# LMI Consensus Condition for Discrete-time Multi-agent Systems

Magdi S. Mahmoud and Gulam Dastagir Khan

***Abstract*—This paper examines a consensus problem in multi-agent discrete-time systems, where each agent can exchange information only from its neighbor agents. A decentralized protocol is designed for each agent to steer all agents to the same vector. The design condition is expressed in the form of a linear matrix inequality. Finally, a simulation example is presented and a comparison is made to demonstrate the effectiveness of the developed methodology.**

***Index Terms*—Consensus algorithms, discrete-time systems, linear matrix inequalities, multi-agent systems.**

## I. INTRODUCTION

RECENTLY the problem of distributed consensus in networked multi-agent systems has received significant attention due to its important applications [1]. The purpose of the consensus problem is to design a distributed protocol in the presence of limited information communication such that a group of agents achieves some agreement between the states [2]. A number of recent papers are devoted to the consensus of multiple LTI systems [3]−[10]. However, most of these results [11]−[18], [19] mainly focus on fixed interaction topology, rather than time-varying topology. How the switches of the interaction topology and agent dynamics jointly affect the collective behavior of the multi-agent system? Attempts to understand this issue had been hampered by the lack of suitable analysis tools. The results of Scardovi *et al.* [20] and Ni and Cheng [21] are mentioned here, because of their contributions dealing with switching topology.

The consensus of multi-agent systems under fixed and switching topology was studied in [21]. In [21], the dynamics of each agent and the leader are considered to be linear and in a continuous time domain. The design technique was based on Riccati inequality, algebraic graph theory and Lyapunov inequality. In [22] distributed consensus problem for multi-agent systems was considered in the discrete-time domain. The interaction topology among the agents was assumed to be switching and undirected.

Manuscript received April 28, 2015; accepted October 23, 2015. This work was supported by Deanship of Scientific research (CDSR) at KFUPM (RG-1316-1). Recommended by Associate Editor Yuanqing Xia. *(Corresponding author: Magdi S. Mahmoud.)*

Citation: M. S. Mahmoud and G. D. Khan, "LMI consensus condition for discrete-time multi-agent systems," *IEEE/CAA J. of Autom. Sinica*, vol. 5, no. 2, pp. 509−513, Mar. 2018.

M. S. Mahmoud and G. D. Khan are with the Systems Engineering Department, KFUPM, P. O. Box 5067, Dhahran 31261, Saudi Arabia (e-mail: msmahmoud@kfupm.edu.sa; dastagir.khan66@gmail.com).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/JAS.2016.7510016

By using Schur orthogonal transformation, the closed-loop system was decomposed into two subsystems: one for the leader and the other for a leader-following subsystem. Common Lyapunov function (CLF) approach was employed to investigate the consensus problem. In [23], authors proposed the control of a multi-agent system with state measurement and input disturbances. Distributed dynamic output feedback protocol and linear matrix inequality (LMI) approach were used to address control problem.

The main contributions and primary distinctions of this paper with other works can be given as follows.

1) A more realistic and accurate discrete time model is proposed which is relevant for many practical sampled data systems;

2) A new form for state-feedback control based on the aggregate Laplacian is proposed in this paper;

3) Sufficient conditions of stabilization are established using Lyapunov stability theory. The solution to the decentralized consensus problem is provided in terms of LMI;

4) A comparison is made between the developed algorithm and the existing results.

The organization of this paper is as follows: Firstly, notations, communication graph and preliminaries are introduced in Section II, which are useful throughout this paper. Then in Section III, stability analysis of a multi-agent system is carried out, where in Theorem 1, decentralized consensus problem in multi-agent system is solved by using Schur complement and controller gain matrix is obtained. Later in Section IV, a simulation example is presented along with some comparison to demonstrate the effectiveness of proposed technique. Finally, the conclusion is made.

