Suboptimal Robust Stabilization of Discrete-time Mismatched Nonlinear System

Niladri Sekhar Tripathy, Indra Narayan Kar, *Senior Member, IEEE*, and Kolin Paul, *Member, IEEE*

***Abstract*—This paper proposes a discrete-time robust control technique for an uncertain nonlinear system. The uncertainty mainly affects the system dynamics due to mismatched parameter variation which is bounded by a predefined known function. In order to compensate the effect of uncertainty, a robust control input is derived by formulating an equivalent optimal control problem for a virtual nominal system with a modified cost-functional. To derive the stabilizing control law for a mismatched system, this paper introduces another control input named as virtual input. This virtual input is not applied directly to stabilize the uncertain system, rather it is used to define a sufficient condition. To solve the nonlinear optimal control problem, a discrete-time general Hamilton-Jacobi-Bellman (DT-GHJB) equation is considered and it is approximated numerically through a neural network (NN) implementation. The approximated solution of DT-GHJB is used to compute the suboptimal control input for the virtual system. The suboptimal inputs for the virtual system ensure the asymptotic stability of the closed-loop uncertain system. A numerical example is illustrated with simulation results to prove the efficacy of the proposed control algorithm.**

***Index Terms*—Discrete-time general Hamilton-Jacobi-Bellman (DT-HJB) equation, discrete-time optimal control, discrete-time robust control, mismatched uncertainty, nonlinear optimal control.**

I. Introduction

REQUIREMENT of exact system model to design a feedback control law is the primary shortcoming of the classical feedback control technique. An uncertain system model is a more realistic representation and has far greater significance over the exact system model. However, there are open problems of designing a control law to deal with system uncertainties. To deal with parametric uncertainty, F. Lin and D. Wang *et al.* have proposed a continuous-time robust control technique for both linear and nonlinear system [1]−[5]. In both the cases, they have formulated an equivalent optimal control problem to derive the proposed robust control input. The optimal control problem is solved based on the nominal dynamics by minimizing a quadratic cost-functional with the knowledge of uncertainty bound. Similar concepts are used for nonlinear continuous system in [6], [7], where a non-quadratic cost-functional is considered. The discrete-time version of the proposed robust-optimal control approach is still an open problem. Recently, Wang *et al.* [8] have extended the Lin's approach [1]−[3] for a discrete-time nonlinear system. To realize the robust control law, the assumption in their work is that the physical system is affected by matched uncertainty (i.e., uncertainty is in the range space of input matrix [9]−[11]). But there are several physical systems like maglev suspension system [12], [13], aircraft engine system [14], the movement control of truck-trailer problem [15], where the so-called matching condition does not hold. Therefore considering mismatched uncertainty in both state and input functions is a more realistic control problem. In general, it is known that the existence of stabilizing control law can be guaranteed for matched uncertainty but not so for mismatched system.

In this paper, a discrete-time robust control technique for uncertain nonlinear system is proposed. The system is primarily affected by mismatched uncertainty due to bounded parametric variation. To stabilize such systems, a robust control law is derived by solving a nonlinear optimal control problem for nominal virtual system with a cost-functional. To solve the nonlinear optimal control problem, the solution of a discrete-time general Hamilton-Jacobi-Bellman (DT-GHJB) equation is approximated using a neural network implementation. Based on the approximated solution of DT-GHJB, the cost-functional and control inputs are estimated. The block diagram representation of proposed control approach is shown in Fig. 1.

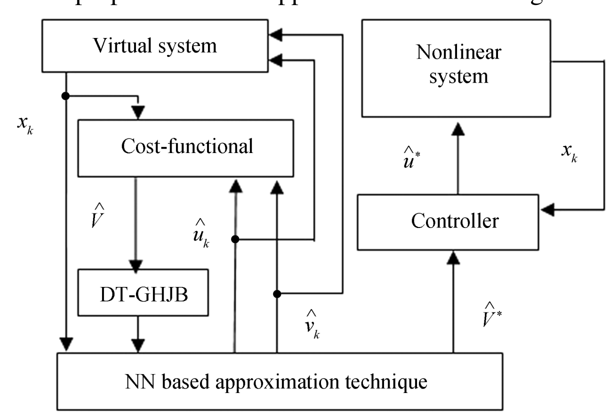

Fig. 1. The block diagram of proposed discrete-time robust control technique is shown in this figure. Here notations x_k, $\hat{u}_k$ and $\hat{v}_k$ represent the system's state and two estimated control inputs, respectively. Using NN based approximation technique, the estimated cost-functional $\hat{V}$ converges to its optimal cost $\hat{V}^*$. Using $\hat{V}^*$, the optimal inputs $\hat{u}^*$ and $\hat{v}^*$ are computed. Input $\hat{u}^*$, is applied to the nonlinear uncertain system to solve the robust control problem.

