An Iterative Relaxation Approach to the Solution of the Hamilton-Jacobi-Bellman-Isaacs Equation in Nonlinear Optimal Control

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***Abstract*—In this paper, we propose an iterative relaxation method for solving the Hamilton-Jacobi-Bellman-Isaacs equation (HJBIE) arising in deterministic optimal control of affine nonlinear systems. Local convergence of the method is established under fairly mild assumptions, and examples are solved to demonstrate the effectiveness of the method. An extension of the approach to Lyapunov equations is also discussed. The preliminary results presented are promising, and it is hoped that the approach will ultimately develop into an efficient computational tool for solving the HJBIEs.**

***Index Terms*—Affine nonlinear system, bounded continuous function, convergence, Hamilton-Jacobi-Bellman-Isaacs equation, Lyapunov equation, relaxation method, Riccati equation.**

I. Introduction

OPTIMAL control problems can be solved using either the minimum principle of Pontryagin [1], [2] or the dynamic programming principle of Bellman, also known as Hamilton-Jacobi theory [1], [2]. The latter approach involves the solution of a nonlinear partial-differential equation also known as the Hamilton-Jacobi equation, which was originally derived by Hamilton [3] in 1834 from a mechanics perspective, and later on improved by Jacobi [3] in 1838. The Hamilton-Jacobi equation (HJE) gives necessary and sufficient conditions for the existence of an optimal control for both constrained and unconstrained problems. Later on, Bellman [1], [2] developed the discrete-time equivalent of the Hamilton-Jacobi equation also known as the dynamic programming principle, and it became known as the Hamilton-Jacobi-Bellman equation. Finally, in 1952, Isaacs [4], [5] further modified it in the context of N-player non-zero sum differential games, and it became known as the Hamilton-Jacobi-Bellman-Isaacs equation (HJBIE).

Unfortunately, a bottle-neck in the practical application of nonlinear optimal control theory is the difficulty in solving the HJBIE [6]−[17] There are no closed-form solutions for it, and no proven established systematic numerical approaches for solving it.

Several attempts have however been made to find computationally sound methods for solving the HJBIE, and there is a vast literature on the subject. The reader can refer to [18], [19] for an excellent literature review of past approaches. In Glad [14], Lukes [15], Isidori [20], [21], and Huang [22], Taylor series approximations are presented. While in [16]−[19], [23] Gallerkin and other basis functions expansions are used. More recently, in [24], [25] policy iterations are used to derive iterative solutions in closed-form. This method is also similar in spirit with the ones presented in [18], [19]. However, the validity of the method has only been demonstrated with scalar systems. A similar recursive approach is utilized in [7].

In addition, attempts to find exact and analytical approaches for solving the HJBIE have also been made in [5], [8]−[10], [26]. The approaches attempt to convert the HJBIEs to algebraic equations, the solution of which can yield the gradient of the desired scalar function. In fact, these were some of the first attempts to derive closed-form solutions to the HJBIEs. However, the success of the approaches in [8], [9] is significantly undermined by the difficulty of solving the resulting discriminant equations. Alternatively, in [26] an attempt is made to find the algebraic gradient from the maximal involutive ideal that contains the Hamiltonian function of the corresponding Hamiltonian system. This approach is mainly useful for Hamiltonians in polynomial form.

On the other hand, in [12], [22] neural network or basis functions and Taylor series approximations respectively, are utilized to obtain recursive solutions to the discrete-time problem. These methods share a lot of spirit with the one originally developed in [23], and are so far some of the most tangible approaches to the discrete-time problem.

The problems with most of the methods so far presented are two fold: 1) they are computationally expensive, requiring the solution of a system of N nonlinear equations, for N basis functions; 2) they do not approximate the scalar function directly, but instead, approximate its gradient. This can lead to undesirable solutions. Consequently, more efficient approaches are still required and desirable.

Thus, in this paper, we present yet a new iterative approach to the solution of the HJBIEs. We apply fixed-point iterations [27], [28] in Banach spaces with a relaxation parameter, to successively approximate the scalar value-function directly, as opposed to its gradient, and we establish convergence of the approach under fairly mild assumptions. The approach is computationally efficient and can easily be automated using symbolic algebra packages such as MAPLE, MATHEMATICA, and MATLAB. It is hoped that the results presented in this paper and subsequent papers will represent the first attempts for establishing a systematic computationally efficient approach for solving the HJBIE which hitherto has been lacking.

The rest of the paper is organized as follows. In Section II, we begin with preliminaries and problem definition. Then in Section III, we develop the iterative relaxation method for the HJBIE in deterministic nonlinear optimal control. Convergence results for the method are established and some examples are presented. Then in Section IV, we extend the results of Section III to Lyapunov equations, and an example is also worked-out. Finally, conclusions and suggestions for

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future work are presented in Section V.

II. Preliminaries

We consider the time-invariant or stationary HJBIEs associated with the infinite-horizon optimal control of the following smooth affine nonlinear state-space system Σ defined over a subset $\mathcal{X} \subset \mathbb{R}^n$ in coordinates $(x_1, \dots, x_n)$:

$$\Sigma : \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u; \quad x(t_0) = x_0 \\ z = \begin{bmatrix} h(x) \\ u \end{bmatrix} \end{cases} \tag{1}$$

where $x = (x_1, \dots, x_n)^T \in \mathcal{X}$ is the state vector; $w \in \mathcal{W} \subset \mathbb{R}^s$ is the disturbance into the system which belongs to the set $\mathcal{W}$ of admissible disturbances; $u \in \mathcal{U}$ is the control input, which belongs to the set $\mathcal{U} \subset \mathbb{R}^p$ of admissible controls; and $z \in \mathbb{R}^r$ is an objective or error function. Whereas $f : \mathcal{X} \to \mathbb{R}^n$, $g_1 : \mathcal{X} \to \mathbb{R}^{n\times s}$, and $g_2 : \mathcal{X} \to \mathbb{R}^{n\times p}$, $h : \mathcal{X} \to \mathbb{R}^m$. We also assume that for $u \in \mathcal{U}$, and any $x(t_0) \in \mathcal{X}$, there exist smooth solutions to the system Σ [29]. In addition, $x^0 = 0$ is an equilibrium point of the system such that for $w = 0$, $u = 0$, $f(x^0) = 0$.