## II. PROBLEM FORMULATION

### A. Notations

Throughout this paper, $\mathbb{R}^n$ is used to denote the n-dimensional Euclidean space equipped with $\|\cdot\|$, the standard $L_2$ norm on vectors or their induced norms on matrices and $\mathbb{R}^{\{m\times n\}}$ is the set of all $m \times n$ real matrices. Let $I_r$ be the unit matrix of order $r$. The superscript "$T$" denotes matrix transposition and "$\cdot$" denotes the transpose of corresponding elements introduced by symmetry. $X > 0$ means that it is real symmetric and positive definite; Moreover, $X > Y$ means $X - Y > 0$ Given a matrix $W$, let $\rho(W)$ denote its spectral radius. For any positive integer $N$, let $IN = \{1, \ldots, N\}, \mathrm{diag}(W_1, \ldots, W_N)$ is a block diagonal matrix with main diagonal block matrices $W_j, j \in IN$ and the off-diagonal blocks are zero matrices. The Kronecker product [24] of $A \in \mathbb{R}^{p\times q} := [a_{ij}]$ and $B \in \mathbb{R}^{m\times n}$ is denoted by $A \otimes B$ and is a $pm \times qn$ matrix defined by

$$A \otimes B := [a_{ij}B].$$

The Kronecker product further facilitates the manipulation of matrices by the following expansion properties

1) $(A \otimes B)(C \otimes D) = AC \otimes BD$
2) $(A \otimes B)^T = A^T \otimes B^T$
3) Let $A \in \mathbb{R}^{r\times s}$ and $B \in \mathbb{R}^{N\times N}$.
Then $(I_N \otimes A)(B \otimes I_s) = (B \otimes I_r)(I_N \otimes A) = B \otimes A$.

### *B. Communication Graph*

Depending on the information flow (unidirectional or bidirectional [24], a graph can be used to express the topology of communication network, either directed or undirected. Let $G = (V, E, A)$ be a weighted directed graph (digraph) of order $N$, where $V = \{v_1, \ldots, v_N\}$ is the set of nodes and $E \subseteq V \times V$ is the set of edges. The node indexes belong to a finite index set $I = \{1, 2, \ldots, N\}$. An edge of $G$ is denoted by $e_{ij} = (v_i, v_j)$, where the first element of $v_i$ of $v_{ij}$ is referred as tail of the edge and the other $v_j$ to be the head. Weighted adjacency matrix is denoted by $A = [a_{ij}] \in \mathbb{R}^{N\times N}$, where the adjacency elements associated with the edges are positive, that is $e_{ij} \in E \Leftrightarrow a_{ij} > 0$. A directed graph is said to be undirected if, it satisfies the condition that $a_{ij} = a_{ji}$ for any $i, j \in I$. The set of neighbors of node $v_i$ is denoted by $IN_i = \{v_j \in V : (v_i, v_j) \in E\}$ which the index is set of the agents from which the $i_{th}$ agent can obtain necessary information. The Laplacian $L = [l_{ij}]_{N\times N}$ associated with the directed graph is defined as

$$l_{ij} = \begin{cases} -1, & \text{if} \quad j \in IN_i \\ |IN_i|, & \text{if} \quad j = i \\ 0, & \text{otherwise} \end{cases}$$

where $IN_i$ denotes the number of neighbors of the $i$th agent (the in-degree of agent $i$). It turns out that $L = \Lambda - A \in \mathbb{R}^{n\times n}$, where $\Lambda = [\Lambda_{ij}]$ is a diagonal matrix with $\Lambda_{ii} = \sum_{j=1}^{n} a_{ij}$. Laplacian matrix $L$ has the property that, all the row sums of $L$ are zero and thus an eigenvector of $L$ associated with the zero eigenvalue is $1_n = [1, 1, 1, \ldots, 1]$.

*Lemma 1:* Given integers $n, N$ and $A \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{N\times N}$. Let $A_0 = I_N \otimes A$ and $R_0 = R \otimes I_n$. Then $R_0A_0 = A_0R_0$.