Mathematical analysis is done to prove the stability of the

Manuscript received September 20, 2016; Accepted March 30, 2017. Recommended by Associate Editor Yingmin Jia. *(Corresponding author: Niladri Sekhar Tripathy.)*

Citation: N. S. Tripathy, I. N. Kar, and K. Paul, "Suboptimal robust stabilization of discrete-time mismatched nonlinear system," *IEEE/CAA J. of Autom. Sinica*, vol. 5, no. 1, pp. 352−359, Jan. 2018.

The authors are all with the Department of Electrical Engineering, Indian Institute of Technology Delhi, New Delhi, Delphi-110016, India (e-mail: niladri.tripathy@yahoo.com; ink@ee.iitd.ac.in; kolin@cse.iitd.ac.in).

Digital Object Identifier 10.1109/JAS.2017.7510676

uncertain system by applying the approximated suboptimal control inputs. Finally, numerical results are reported to prove the efficacy of the proposed control algorithm. The key contributions of this work are:

1) A robust control algorithm is proposed for a discrete time nonlinear system with mismatched uncertainty. A robust control law is derived by formulating an equivalent optimal control problem for a nominal virtual system with a quadratic cost-functional. The virtual dynamics have two control inputs u and v. The concept of virtual input v is used to derive the existence of stabilizing control input u. The virtual input v helps to tackle the mismatched uncertainty. The proposed robust control law ensures asymptotic convergence of uncertain closed-loop system.

2) An optimal solution of a DT-GHJB equation is approximated through a NN implementation, to solve the nonlinear optimal control problem. The approximated inputs ensure the asymptotic convergence of uncertain states both analytically and numerically. The convergence of both the NN weight and cost-functional are also shown through the simulation results.

3) This paper also shows that some of the existing results [8] of matched system are special cases of the proposed results.

Notation & Definitions: The symbol $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. The $\mathbb{R}^n$ represents the n dimensional Euclidean real space and $\mathbb{R}^{n\times m}$ is a set of all $(n \times m)$ real matrices. The notations $X \leq 0$, X^{-1} and X^T denote the negative definiteness, inverse and transpose of matrix X, respectively. The I is used to represent an identity matrix. The minimum and maximum eigenvalue of symmetric matrix $P \in \mathbb{R}^{n\times n}$ are represented by the notations $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$, respectively. The number of iteration for discrete-time system is represented by k. The k^{th} instant state and control input for a discrete-time system are denoted by x_k and u_k. A set Ω is used to denote a continuous Lipschitz compact set where state x_k (including the initial points) satisfy the condition $x_k \in \Omega$ [16]. To prove the theoretical results, following definition is used in this paper.

Definition 1 [17], [18]: Consider a nonlinear discrete-time system as

$$x_{k+1} = f(x_k) + g(x_k)u_k(x_k) \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are system state and input vector respectively. The functions $f(x_k)$ and $g(x_k)$ are continuous nonlinear functions and $f(x_k) + g(x_k)u_k(x_k)$ is Lipschitz continuous on a set Ω including the origin. The control input $u_k(x_k)$ ensures the asymptotic convergence of closed loop system (1), $\forall x_k \in \Omega$. Let Ω_u is a set of admissible control inputs and input u_k minimizes the cost-functional

$$J_k = \frac{1}{2}\sum_{k=0}^{\infty}\left\{x_k^T Q x_k + u_k^T R u_k\right\}. \tag{2}$$

Then, the control input u_k is considered as an admissible (i.e., $u_k \in \Omega_u$) with-respect to its state penalty function $x_k^T Q x_k$ and control energy penalty function $u_k^T R u_k$, $\forall x_k \in \Omega$, if the following conditions hold:

1) $\forall x_k \in \Omega$, input $u_k(x_k)$ is continuous;
2) $u_k(0) = 0$;
3) u_k must stabilizes (1) for $\forall x_k \in \Omega$;
4) $\sum_{k=0}^{\infty}(x_k^T Q x_k + u_k^T R u_k) \leq \infty$, $\forall x_0 \in \Omega$.