The time-invariant HJBIE associated with the above system either for the $\mathcal{H}_2$ optimal control [2], [4], [18] or for the $\mathcal{H}_\infty$ optimal control [5], [21], can generally be represented by

$$HJBI(V) := V_x(x)f(x) + \frac{1}{2}V_x(x)Q(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) = 0, \quad V(x^0) = 0 \tag{2}$$

for some smooth function $V : \mathcal{X} \to \mathbb{R}$, where V_x represents the row vector of partial derivatives of V with respect to x, a smooth matrix function $Q : \mathcal{X} \to \mathcal{M}^{n\times n}(\mathcal{X})$, where $\mathcal{M}^{n\times n}$ is the ring of $n \times n$ matrices over $\mathcal{X}$, and for some smooth output function $h : \mathcal{X} \to \mathbb{R}^m$. For instance, in the case of the state-feedback nonlinear $\mathcal{H}_\infty$ control problem [5], the matrix function $Q(x) = [g_1(x)g_1^T(x)/\tilde{\gamma}^2 - g_2(x)g_2^T(x)]$, $\tilde{\gamma} > 0$, while for the $\mathcal{H}_2$ problem [14], [18], [19], $Q(x) = -g_2(x)g_2^T(x)$.

Our aim in this paper is to find iteratively an approximate solution of the HJBIE (2) associated with the optimal control of system (1) in a region $\Omega \subset \mathcal{X}$. We consider the Banach space of bounded real continuous functions from Ω to $\mathbb{R}$ with the supremum norm, $\mathbf{BC}((\Omega, \mathbb{R}), \sup|.|)$, which for brevity we shall simply denote by $\mathbf{BC}(\Omega)$. However, we shall focus particular attention to a subset of this set containing functions that are also smooth, i.e., $\mathcal{V}(\Omega) := C^\infty \cap \mathbf{BC}(\Omega)$.

In the sequel, we construct smooth maps of the form $\overline{HJBI} : \mathcal{V}(\Omega) \to \mathcal{V}(\Omega)$ such that $\overline{HJBI}$ has a fixed-point in $\mathcal{V}(\Omega)$. We also show that starting from any element $V^0 \in \mathcal{V}(\Omega)$, a relaxation method can be applied to find the fixed-point V^*, and moreover, convergence to this fixed-point is shown to be quadratic.

III. Relaxation Method for the HJBIE

Our aim in this section is to develop a gradient-free iterative or successive approximation method for solving HJBIE arising in optimal control problems for affine nonlinear systems. Notice that, since V does not appear explicitly in (2), a gradient-based method such as the steepest-descent or Newton's method [11], [27], [28], their variants will not be suitable to use at this point. However, the relaxation method becomes very handy in this respect. Accordingly, define the following iterative inverse map by

$$V^{k+1}(x) = V^k(x) - \gamma(k)\left(HJBI(V^k)(x) - HJBI(V^{k-1})(x)\right) \quad k = 1, 2, 3, \dots \tag{3}$$

where $0 < \gamma(k) < 1$ is the relaxation parameter which is chosen carefully to improve convergence.

Based on the iterative formula (3), we proceed to establish convergence results for the approximation error $|V^{k+1}(x) - V^*(x)|$, and for V^k, $k = 0, 1, \dots$ to a smooth solution of the HJBIE (2). The following assumption on the system (1) will be essential.

Assumption 1: For the nonlinear system Σ (1), the following hold:

1) there exists a solution $V^* \in \mathcal{V}(\Omega)$ to the HJBIE (2) for the system, i.e., $V^* \in C^\infty(\Omega)$ and $\sup_\Omega |V^*(x)| < \infty$;

2) $\exists\, 0 < \kappa_2, \kappa_3 < \infty$ (real constants) such that

$$\sup_\Omega \|f(x)\| \le \kappa_2 \tag{4}$$

$$\sup_\Omega \|Q(x)\| \le \kappa_3. \tag{5}$$

Proposition 1: Consider the HJBIE (2) and let Assumption 1 be satisfied by the system. Suppose in addition, the solution V^* to the HJBIE (2) is such that

$$\sup_\Omega \|V_x^*(x)\| \le c_0. \tag{6}$$

Then, starting with an approximation $V^0 \in \mathcal{V}(\Omega)$, the approximation error at every iteration of the formula (3) remains point-wise bounded for all $x \in \bar{\Omega}_r := \{x : \|x - x^0\| \le r\} \subset \Omega$ (r small).