*Proof:* Using the expansion properties, we obtain

$$\begin{aligned} R_0A_0 &= (R \otimes I_n)(I_N \otimes A) = (RI_N) \otimes (I_nA) \\ &= (I_NR) \otimes (AI_n) = (I_N \otimes A)(R \otimes I_n) \\ &= A_0R_0. \end{aligned}$$ ■

### *C. Preliminaries*

The multi-agent system (MAS) under study is a group of $n$ agents $(1, \ldots, N)$ with the same dynamics:

$$x_i(k+1) = Ax_i(k) + Bu_i(k), i = 1, \ldots, N \tag{1}$$

where $x_i(k) \in \mathbb{R}^n$ is the state of agent $i$ and $u_i(k) \in \mathbb{R}^m$ is the associated control input, through which the interactions or coupling between agent $i$ and other agents are realized. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the state and input matrices, where $B$ is of full column rank. The state information is transmitted among these agents, and the agents together with the transmission channels form a network. The following assumption is used throughout the paper.

*Assumption 1:* The pair $(A,\ B)$ is stabilizable [4].

*Assumption 2:* The interconnection graph of system (1) has a directed spanning tree to guarantee that consensus among agents is possible [9].

Observe that the models of all $N$ agents can be stacked together in the form:

$$\begin{aligned} &x(k+1) = A_0x(k) + B_0u(k) \\ &A_0 = I_N \otimes A, B_0 = I_N \otimes A \end{aligned} \tag{2}$$

where $x = [x_1^T, x_2^T, \ldots, x_N^T] \in \mathbb{R}^{nN}, u = [u_1^T, u_2^T, \ldots, u_N^T] \in \mathbb{R}^{mN}$ are the group state and group control input, respectively. The consensus problem is to design the individual state feedback controls $u_i(k)$ such that all agents state converge to the same vector, that is

$$\sum_{k\to\infty} \|x_i(k) - x_j(k)\| \to 0, \forall i, j. \tag{3}$$

To facilitate further development, we introduce a decentralized control input as

$$\begin{aligned} u(k) = K_0L_0x(k), L_0 = L \otimes I_n \\ K_0 = \mathrm{diag}\{K_1, K_2, \ldots, K_N\}, K_i = \mathbb{R}^{n\times n} \end{aligned} \tag{4}$$

which ensures that the control input of the $i$th agent depends only on states of its neighbor agents and itself. It is important to emphasize that designing $K_0$ aims at steering all states to the same vector.

## III. STABILIZATION

On combining (2) and (4), the closed-loop system becomes:

$$x(k+1) = [A_0 + B_0K_0L_0](k). \tag{5}$$

Recalling the properties of $L$, there exists a vector $x_s(k)$ such that

$$x_s(k) = L_0(k) = 0 \Leftrightarrow x(k) = \alpha 1_n \tag{6}$$

for some $\alpha$. It follows with the help of Lemma 1, replacing $R_0$ by $L_0$, that

$$\begin{aligned} x_s(k+1) &= L_0x(k+1) = L_0[A_0 + B_0K_0L_0]x(k) \\ &= A_0L_0x(k) + L_0B_0K_0x_s(k) \\ &= [A_0 + L_0B_0K_0]x_s(k). \end{aligned} \tag{7}$$

Considering the stabilization of system (7), we invoke Lyapunov stability theory and reach the following result:

*Theorem 1:* The controller (4) solves the decentralized consensus problem in system (2) if and only if there exists matrices $0 < X, 0 < R, Y$ satisfying

$$\begin{bmatrix} -X + R & XA_0^t + Y^tB_0^tL_0^t \\ * & -X \end{bmatrix} < 0 \tag{8}$$

where $K_0$ the gain matrix is given by $K_0 = YX^{-1}$.

