& -(1-2\alpha)\tilde{q}_{k+1} - \tilde{q}_k{}^2 \\ & -2\alpha(\tilde{q}_{k+1} - \tilde{q}^*)^T L(\tilde{y}_k - \tilde{y}^* + \tilde{q}_k - \Delta) \end{aligned}$$

and

$$\begin{aligned} M_2(\tilde{y}_{k+1}) - M_2(\tilde{y}_k) =& 2(\tilde{q}_{k+1} - \tilde{q}^*)^T L(\tilde{y}_k - \tilde{y}^* + \tilde{q}_k - \Delta) \\ & + (\tilde{q}_{k+1} - \tilde{q}_k)^T L(\tilde{q}_{k+1} - \tilde{q}_k) - (\tilde{q}_k - \tilde{q}^*)^T L(\tilde{q}_k - \tilde{q}^*). \end{aligned}$$

Proof: Since the proof is similar with that of Lemma 4 in [11], we omit it here due to the page limit. ■

Theorem 1: Assume the graph $\mathcal{G}$ of the multi-robot system to be undirected and connected. The algorithm in (6) is convergent if $\alpha < 1/[2+\lambda_1(L)]$, where λ_1 is the maximum eigenvalue of matrix.

Proof: We construct a Lyapunov function as

$$V(\tilde{q}_k, \tilde{y}_k) = M_1(\tilde{q}_k) + \alpha M_2(\tilde{y}_k).$$

The definition of the M_1 and M_2 in V is the same as that in Lemma 4. From the inequalities in Lemma 4, assuming that I is the identity matrix, one gets

$$\begin{aligned} & V(\tilde{q}_{k+1}, \tilde{y}_{k+1}) - V(\tilde{q}_k, \tilde{y}_k) \\ & \le -(\tilde{q}_{k+1} - \tilde{q}_k)^T [I - \alpha(2I + L)](\tilde{q}_{k+1} - \tilde{q}_k) \\ & \quad - \alpha(\tilde{q}_k - \tilde{q}^*)^T L(\tilde{q}_k - \tilde{q}^*). \end{aligned}$$

The rest of the proof is similar with that of Theorem 2 in [11], so we omit it here due to the page limit. ■

Multi-robot task assignment: In the process of seeking the optimal solution, the idea of assignment is also used to find a way to minimize the total path length of all robots. To solve the problem, we choose the Hungarian algorithm (HA) [12], through which the optimal solution of the problem can be obtained just by matrix transformation.

We construct an efficiency matrix C in order to find the minimum value of f in (1) under the constraints (2) and (3), and $C_{ij} = \|p^i - q^j\|^2$. According to the basic principle of the Hungarian algorithm, a new matrix, which is obtained by subtracting or adding the same constant to each element of a row or column of C, has the same optimal solution as the original matrix C.

For solving the problem, we first subtract the minimum element of each row in C, and subtract the minimum element of each column in C. Then, we can find out the independent zero elements in the matrix (that is, only one 0 element in each row and column). If the number of independent zero elements is the same as the order of the matrix, the assignment is completed. Otherwise, the matrix needs to be adjusted and we need to keep calculating until the number of independent zero elements are equal to the order of the matrix. Finally, we transform all zero elements in matrix into one and the other elements into zero to obtain the optimal matrix. The position of the non-zero element represents the assigned result. For example, if $\chi_{ij} = 1$, it means that the ith robot matches the jth target point.

After obtaining the optimal solution, there is only one element in each row and column of the matrix with the value of 1 and the rest is 0. The position of the non-zero element represents the assigned result. For example, if $\chi_{ij} = 1$, it means that the ith robot matches the jth target point. Then we give the Hungarian task assignment algorithm combining with the proposed formation formulas in Algorithm 1.

Algorithm 1 Distributed Formation Algorithm with Virtual Reference Center

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Initialization:
Set p, q, h, q_0, (virtual reference center's position).
Calculate Δ, L and α.
Initialize the task assignment matrix χ.
Iterations:
while The objective function f is not convergent **do**
  for k = 1: Maximum number of iterations **do**
    for i = 1: m **do**
      the iterations in (7) in distributed manner.
    end for
  end for
  Calculate the assignment matrix χ using the Hungarian algorithm.
  Update q and h according to the new assignment matrix χ.
  Calculate the values of the objective function and output.
end while
return q

Simulation results: In this section, we will show the optimal matching between the initial formation and the optimal formation of the multi-robot system by a simulation example. We consider the formation problem on a two-dimensional plane. We set the target formation is composed of three concentric circles, the number of robots is 48, and the virtual reference center is the origin point. We obtain the optimal solution in (4) by the iterations in (6).

The final result is shown in Fig.1, in which the green points represent the initial positions of robots, the red points form the final formation, and the dotted lines represent the relationship and moving tracks between each robot and the target point. Furthermore, the value of the objective function f in (1) is shown in Fig.2, from which we can see that the objective function decreases as the algorithm iterates.

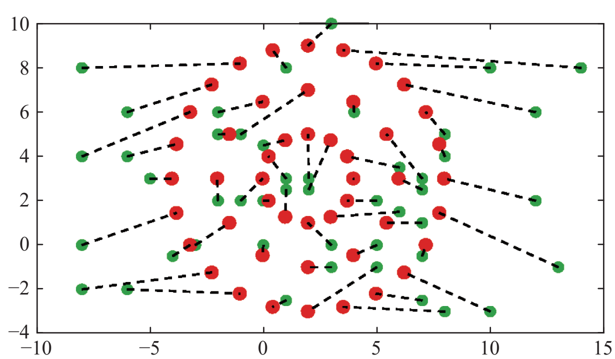

Fig. 1. Optimal formation solved by the proposed algorithm.

Compared with the algorithm in [11] for solving the above formation problem, the proposed algorithm gets the optimal formation after 10 projection iterations and 1 task assignment with 1 outer loop, but the algorithm in [11] needs 20 projection iterations and 1 task assignment with 300 outer loop. Therefore, the proposed algorithm simplifies the iterations with better performance for the scenario of large-scale formation.

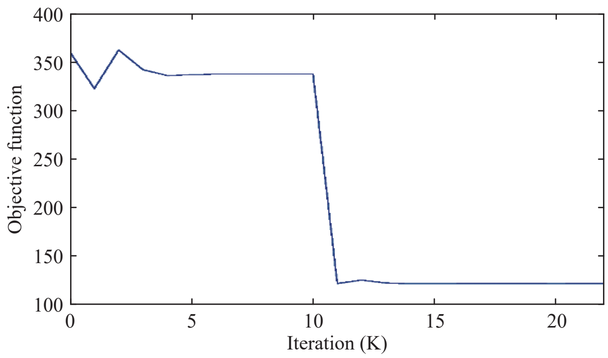

Fig. 2. The evolution of the objective function using the proposed algorithm.

Conclusions: In this letter we propose a fixed step algorithm for multi-robot formation with virtual reference center and analyze its convergence to obtain the global optimal solution. Furthermore, we combine the Hungarian task assignment algorithm with the proposed formation algorithm and apply it to the global formation optimization. The simulation shows the results and some properties of the proposed algorithm. The convergence of the algorithm can be seen to be fast for large-scale formations, which is a potential advantage of the algorithm.

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