Proof: From (3) and noting that $HJBI(V^*) = 0$, we have

$$|V^{k+1}(x) - V^*(x)| \le |V^k(x) - V^*(x)| + \gamma(k)\Big(|HJBI(V^k)(x) - HJBI(V^*)(x)| + |HJBI(V^{k-1})(x) - HJBI(V^*)(x)|\Big). \tag{7}$$

Now from (2),

$$|HJBI(V^k)(x) - HJBI(V^*)(x)| \le |V_x^k(x)f(x) - V_x^*(x)f(x)| + \frac{1}{2}\left|V_x^k(x)Q(x)(V_x^k)^T(x) - V_x^*(x)Q(x)(V_x^*)^T(x)\right|. \tag{8}$$

Observe also that

$$\frac{1}{2}V_x^k(x)Q(x)(V_x^k)^T(x) - \frac{1}{2}V_x^*(x)Q(x)(V_x^*)^T(x) = V_x^*(x)Q(x)(V_x^k(x) - V_x^*(x)) + \frac{1}{2}(V_x^k(x) - V_x^*(x))Q(x)(V_x^k(x) - V_x^*(x))^T. \tag{9}$$

Therefore, using (9), (8) in (7), we have

$$\begin{aligned}
&|V^{k+1}(x) - V^*(x)| \\
&\le |V^k(x) - V^*(x)| \\
&\quad + \gamma(k)\Big\{\|V_x^k(x) - V_x^*(x)\|\|f(x)\| \\
&\quad + \|V_x^{k-1}(x) - V_x^*(x)\|\|f(x)\| \\
&\quad + \|Q(x)\|V_x^*(x)\|\|V_x^k(x) - V_x^*(x)\| \\
&\quad + \|Q(x)\|V_x^*(x)\|\|V_x^{k-1}(x) - V_x^*(x)\| \\
&\quad + \frac{\|Q(x)\|}{2}\|V_x^k(x) - V_x^*(x)\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\|Q(x)\|}{2}\|V_x^{k-1}(x) - V_x^*(x)\|^2\Big\} \\
&\le |V^k(x) - V^*(x)| \\
&\quad + \gamma(k)\Big\{(c_0\kappa_3 + \kappa_2)\|V_x^k(x) - V_x^*(x)\| \\
&\quad + \frac{\kappa_3}{2}\|V_x^k(x) - V_x^*(x)\|^2 \\
&\quad + (c_0\kappa_3 + \kappa_2)\|V_x^{k-1}(x) - V_x^*(x)\| \\
&\quad + \frac{\kappa_3}{2}\|V_x^{k-1}(x) - V_x^*(x)\|^2\Big\}. \qquad (10)
\end{aligned}
$$

It is desired to compute smooth successive approximations $V^k, k = 1, \dots$ to the solution V^* of (2) in the neighborhood Ω_r. Thus, the difference $V_x^k(x) - V_x^*(x)$ can be estimated as

$$
\begin{aligned}
\|V_x^k(x) - V_x^*(x)\| \le\ & \|V_x^k(x) - V_x^k(x^0)\| \\
&+\|V_x^k(x^0) - V_x^*(x^0)\| \\
&+\|V_x^*(x^0) - V_x^*(x)\|. \qquad (11)
\end{aligned}
$$

If $V^0(x)$ is smooth, then the iterative formula (3) generates smooth (except possibly at isolated points) successive approximations V^k to the solution V^* of (2). Thus, for $\|x-x^0\| < r$, $\exists\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that

$$
\begin{aligned}
\|V_x^k(x) - V_x^*(x)\| &\le \varepsilon_1 + \varepsilon_2 + \|V_x^k(x^0) - V_x^*(x^0)\| \\
&\le \varepsilon + \|V_x^k(x^0) - V_x^*(x^0)\| \qquad (12)
\end{aligned}
$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$. The last term in (10) can be estimated from a first-order Taylor approximation of the difference $V^k - V^*$ around x^0, as

$$
\begin{aligned}
V^k(x) - V^*(x) =\ & V^k(x^0) - V^*(x^0) \\
&+(V_x^k(x^0) - V_x^*(x^0))(x - x^0) \\
&+O(\|x - x^0\|^2)
\end{aligned}
$$

for all x in the neighborhood Ω_r. Therefore, by the triangle-inequality

$$
\begin{aligned}
&\sup_{\bar\Omega_r}|V^k(x) - V^*(x)| + |V^k(x^0) - V^*(x^0)| \\
&\quad \ge r\|V_x^k(x^0) - V_x^*(x^0)\| \qquad (13)
\end{aligned}
$$

Consequently, using (13) in (12), we have

$$
\begin{aligned}
&\|V_x^k(x) - V_x^*(x)\| \\
&\le \varepsilon + \frac{1}{r}\sup_{\bar\Omega_r}\Big(|V^k(x) - V^*(x)| + |V^k(x^0) - V^*(x^0)|\Big). \quad (14)
\end{aligned}
$$

Finally, using (14) in (10), we get

$$
\begin{aligned}
&|V^{k+1}(x) - V^*(x)| \\
&\le |V^k(x) - V^*(x)| \\
&\quad +\gamma(k)\sup_{\bar\Omega_r}\Big\{K_0\Big(\varepsilon + \frac{1}{r}[|V^k(x) - V^*(x)| + c_1(k)]\Big) \\
&\quad +\frac{\kappa_3}{2}\left(\varepsilon + \frac{1}{r}[|V^k(x) - V^*(x)| + c_1(k)]\right)^2\Big\} \\
&\quad +\gamma(k)\sup_{\bar\Omega_r}\Big\{K_0\Big(\varepsilon + \frac{1}{r}[|V^{k-1}(x) - V^*(x)| + c_1(k)]\Big) \\
&\quad +\frac{\kappa_3}{2}\left(\varepsilon + \frac{1}{r}[|V^{k-1}(x) - V^*(x)| + c_1(k)]\right)^2\Big\} \\
&= 2\gamma(k)\left(K_0(\varepsilon + \frac{c_1(k)}{r}) + \frac{\kappa_3}{2}(\varepsilon + \frac{c_1(k)}{r})^2\right) \\
&\quad + \left[1 + \gamma(k)\left(\kappa_3(\varepsilon + \frac{c_1(k)}{r}) + \frac{K_0}{r}\right)\right] \\
&\quad \times \sup_{\bar\Omega_r}|V^k(x) - V^*(x)| + \frac{\gamma(k)\kappa_3}{2r^2}\sup_{\bar\Omega_r}|V^k(x) - V^*(x)|^2 \\
&\quad + \left[1 + \gamma(k)\left(\kappa_3(\varepsilon + \frac{c_1(k)}{r}) + \frac{K_0}{r}\right)\right] \\
&\quad \times \sup_{\bar\Omega_r}|V^{k-1}(x) - V^*(x)| + \frac{\gamma(k)\kappa_3}{2r^2}\sup_{\bar\Omega_r}|V^{k-1}(x) - V^*(x)|^2 \\
&= \sup_{\bar\Omega_r}\Big\{K_1(k) + K_2(k)|V^k(x) - V^*(x)| \\
&\qquad +K_3(k)|V^k(x) - V^*(x)|^2 + K_4(k) \\
&\qquad +K_5(k)|V^{k-1}(x) - V^*(x)| \\
&\qquad +K_6(k)|V^{k-1}(x) - V^*(x)|^2\Big\} \qquad (15)
\end{aligned}
$$