*Proof:* The stability of the closed-loop system (7) implies the existence of matrices $0 < P, 0 < Q$ such that

$$-P + [\hat{A}_0 + \hat{L}_0\hat{K}_0]^T P[\hat{A}_0 + \hat{L}_0 B\hat{K}_0] + Q < 0. \quad (9)$$

Invoking the Schur complements, inequality (9) is equivalent to

$$\begin{bmatrix} -P+Q & A_0^t + K_0^t B_0^t L_0^t \\ * & -P \end{bmatrix} < 0. \quad (10)$$

Introducing $X = P^{-1}, R = XQX$ and applying the congruent transformation $T = \mathrm{diag}\{X, X\}$ to (10), we readily obtain inequality (8) with $K_0 X = Y$.

## IV. Numerical Simulation

In this section, a numerical simulation is presented to illustrate the effectiveness of the developed methodology. Multi-agent system with six agents is considered, where each agent is modeled by following linear dynamics:

$$\begin{aligned} x_i(k+1) &= \begin{bmatrix} -2.42 & 1.423 & 5.2343 \\ -7.23 & 3.483 & 9.454 \\ 5.33 & 6.23 & -5.23 \end{bmatrix} x_i(k) \\ &+ \begin{bmatrix} -4.24 & -1.832 \\ -0.23 & 0.354 \\ 3.23 & 1.097 \end{bmatrix} u_i(k), \quad (11) \\ i &= 1, \ldots, 6. \end{aligned}$$

It is simple to test that matrix pair $(A, B)$ is stabilizable which satisfies Assumption 1. Consider the communication topology graph as given by Fig. 1. Accordingly the Laplacian matrix $L$ is defined as

$$\begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

whose non-zero eigenvalues are given by 1, 1.33760, $+0.5623j, -0.5623j, 2, 3.2347$. For the purpose of simulation, the following initial conditions were selected

$$x_i(0) = (0.3, 0.15, 0.002, -0.1, 0.05, -0.09)^T.$$

Considering Theorem 1, it turns out that the feasible solution of LMI (8) yields that gain matrix:

$$K_0 = \begin{bmatrix} -0.00724 & -0.002751 & 0.00153 \\ -0.0086311 & -0.081 & -0.02343 \end{bmatrix}.$$

The decentralized control input $u(k)$ (4) with feedback gain matrices given as above solves the consensus problem for the communication graph in Fig. 1. The states of the network (11) with the decentralized control input protocol (4) is depicted in Fig. 2. A similar problem has been discussed in [25], [26] where authors investigate average consensus problems in a class of second-order continuous-time multi-agent systems with switching and jointly connected topologies respectively with time-delay, in terms of linear matrix inequalities (LMIs). The simulation results of the state responses obtained in [25], [26] are shown in Figs. 3 and 4 respectively. Also in [27], average event triggered discrete consensus control for discrete-time multi-agent systems (MASs) is investigated. Stability criteria were established using Lyapunov matrix inequality. In order to avoid Zeno-behavior certain restriction has been imposed on event condition, which led to conservativeness in the developed algorithm. The simulation results of the state responses obtained in this case are shown in Fig. 5.

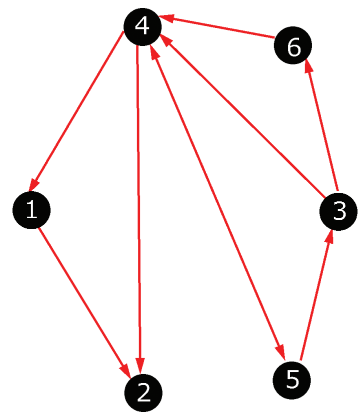



Fig. 1: The communication topology.

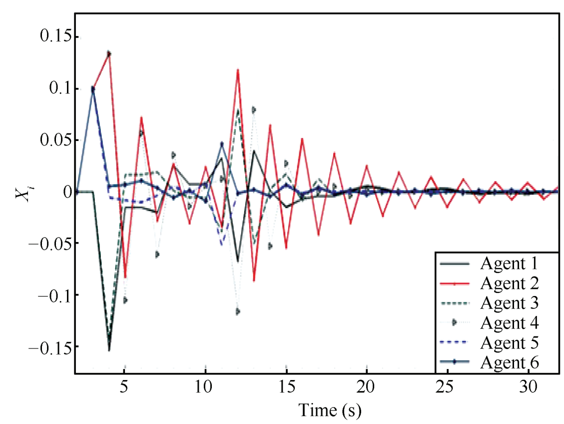



Fig. 2: The state of the network for $i$th agents where $i = 1, 2, 3, \ldots, 6$ under the decentralized control input.