II. Robust Control Design

System Description: A discrete-time uncertain nonlinear system is described by the state equation in the form

$$x_{k+1} = f(x_k) + g(x_k)u_k + d(x_k) \tag{3}$$

where $x_k \in \mathbb{R}^n$ is the state and $u_k \in \mathbb{R}^m$ is the periodic control input and $f \in \mathbb{R}^n$, $g \in \mathbb{R}^{n\times m}$ are the nonlinear functions. It is assumed that (3) is Lipschitz continuous on a compact set $\Omega \in \mathbb{R}^n$ and origin is the equilibrium point, i.e., $f(0) = 0$ and $g(0) = 0$. The unknown function $d(x_k) \in \mathbb{R}^n$ is used to represent the system uncertainty and it is always upper bounded by a known function $d_{\max}(x_k)$, that is

$$\| d(x_k) \| \leq d_{\max}(x_k) \qquad \forall k. \tag{4}$$

Generally system uncertainties are classified as matched and mismatched uncertainty and they are defined as follows [3], [8]–[10].

Definition 2: System (3) suffers through the matched uncertainty if the uncertainty $d(x_k)$ satisfy the following

$$d(x_k) = g(x_k)\phi(x_k) \tag{5}$$

$$\|\phi(x_k) \leq U_{\text{matched}} \qquad \forall k \tag{6}$$

where $\phi(x_k)$ is the unknown function and U_{matched} is the upper bound of $\|\phi(x_k)\|$. In other words, $d(x_k)$ is in the range space of $g(x_k)$.

Definition 3: System (3) has mismatched uncertainty if the uncertain component $d(x_k)$ is not in the range space of input matrix $g(x_k)$.

For the simplification, uncertainty can be decomposed in matched and mismatched component as follows

$$\begin{aligned} d(x_k) &= g(x_k)g(x_k)^+ S\phi(x_k) \\ &\quad + (I - g(x_k)g(x_k)^+)S\phi(x_k) \qquad \forall k \end{aligned} \tag{7}$$

where $g(x_k)g(x_k)^+S\phi(x_k)$ and $(I - g(x_k)g(x_k)^+)S\phi(x_k)$ are the matched and mismatched components respectively. The matrix $g(x_k)^+ = (g^T(x_k)g(x_k))^{-1}g(x_k)^T$ denotes the left pseudo inverse of matrix $g(x_k)$ [19] and S is a scaling matrix where $S \neq g(x_k)$. For a matrix $S = g(x_k)$, the uncertainty (7) reduces to a matched one as defined in (5). The decomposition of uncertainty into a matched and mismatched components will be used to define a nominal virtual system for (3) which is discussed in the subsequent subsection.

Problem Statement: Design a state feedback control law $u_k = K(x_k)$, to stabilize the discrete-time uncertain nonlinear system (3), such that the closed-loop system is asymptotically stable in the presence of uncertainty (7).

Proposed Solution: This problem is solved in two steps. First, the controller is designed by adopting nonlinear optimal control theory and then an algorithm is used to approximate the solution of DT-GHJB equation. The approximate solution of DT-GHJB equation is used to compute the stabilizing and virtual control inputs u_k and v_k, respectively.

Robust Control Problem: Design a state feedback control law $u_k = K(x_k)$ such that the uncertain closed-loop system (3) is asymptotically stable $\forall \parallel\ d(x_k) \parallel \leq d_{\max}(x_k)$. In order to stabilize (3), the robust control law u_k is designed using an optimal control approach.

Optimal Control Approach: The key idea is to design a discrete-time nonlinear optimal control law for virtual nominal system by minimizing a cost-functional J, which depends on the upper-bound of system uncertainty. An extra term $(I - g(x_k)g(x_k)^+)Sv(k)$ is added with the nominal dynamics of (3) to define a virtual system (8). The derived optimal input for virtual system is shown to be a robust input for original uncertain system. The virtual nominal dynamics and cost-functional for solving robust control problem are given below:

$$x_{k+1} = f(x_k) + g(x_k)u_k + M(x_k)v_k \tag{8}$$

$$J_k = \frac{1}{2}\sum_{k=0}^{\infty}\Bigg\{d^2_{\max}(x_k) + v^2_{\max}(x_k) + x_k^TQx_k + \begin{bmatrix}u_k^T & v_k^T\end{bmatrix}\begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix}\begin{bmatrix}u_k\\ v_k\end{bmatrix}\Bigg\} \tag{9}$$

where matrices $M = (I - g(x_k)g(x_k)^+)S$, $Q \geq 0$, $R_1 > 0$ and $R_2 > 0$. Here $v_{\max}$ is a scalar.