where $c_1(k) = |V^k(x^0) - V^*(x^0)|$

$$
\begin{aligned}
K_0 &= (c_0\kappa_3 + \kappa_2) \\
K_1(k) &= \gamma(k)\left(K_0(\varepsilon + c_1(k)) + \frac{\kappa_3}{2}(\varepsilon + \frac{c_1(k)}{r})^2\right) \\
K_2(k) &= \left[1 + \gamma(k)\left(\kappa_3(\varepsilon + \frac{c_1(k)}{r}) + \frac{K_0}{r}\right)\right] \\
K_3(k) &= \frac{\gamma(k)\kappa_3}{2r^2} \\
K_4(k) &= K_1(k) \\
K_5(k) &= K_2(k) - 1 \\
K_6(k) &= K_3(k).
\end{aligned}
$$

This shows that the iteration error is bounded; for if we start with $k = 1$, we see that the error $|V^2(x) - V^*(x)|$ is point-wise bounded by $|V^1(x) - V^*(x)|$ and $|V^0(x) - V^*(x)|$. Similarly, the error $|V^2(x) - V^*(x)|$ is point-wise bounded by $|V^1(x) - V^*(x)|$, and so on. Note also that, the above result holds for $\tilde r = r + \epsilon$, ϵ small, and thus for $\bar\Omega_r$. ■

We summarize next the main convergence result of the method.

Theorem 1: Consider the HJBIE (2) and the problem of finding the scalar function $V : \mathbb{R}^n \to \mathbb{R}$ that solves it. Suppose all the assumptions of Proposition 1 hold, and in addition, suppose $\alpha_2(k) = \beta_2(k) = \sqrt{K_3(k)} < 1$ in $\bar\Omega_r$. Then, the iterative formula (3) starting with smooth approximations $V^0, V^1 \in \mathcal{V}(\Omega)$ converges uniformly and quadratically to a smooth solution $V^* \in \mathcal{V}(\bar\Omega_r)$ of (2).

Proof: From the proof of Proposition 1, inequality (15) can without any loss of generality be represented as

$$
\begin{aligned}
&|V^{k+1}(x) - V^*(x)| \\
&\le \sup_{\bar\Omega_r}\Big(\alpha_1(k) + \alpha_2(k)|V^k(x) - V^*(x)|\Big)^2 - \alpha_0(k) \\
&\quad + \sup_{\bar\Omega_r}\Big(\beta_1(k) + \beta_2(k)|V^{k-1}(x) - V^*(x)|\Big)^2 - \beta_0(k) \quad (16)
\end{aligned}
$$

for some constants $\alpha_1(k) = K_2(k)/2\sqrt{K_3}$, $\alpha_0(k) = K_2^2(k)/(2K_3(k)) - K_1(k)$, $\beta_1(k) = K_5(k)/2\sqrt{K_6(k)}$,

$\beta_0(k)K_5^2(k)/(2K_6) - K_4(k)$. Applying now (16) inductively for $k, k-1, \ldots, 1, 0$, we have

$$
\begin{aligned}
&|V^{k+1}(x) - V^*(x)| \\
\leq\ & \sup_{\bar{\Omega}_r}\Big[\alpha_1(k) - \alpha_2(k)\alpha_0(k-1) + \alpha_2(k)\Big(\alpha_1(k-1) \\
&+\alpha_2(k-1)|V^{k-1}(x) - V^*(x)|\Big)^2\Big]^2 - \alpha_0(k) \\
&+\sup_{\bar{\Omega}_r}\Big[\beta_1(k) - \beta_2(k)\beta_0(k-1) + \beta_2(k)\Big(\beta_1(k-1) \\
&+\beta_2(k-1)|V^{k-2}(x) - V^*(x)|\Big)^2\Big]^2 - \beta_0(k) \\
\leq\ & \sup_{\bar{\Omega}_r}\Big[\alpha_1(k) - \alpha_2(k)\alpha_0(k-1) \\
&+\alpha_2(k)\Big(\alpha_1(k-1) - \alpha_2(k)\alpha_0(k-2) \\
&+\alpha_2(k-1)\{\alpha_1(k-2) - \alpha_2(k-1)\alpha_0(k-3) \\
&+\alpha_2(k-1)|V^{k-2}(x) - V^*(x)|\}^2\Big)^2\Big]^2 - \alpha_0(k) \\
&+\Big[\beta_1(k) - \beta_2(k)\beta_0(k-1) \\
&+\beta_2(k-1)\Big(\beta_1(k-1) - \beta_2(k-1)\beta_0(k-2) \\
&+\beta_2(k-1)\{\beta_1(k-2) - \beta_2(k-1)\beta_0(k-3) \\
&+\beta_2(k-2)|V^{k-3}(x) - V^*(x)|\}^2\Big)^2\Big]^2 - \beta_0(k) \\
\leq\ & \sup_{\bar{\Omega}_r}\Big[\alpha_1(k) - \alpha_2(k)\alpha_0(k-1) \\
&+\alpha_2(k)\Big(\alpha_1(k-1) - \alpha_2(k)\alpha_0(k-2) \\
&+\alpha_2(k)\Big(\alpha_1(k-2) - \alpha_2(k)\alpha_0(k-3) + \cdots \\
&+\alpha_2(k)\Big(\alpha_1(1) - \alpha_2(k)\alpha_0(0) + \alpha_2(0)|V^0(x) - V^*(x)|\Big)^2 \\
&\ldots\Big)^2\Big)^2\Big]^2 - \alpha_0(k) \\
&+\sup_{\bar{\Omega}_r}\Big[\beta_1(k) - \beta_2(k)\beta_0(k-1) + \beta_2(k)\Big(\beta_1(k-1) \\
&-\beta_2(k-1)\beta_0(k-2) + \beta_2(k-1)\Big(\beta_1(k-2) \\
&-\beta_2(k)\beta_0(k-3) + \ldots + \beta_2(3)\Big(\beta_1(2) - \beta_2(2)\beta_0(1) \\
&+\beta_2(1)|V^0(x) - V^*(x)|\Big)^2\ldots\Big)^2\Big)^2\Big]^2 - \beta_0(1). \qquad (17)
\end{aligned}
$$