Several observations can be made from Figs. 3–5. Firstly, our proposed control algorithm demonstrates that the consensus condition can be achieved asymptotically with six agents being steered to the same vector. Secondly, the proposed algorithm takes less time to achieve consensus of the multi-agent network. Thirdly, the MAS oscillates with smaller magnitude. Therefore, from the above simulation results, it can be concluded that the proposed technique of decentralized control input can be successfully employed to achieve a consensus of multi-agent network steered to the same vector.

## V. Conclusions

In this paper, a state feedback protocol is designed to solve the consensus problem in discrete-time multi-agent systems in terms of a linear matrix inequality. Feedback gain matrix is obtained by solving the simple LMI. Consensus condition

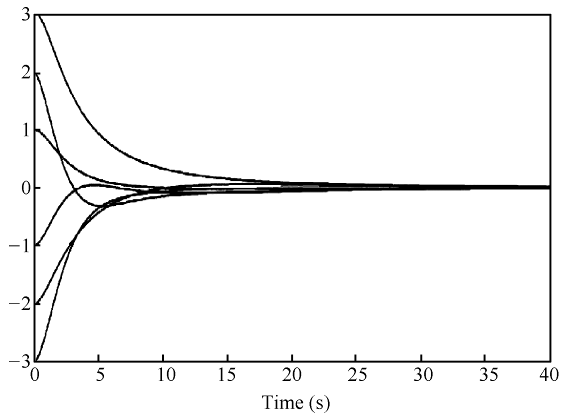



Fig. 3: The state of the network for $i$th agents, using the algorithm developed in [25].

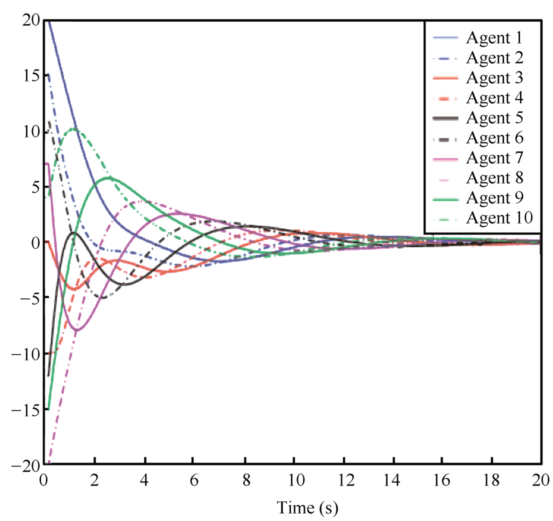



Fig. 4: The state of the network for $i$th agents, using the algorithm developed in [26].

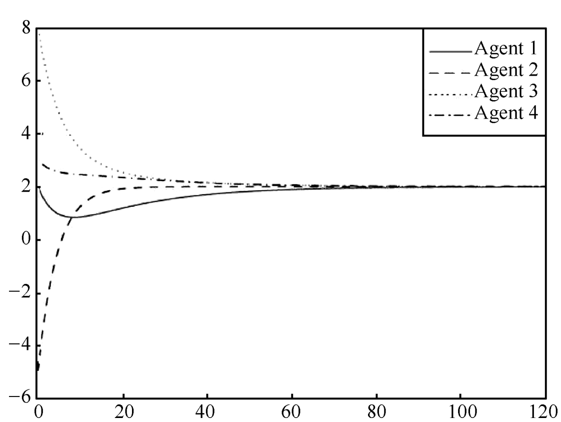



Fig. 5: The state of the network for $i$th agents, using the algorithm developed in [27].

is achieved with six agents being steered to the same vector. In order to show the improved performance of the proposed methodology, the simulation results obtained in this paper are compared with those in [25]−[27]. Future work will focus on solving observer based model where the dynamics of the followers are subjected to perturbations as well as leader-following tracking for multi-agent systems with nonlinear dynamics.