Inspired by the results reported in [20] and [21], the discrete-time HJB (DT-HJB) equation for (8) with the optimal cost-functional J_k^* of (9) is

$$J_k^* = \min_{u(x_k),v(x_k)} \frac{1}{2}\Bigg\{d^2_{max}(x_k) + v^2_{max}(x_k) + x_k^TQx_k + \begin{bmatrix}u_k^T & v_k^T\end{bmatrix}\begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix}\begin{bmatrix}u_k\\ v_k\end{bmatrix}\Bigg\} + J^*_{k+1}. \tag{10}$$

Using (10), the optimal control input for (8) is

$$\begin{bmatrix}u_k^*\\ v_k^*\end{bmatrix} = \begin{bmatrix}R_1^{-1}g(x_k)^T\frac{\partial J^*_{k+1}}{\partial x_k}\\ R_2^{-1}M(x_k)^T\frac{\partial J^*_{k+1}}{\partial x_k}\end{bmatrix}. \tag{11}$$

Let $V(x_k)$ be a positive definite continuously differentiable function, which satisfies $V(x_0) = J(x_0, u)$. Applying Taylor series expansion of the cost-functional, the DT-HJB (10) reduces to discrete-time general HJB as in [21]

$$d^2_{\max}(x_k) + v^2_{\max}(x_k) + x_k^TQx_k + u_k^TR_1u_k + v_k^TR_2v_k + \nabla V^T[x_{k+1} - x_k] + \frac{1}{2}[x_{k+1} - x_k]^T\nabla^2V[x_{k+1} - x_k] = 0 \tag{12}$$

where $\nabla^2 V = \begin{bmatrix}\frac{\partial^2V(x_k)}{\partial x_1^2} & \frac{\partial^2V(x_k)}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2V(x_k)}{\partial x_1\partial x_n}\\ \frac{\partial^2V(x_k)}{\partial x_2\partial x_1} & \frac{\partial^2V(x_k)}{\partial x_2^2} & \cdots & \frac{\partial^2V(x_k)}{\partial x_2\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^2V(x_k)}{\partial x_n\partial x_1} & \frac{\partial^2V(x_k)}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2V(x_k)}{\partial x_n^2}\end{bmatrix}$ and $\nabla V = \frac{\partial V(x_k)}{\partial x_k}$. The notation $x_{i_{i\in 1\ldots n}}$ represents the ith element of state vector x_k. The hermitian matrix ∇^2V is positive-definite $\forall\ x_k \in \Omega$. In the Taylor series expansion, the third and higher order terms are dropped to make it computationally feasible. This is made possible by adopting the small gain perturbation assumption around the equilibrium point. Using equations (9) and (12), it can be proved very easily that $V(x_k) = J(x_k, u)$ [17]. Now according to optimal control theory [22], the optimal inputs u_k^* and v_k^* satisfy the DT-GHJB and also minimize the following Hamiltonian:

$$\begin{aligned}H(x_k, u_k, v_k, \nabla V) &= d^2_{\max}(x_k) + v^2_{\max}(x_k) + x_k^TQx_k\\ &+ u_k^TR_1u_k + v_k^TR_2v_k + \nabla V_k^T(x_{k+1} - x_k)\\ &+ \frac{1}{2}(x_{k+1} - x_k)^T\nabla^2V_k(x_{k+1} - x_k).\end{aligned} \tag{13}$$

That means $\frac{\partial H}{\partial u^*} = 0$ and $\frac{\partial H}{\partial v^*} = 0$ which correspond to

$$g^T\nabla^2V^*(f + gu^* + Mv^* - x) + (2R_1u^* + g^T\nabla V^*) = 0 \tag{14}$$

$$M^T\nabla^2V^*(f + gu^* + Mv^* - x) + (2R_2v^* + M^T\nabla V^*) = 0. \tag{15}$$

The scalar $V^*(x_k)$ is the optimal value of $V(x_k)$ and it satisfies equation (12). After further simplification, from (14) and (15), the optimal inputs are

$$\begin{bmatrix}u_k^*\\ v_k^*\end{bmatrix} = -\begin{bmatrix}(2R_1 + g^T\nabla^2V^*g) & g^T\nabla^2V^*M\\ M^T\nabla^2V^*g & (2R_2 + M^T\nabla^2V^*M)\end{bmatrix}^{-1} \times \begin{bmatrix}g^T(\nabla V^* + \nabla^2V^*(f - x))\\ M^T(\nabla V^* + \nabla^2V^*(f - x))\end{bmatrix}. \tag{16}$$

To address the stability issue of virtual nominal system (8) by applying the optimal inputs (16), following Lemma is used.