Taking the limit as $k \to \infty$ in the above inequality (17) and since $\alpha_2(k) = \beta_2(k) < 1$, all terms of $|V^0(x) - V^*(x)|$ go to 0 and we have

$$
\lim_{k\to\infty} |V^{k+1}(x) - V^*(x)| \leq \Upsilon(\alpha_0(\infty), \alpha_1(\infty), \alpha_2(\infty), \beta_0(\infty), \beta_1(\infty), \beta_2(\infty))
$$

a constant. This implies uniform convergence of the approximations V^k to the solution V^*, which may differ from it by a constant. However, application of the boundary condition in (2) guarantees that this constant is zero. Finally, by (3), $V^k(x)$ is smooth on $\partial\Omega_r$ and therefore $V^k \in \mathcal{V}(\bar{\Omega}_r)$. Moreover, since $\mathcal{V}(\bar{\Omega}_r)$ is a Banach space, then V^k converges to a smooth solution $V^* \in \mathcal{V}(\bar{\Omega}_r)$. ■

Remark 1: We notice also from inequality (15) that, if we let $r \to \infty$, then the iteration error satisfies

$$
\begin{aligned}
&|V^{k+1}(x) - V^*(x)| \\
\leq\ & \sup_{\bar{\Omega}_r}\Big\{K_1(k) + K_4(k) + K_2(k)|V^k(x) - V^*(x)| \\
&+K_5(k)|V^{k-1}(x) - V^*(x)|\Big\}
\end{aligned}
$$

with

$$
K_1(k) \to \gamma(k)\left(K_0(\varepsilon + c_1(k)) + \frac{\kappa_3\varepsilon}{2}\right)
$$
$$
K_2(k) \to (1 + \gamma(k)\kappa_3\varepsilon).
$$

However, since $K_2(k) > 1$, it means that the algorithm does not converge.

Remark 2: The relaxation parameter $\gamma(k)$ is chosen to improve the convergence. Usually, $0 < \gamma(k) < 2$. If $0 < \gamma(k) < 1$ we have under-relaxation, and this makes a non-convergent system converges. Alternatively, if $1 < \gamma(k) < 2$ we have over-relaxation, and this used to speed up the convergence of algorithm.

Remark 3: The iterative formula (2) requires one function evaluation, i.e., $HJBIE(V^k)$ in each iteration. This requires the evaluation of one quadratic-form (see 2) together with two vector scalar-products and polynomial addition operations. Hence, the computational time of the algorithm is of the order of $O(n^2)$.

We specialize the above results to linear systems and the corresponding Riccati equation. Consider the following linear system:

$$
\Sigma_l : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u; \quad x(t_0) = x_0 \\ z = \begin{bmatrix} Hx \\ u \end{bmatrix} \end{cases} \qquad (18)
$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times s}$, $B_2 \in \mathbb{R}^{n\times p}$, $H \in \mathbb{R}^{r\times n}$, and the corresponding Riccati equation arising in the quadratic optimal control of the system [1], [2], [30], [31]

$$
RIC(P) := A^T P + PA + PQP + H^T H = 0 \qquad (19)
$$

where $Q = -B_2^T B_2^T$. Let now $\mathcal{P} := \{n \times n$ real symmetric matrices$\}$. Then, application of the formula (3) with $V^k(x) = x^T P^k x/2$, leads to the following recursive inverse map $\overline{RIC} : \mathcal{P} \to \mathcal{P}$ for (19):

$$
\begin{aligned}
P^{k+1} = P^k - \gamma(k)\Big[&(A^T P^k + P^k A + P^k Q P^k + H^T H) \\
&-(A^T P^{k-1} + P^{k-1}A + P^{k-1}QP^{k-1} + H^T H)\Big] \qquad (20)
\end{aligned}
$$