## REFERENCES

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215−233, Jan. 2007.

[2] M. S. Mahmoud, *Distributed Control and Filtering for Industrial Systems*. UK: IET Press, 2012.

[3] W. Ni and D. Z. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *Syst. Control Lett.*, vol. 59, no. 3−4, pp. 209−217, Mar. 2010.

[4] C. R. Wang and H. B. Ji, "Leader-following consensus of multi-agent systems under directed communication topology via distributed adaptive nonlinear protocol," *Syst. Control Lett.*, 2014, vol. 70, pp. 23−29, Aug. 2014.

[5] M. Defoort, T. Floquet, A. Kokosy, and W. Perruquetti, "Sliding-mode formation control for cooperative autonomous mobile robots," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 3944−3953, Nov. 2008.

[6] D. Cruz, J. McClintock, B. Perteet, O. A. A. Orqueda, Y. Cao, and R. Fierro, "Decentralized cooperative control — A multivehicle platform for research in networked embedded systems," *IEEE Control Syst.*, vol. 27, no. 3, pp. 58−78, Jun. 2007.

[7] X. H. Wang, V. Yadav, and S. N. Balakrishnan, "Cooperative UAV formation flying with obstacle/collision avoidance," *IEEE Trans. Control Syst. Technol.*, vol. 15, no. 4, pp. 672−679, July 2007.

[8] J. M. Chen, X. H. Cao, P. Cheng, Y. Xiao, and Y. X. Sun, "Distributed collaborative control for industrial automation with wireless sensor and actuator networks," *IEEE Trans. Industrial Electron.*, vol. 57, no. 12, pp. 4219−4230, Dec. 2010.

[9] R. M. Murray, "Recent research in cooperative control of multivehicle systems," *J. Dyn. Syst., Meas., Control*, vol. 129, no. 5, pp. 571−583, May 2007.

[10] W. Ren and C. Cao, *Distributed Coordination of Multi-agent Networks: Emergent Problems, Models, and Issues*. London: Springer-Verlag, 2011.

[11] X. Li, X. F. Wang, and G. R. Chen, "Pinning a complex dynamical network to its equilibrium," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 10, pp. 2074−2087, Oct. 2004.

[12] H. W. Zhang, F. L. Lewis, and A. Das, "Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback," *IEEE Trans. Automatic Control*, vol. 56, no. 8, pp. 1948−1952, Aug. 2011.

[13] R. X. Cui, S. S. Ge, and B. B. Ren, "Synchronized tracking control of multi-agent system with limited information," in *Proc. 49th IEEE Conference on Decision and Control*, Atlanta, GA, 2010. pp. 5480−5485

[14] Y. G. Hong, G. R. Chen, and L. Bushnell, "Distributed observers design for leader-following control of multi-agent networks," *Automatica*, vol. 44, no. 3, pp. 846−850, Mar. 2008.

[15] W. Ni and D. Z. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *Syst. Control Lett.*, vol. 59, no. 3−4, pp. 209−217, Mar. 2010.

[16] L. Moreau, “Stability of multiagent systems with time-dependent communication links,” *IEEE Trans. Automatic Control*, vol. 50, no. 2, pp. 169−182, Feb. 2005.

[17] W. Ren, R. W. Beard, and E. M. Atkins, “Information consensus in multivehicle cooperative control,” *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71−82, Apr. 2007.

[18] J. H. Wang and D. Z. Cheng, “Consensus of multi-agent systems with higher order dynamics,” in *Proc. Chinese Control Conference*, Hunan, China, 2007. pp. 761−765.