It then follows that, if $P^* \in \mathcal{P}$ is a solution of (19), then

$$
\begin{aligned}
&P^{k+1} - P^* \\
=\ & P^k - \gamma(k)(A^T P^k + P^k A + P^k Q P^k + H^T H) \\
&-P^* + \gamma(k)(A^T P^* + P^* A + P^* Q P^* + H^T H) \\
&-\gamma(k)(A^T P^{k-1} + P^{k-1} A + P^{k-1} Q P^{k-1} + H^T H) \\
&+\gamma(k)(A^T P^* + P^* A + P^* Q P^* + H^T H) \\
=\ & (P^k - P^*) - \gamma(k)\Big[A^T(P^k - P^*) \\
&+(P^k - P^*)A + (P^k - P^*)Q(P^k - P^*) \\
&+(P^k - P^*)QP^* + P^* Q(P^k - P^*)\Big] \\
&-\gamma(k)\Big[A^T(P^{k-1} - P^*) + (P^{k-1} - P^*)A \\
&+(P^{k-1} - P^*)Q(P^{k-1} - P^*)
\end{aligned}
$$

$$+(P^{k-1}-P^*)QP^*+P^*Q(P^{k-1}-P^*)\Big] \quad (21)$$

Therefore,

$$\begin{aligned}&\|P^{k+1}-P^*\|\\ \leq\ & \|P^k-P^*\|\Big(1+2|\gamma(k)|\|A\|+2|\gamma(k)|\|Q\|\|P^*\|\Big)\\ &+\|Q\||\gamma(k)|\|P^k-P^*\|^2\\ &+\|P^{k-1}-P^*\|\Big(2|\gamma(k)|\|A\|+2|\gamma(k)|\|Q\|\|P^*\|\Big)\\ &+|\gamma(k)|\|Q\|\|P^{k-1}-P^*\|^2. \quad (22)\end{aligned}$$

The above inequality (22), can further be represented as

$$\begin{aligned}\|P^{k+1}-P^*\| &= (\bar\alpha_1(k)+\bar\alpha_2(k)\|P^k-P^*\|)^2-\bar\alpha_0(k)\\ &\quad+(\bar\beta_1(k)+\bar\beta_2(k)\|P^{k-1}-P^*\|)^2-\bar\beta_0(k)\end{aligned}$$

where

$$\begin{aligned}\bar\alpha_2(k)&=\sqrt{|\gamma(k)|\|Q\|}\\ \bar\alpha_1(k)&=\frac{[1+2|\gamma(k)|\|A\|+2|\gamma(k)|\|Q\|\|P^*\|]}{2\bar\alpha_2(k)}\\ \bar\alpha_0(k)&=\bar\alpha_1^2(k)\\ \bar\beta_2(k)&=\bar\alpha_2(k)\\ \bar\beta_1&=\frac{[2|\gamma(k)|\|A\|+2|\gamma(k)|\|Q\|\|P^*\|]}{2\bar\alpha_2(k)}\\ \bar\beta_0(k)&=\bar\beta_1^2(k)\end{aligned}$$

Thus, inequality (23) is the linear equivalent of (17), and if $\bar\alpha_2=\bar\beta_2<1$, then convergence of the approximations $\{P^k\}$ to P^* can be established from the result of Theorem 1. This result is now summarized in the following corollary to the Theorem.

Corollary 1: Consider the Riccati equation (19), and suppose there exists a symmetric solution P^* to it. In addition, suppose for the system Σ_l, $\gamma(k)$ can be chosen so that $\bar\alpha_2(k)=\bar\beta_2<1$. Then, starting with an initial approximation $P^0\in\mathcal{P}$, the iterative formula (20) converges quadratically to a solution $P^*\in\mathcal{P}$ of the Riccati equation (19).

Remark 4: The above recursive formula (20) and algorithm is similar in spirit to the ones proposed in [32]−[34].

IV. Extension to Lyapunov Equations

It is well-known that Lyapunov equations are special cases of HJBIEs [1], [2], [35]. In this section, we discuss how the basic relaxation algorithm (3) can be extended to solve Lyapunov equations that arise in certain factorization problems for nonlinear systems [36]. For the nonlinear system (1), we consider the Lyapunov equation [36]:

$$\begin{aligned}LYAP(\widetilde V)(x) &:= \widetilde V_x(x)f(x)+\frac{1}{2}h^T(x)h(x)=0\\ &\widetilde V(x)>0,\ x\neq x^0,\ \widetilde V(x^0)=0 \quad (23)\end{aligned}$$

for some smooth function $\widetilde V:\Omega\to\mathbb{R}$.

Adapting the iterative formula (3) to the above Lyapunov equation (23), we have the following recursion

$$\begin{aligned}&\widetilde V^{k+1}(x)\\ &=\widetilde V^k(x)-\tilde\gamma(k)[LYAP(\widetilde V^k)(x)-LYAP(\widetilde V^{k-1})(x)].\end{aligned}$$

Consequently, we have the following result on the convergence of this iterative procedure.

Proposition 2: Consider the Lyapunov equation (23) and let Assumption 3.1 (b) hold for the system. In addition, let $\bar\Omega_r:=\{x:\|x-x^0\|\leq r\}\subset\Omega$ (r small), and suppose $\frac{|\gamma(k)|\kappa_2}{r}<1$. Then, starting with an approximation $\widetilde V^0\in\mathcal{V}(\Omega)$, the successive approximation (24) converges to a smooth solution $\widetilde V^*\in\mathcal{V}(\bar\Omega_r)$.

Proof: From (24), we have

$$\begin{aligned}|\widetilde V^{k+1}(x)-\widetilde V^k(x)|&\leq|\tilde\gamma(k)||\widetilde V_x^k(x)f(x)-\widetilde V_x^{k-1}(x)f(x)|\\ &\leq|\tilde\gamma(k)|\|\widetilde V_x^k(x)-\widetilde V_x^{k-1}(x)\|\|f(x)\|.\end{aligned}$$