[19] W. Ren, R. W. Beard, and E. M. Atkins, “A survey of consensus problems in multi-agent coordination,” in *Proc. 2005 American Control Conference*, Portland, OR, pp. 1859−1864, 2005.

[20] L. Scardovi and R. Sepulchre, “Synchronization in networks of identical linear systems,” *Automatica*, vol. 45, pp. 2557−2562, Nov. 2009.

[21] W. Ni and D. Z. Cheng, “Leader-following consensus of multi-agent systems under fixed and switching topologies,” *Syst. Control Lett.*, vol. 59, no. 3−4, pp. 209−217, Mar. 2010.

[22] L. X. Gao, C. F. Tong and L. Y. Wang, “$H_\infty$ dynamic output feedback consensus control for discrete-time multi-agent systems with switching topology,” *Arab. J. Sci. Eng.*, vol. 39, no.2, pp. 1477−1478, Feb. 2013.

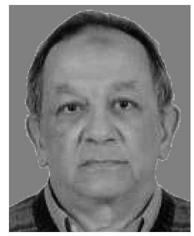
[23] L. Yang, Y. M. Jia, J. P. Du, and S. Y. Yuan, “Dynamic output feedback control for consensus of multi-agent systems: an $H_\infty$ approach,” in *Proc. American Control Conference*, St. Louis, MO, 2009. pp. 4470−4475.

[24] C. Godsil and G. F. Royle, *Algebraic Graph Theory*. New York: Springer-Verlag, 2001.

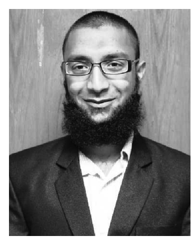
[25] P. Lin and Y. M. Jia, “Consensus of a class of second-order multi-agent systems with time-delay and jointly-connected topologies,” *IEEE Trans. Automatic Control*, vol. 55, no. 3, pp. 778−784, Mar. 2010.

[26] P. Lin and Y. M. Jia, “Average consensus in networks of multi-agents with both switching topology and coupling time-delay,” *Physica A*, vol. 387, no. 1, pp. 303−313, Jan. 2008.

[27] X. Chen and F. Hao, “Event-triggered average consensus control for discrete-time multi-agent systems,” *IET Control Theory Appl.*, vol. 6, no. 16, pp. 2493−2498, Nov. 2012.



**Magdi S. Mahmoud** obtained B. Sc. (Honors) in Communication Engineering, M. Sc. in Electronic Engineering and Ph. D. in Systems Engineering, all from Cairo University in 1968, 1972 and 1974, respectively. He has been Professor of Engineering since 1984. He is now a Distinguished Professor at KFUPM, Saudi Arabia. He was on the faculty at different universities worldwide including Egypt (CU, AUC), Kuwait (KU), UAE (UAEU), UK (UMIST), USA (Pitt, Case Western), Singapore (Nanyang) and Australia (Adelaide). He lectured in Venezuela (Caracas), Germany (Hanover), UK ((Kent), USA (UoSA), Canada (Montreal) and China (BIT, Yanshan). He is the principal author of thirty-six (36) books, inclusive book-chapters and the author/co-author of more than 520 peer-reviewed papers. He is the recipient of two national, one regional and several university prizes for outstanding research in engineering and applied mathematics. He is a fellow of the IEE, a senior member of the IEEE, the CEI (UK), and a registered consultant engineer of information engineering and systems (Egypt). He is currently actively engaged in teaching and research in the development of modern methodologies to distributed control and filtering, networked-control systems, triggering mechanisms in dynamical systems, fault-tolerant systems and information technology.



**Gulam Dastagir Khan** received his bachelor degree in Electronic and Instrumentation Engineering from Osmania university, Hyderabad, India in 2009. He did his master degree in Control Instrumentation and Systems Engineering from King Fahd University of Petroleum and Minerals, Saudi Arabia in 2013. His research interests include Markov models for buffered packet, game theory, and multiagent networked controlled systems.