By inequality (14), for any two successive approximations $\widetilde V^k$, $\widetilde V^{k-1}$,

$$\begin{aligned}&\|\widetilde V_x^k(x)-\widetilde V_x^{k-1}(x)\|\\ &\leq\varepsilon+\frac{1}{r}\sup_{\bar\Omega_r}\Big(|\widetilde V^k(x)-\widetilde V^{k-1}(x)|+|\widetilde V^k(x^0)-\widetilde V^{k-1}(x^0)|\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}&|\widetilde V^{k+1}(x)-\widetilde V^k(x)|\\ &\leq|\tilde\gamma(k)|\varepsilon\kappa_2+\frac{|\tilde\gamma(k)|\kappa_2}{r}\sup_{\bar\Omega_r}\Big(|\widetilde V^k(x)-\widetilde V^{k-1}(x)|+\tilde c_1(k)\Big)\\ &\leq\frac{|\tilde\gamma(k)|\kappa_2}{r}\sup_{\bar\Omega_r}|\widetilde V^k(x)-\widetilde V^{k-1}(x)|+|\tilde\gamma(k)|\kappa_2\left(\frac{1}{r}\tilde c_1(k)+\varepsilon\right)\end{aligned}$$

where $\tilde c_1(k)=|\widetilde V^k(x^0)-\widetilde V^{k-1}(x^0)|$. The above inequality (25) implies $\widetilde V^k\to\widetilde V^*$ linearly. Moreover, by (23), $\widetilde V^k$ is smooth on $\bar\Omega_r$, and since $\mathcal{V}(\bar\Omega_r)$ is a Banach space, then $\{\widetilde V^k\}$ converges in $\mathcal{V}(\bar\Omega_r)$, i.e., $\widetilde V^*\in\mathcal{V}(\bar\Omega_r)$. ■

V. Computational Examples and Simulations

In this section, we present some simple examples and simulation results to demonstrate the effectiveness of the methods developed.

Example 1: Consider the following system and the example:

$$\begin{aligned}\dot x_1&=-x_1^3+x_2\\ \dot x_2&=-x_1-x_2+u\\ z&=[x_1\ \ x_2\ \ u]^T.\end{aligned}$$

The resulting HJBIE for the $\mathcal{H}_2$ problem is

$$\begin{aligned}HJBI(V)&=V_{x_1}(x)(-x_1^3+x_2)+V_{x_2}(x)(-x_1-x_2)\\ &\quad-\frac{1}{2}V_{x_2}^2(x)+\frac{1}{2}(x_1^2+x_2^2)=0, V(0)=0 \quad (24)\end{aligned}$$

where

$$Q(x)=-\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

Taking $\Omega=\{x|x_1^2+x_2^2\leq1\}$, $r=1$, $\kappa_2=2$, $\kappa_3=1$, $\gamma(k)=1/2$ then, $\alpha_2=1/2<1$. Now taking $V^0(x)=0$, $V^1(x)=(x_1^2+x_2^2)/2$, and applying three iterations of formula (3), we get

$$\begin{aligned}HJBI(V^0)(x)&=\frac{1}{2}(x_1^2+x_2^2)\\ HJBI(V^1)(x)&=\frac{1}{2}x_1^2-x_1^4-x_2^2\\ V^2(x)&=\frac{1}{2}x_1^2+\frac{1}{2}x_1^4+\frac{5}{4}x_2^2\\ HJBI(V^2)(x)&=-2x_1^6-x_1^4-\frac{41}{8}x_2^2+2x_1^3x_2\\ &\quad-\frac{3}{2}x_1x_2+\frac{1}{2}x_1^2\end{aligned}$$

$$
\begin{aligned}
V^3(x) &= x_1^6 + \frac{1}{2}x_1^4 - x_1^3 x_2 + \frac{1}{2}x_1^2 + \frac{3}{4}x_1 x_2 + \frac{53}{16}x_2^2,\\
HJBI(V^3)(x) &= -6x_1^8 - \frac{5}{2}x_1^6 + 6x_1^5 + 3x_1^5 x_2 + \frac{3}{4}x_1^4\\
&\quad + \frac{71}{8}x_1^3 x_2 - 3x_1^2 x_2^2 - \frac{13}{16}x_1^2\\
&\quad - \frac{363}{32}x_1 x_2 - \frac{43}{8}x_2^2 - \frac{2809}{128}x_2^4\\
V^4(x) &= 3x_1^8 + \frac{5}{4}x_1^6 - \frac{1}{3}x_1^5 - \frac{3}{8}x_1^4 - \frac{71}{16}x_1^3 x_2\\
&\quad + \frac{3}{2}x_1^2 x_2^2 + \frac{47}{32}x_2^2 + \frac{363}{64}x_1 x_2 + \frac{57}{16}x_2^2 + \frac{2809}{128}x_2^4.
\end{aligned}
$$

Remark 5: What we see in the above example is that, all the approximations V^2, V^3, V^4, are locally positive-definite. Thus, starting with a positive-definite initial approximations V^0, V^1, the algorithm has the tendency to maintain the sign definiteness of the successive approximations. Whereas, the values of the function $HJBI(V^k)$, $k = 1, 2, 3$ are increasingly negative-semidefinite. That is, the successive approximations $V^k, k = 2, 3, 4$, try to satisfy $HJBI(V) \le 0$ or the inequality form of the HJBIE. It is well-known that a solution for the latter is also a solution for the former [5].

The corresponding control laws for the above approximations V^2, V^3, V^4 are given by

$$u^i = -g_2^T(x){V^i_x}^T(x), \quad i = 2, 3, 4$$

The system was simulated with the above control laws and the results of the simulation are shown respectively on Figs. 1–3. The result of the simulation shows that the new iterative method can indeed find stabilizing solutions of the HJBIE.

We consider the following example to solve the Lyapunov equation.

Example 2: Reconsider the system of Example 1 above. The corresponding Lyapunov equation (23) for the system is

$$
\begin{aligned}
LYAP(\widetilde{V}) = \widetilde{V}_{x_1}(x)(-x_1^3 + x_2) + \widetilde{V}_{x_2}(x)(-x_1 - x_2)\\
+\frac{1}{2}(x_1^2 + x_2^2) = 0
\end{aligned}
$$

where taking $\Omega = \{x | x_1^2 + x_2^2 \le 1\}$, $r = 1$, $\kappa_2 = 2$, $\gamma(k) = 1/4$. Then, $|\gamma(k)|\kappa_2/r = 1/2 < 1$. Now taking $\widetilde{V}^0(x) = 0$, $\widetilde{V}^1(x) = (x_1^2 + x_2^2)/2$, and applying three iterations of formula (3), we get

$$
\begin{aligned}
LYAP(\widetilde{V}^0)(x) &= \frac{1}{2}(x_1^2 + x_2^2)\\
LYAP(\widetilde{V}^1)(x) &= \frac{1}{2}x_1^2 - x_1^4 - x_2^2\\
\widetilde{V}^2(x) &= \frac{1}{4}x_1^4 + \frac{3}{4}x_1^2 + \frac{1}{2}x_2^2\\
LYAP(\widetilde{V}^2)(x) &= -2x_1^6 - x_1^4 - \frac{41}{8}x_2^2 + 2x_1^3 x_2\\
&\quad - \frac{3}{2}x_1 x_2 + \frac{1}{2}x_1^2\\
\widetilde{V}^3(x) &= \frac{1}{4}x_1^6 + \frac{3}{2}x_1^4 - \frac{1}{4}x_1^3 x_2 + \frac{1}{2}x_1^2 - \frac{1}{8}x_1 x_2\\
&\quad + \frac{3}{4}x_2^2.
\end{aligned}
$$

Remark 6: Notice in the above example, the approximations $\widetilde{V}^2$, $\widetilde{V}^3$, $\widetilde{V}^4$, are locally positive-definite.

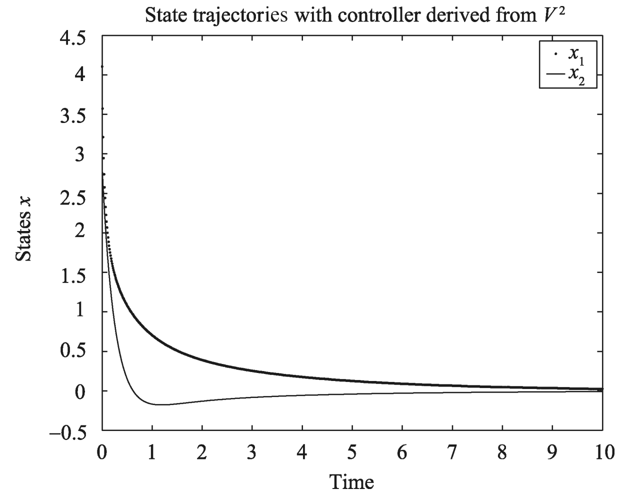

Fig. 1. Closed-loop state trajectories with control law u^2.

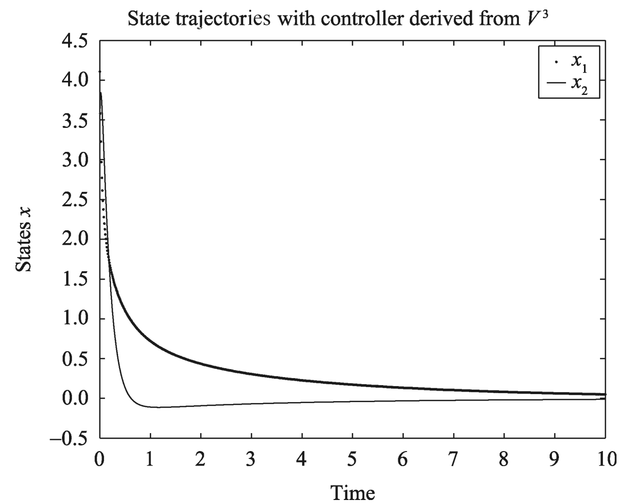

Fig. 2. Closed-loop state trajectories with control law u^3.

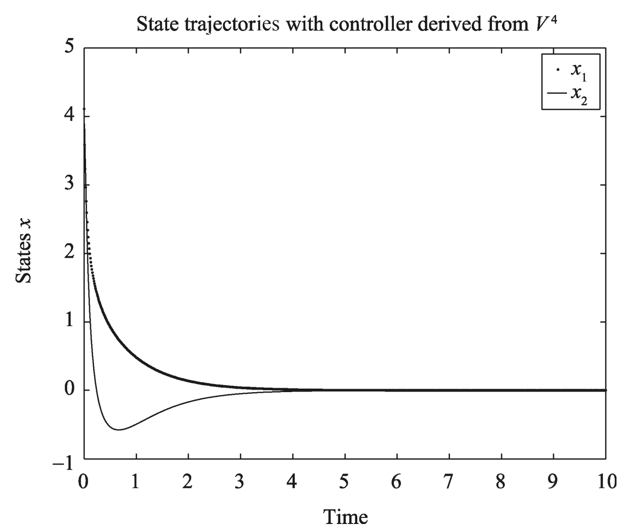

Fig. 3. Closed-loop state trajectories with control law u^4.

VI. CONCLUSION

In this paper, we have presented a new iterative approach for solving the HJBIE arising in the optimal control of affine nonlinear systems. Fixed-point iterations in Banach spaces and a relaxation method are combined to successively approximate the scalar value-function directly, and convergence results for the approach have been established under fairly mild conditions. Some examples have also been worked-out to demonstrate the effectiveness of the approach. In addition, the approach can easily be automated using symbolic algebra packages. Applications or extensions to Lyapunov equations have also been discussed.

However, the results presented are really preliminary, and it will require many experimentation to establish conclusively it's usefulness and computational efficiency. It is sufficient here to observe that from the few examples that have been solved, the approach will be suited for affine nonlinear systems with polynomial nonlinearities. As such, future efforts will go into computational experimentations with the method on practical nonlinear systems such as the nonlinear bench-mark problem [16], as well as seeking improvements and refinements of the algorithm. It will also be worth-while to see if convergence of the algorithm can be established under much weaker and more general assumptions.

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