Lemma 1: Suppose there exists a Lyapunov function $V(x_k)$ for (8) and DT-GHJB (12) is satisfied. Then the optimal inputs u_k^* and v_k^* defined in (16) ensure the asymptotic convergence of virtual nominal system (8).

Proof: Consider $V(x_k)$ is a Lyapunov function for (8). Using (12), the $\Delta V(x_k) = V_{k+1} - V_k$ reduces to

$$\begin{aligned}\Delta V = &-(d^2_{\max}(x_k) + v^2_{\max}(x_k) + u_k^{*T}R_1u_k^*\\ &+ v_k^{*T}R_2v_k^*) + x_k^TQx_k.\end{aligned} \tag{17}$$

The negative-definiteness of ΔV along the solution of (8) proves the asymptotic stability of (8) through the inputs (16). ■

Remark 1: In DT-GHJB, the derivative of cost-functional is linearly related but it is nonlinear for DT-HJB. As a result, solving DT-GHJB corresponds to solving a linear partial difference equation. This makes the DT-GHJB computationally easier to solve than the DT-HJB. However it is still difficult to achieve a closed form solution as it is a partial difference equation.

Remark 2: A block matrix $\begin{bmatrix}A & B\\ B^T & C\end{bmatrix}$ is invertible if the following conditions are satisfied [23]:

1) det$(A) \neq 0$;

2) det$(C - BA^{-1}B^T) \neq 0$.

Now, the control inputs (16) can be computed if the matrix

$$\left[\begin{array}{c|c}A & B\\ \hline B^T & C\end{array}\right] = \left[\begin{array}{c|c}2R_1 + g^T\nabla^2V^*g & g^T\nabla^2V^*M\\ \hline M^T\nabla^2V^*g & 2R_2 + M^T\nabla^2V^*M\end{array}\right]$$

is invertible. Here R_1, R_2 and $\nabla^{*2}V$ are the positive definite matrices. So the sub-matrix $A(= 2R_1 + g^T\nabla^2V^*g)$ is positive definite as $(2R_1 + g^T\nabla^2V^*g) > 0$ and hence det$(A) \neq 0$. Now a suitable selection of design matrices R_1 and R_2 helps to satisfy condition 2).

The realization of optimal control inputs (16) depend on the solution of DT-GHJB (12). In the next section, a brief description of NN based approximation technique is discussed to achieve the estimated solution of (12) which helps to design the optimal inputs (16).

A. NN Based Approximation Using Least Squares Approach

Neural network (NN) has universal function approximation property. Using this approximation property, several researchers have used NN to approximate the solution of HJB or GHJB as reported in [6], [20] and [21]. The key aim of this section is to approximate the optimal cost functional $V^*(x_k)$, using a NN based algorithm. Applying NN based algorithm, the cost-functional $V(x_k)$ is approximated as $\hat{V}(x_k)$. The estimated cost functional $\hat{V}(x_k)$ is used to compute the approximate control inputs $\hat{u}_k$ and $\hat{v}_k$. To estimate $\hat{V}(x)$ using NN, the basis function $\sigma(x_k) = [\sigma_1(x_k)\ \sigma_2(x_k) \cdots \sigma_l(x_k)]^T$ and weight vector $\hat{w} = [\hat{w}_1, \hat{w}_2, \hat{w}_3, \ldots, \hat{w}_l]^T$ are selected. The scalar l denotes the number of hidden layers in the NN. The selection of activation function depends on the following polynomial [17], [18]

$$\sum_{j=1}^{\frac{L}{2}} \left(\sum_{k=1}^{n} x_k \right)^{2j} \tag{18}$$

where L and n represent the order of approximation and the dimension of the system respectively. The equation (19) corresponds to the activation function for a 2-dimensional system as

$$\sigma(x_k) = \{x_1^2, x_1x_2, x_2^2, x_1^4, \ldots, x_2^L\}. \tag{19}$$

The selected basis function $\sigma(x_k)$ is smooth and continuous moreover it also holds the property $\sigma(0) = 0, \forall x_k = 0$. Applying the basis function $\sigma(x_k)$ and NN weight $\hat{w}$, the estimated cost functional reduces to

$$\hat{V}(x_k) = \sum_{j=1}^{l} \hat{w}_j^T \sigma_l(x_k) \tag{20}$$

with a residual error (e_r)

$$\text{DT-GHJB}\left(\hat{V} = \sum_{j=1}^{l} \hat{w}_j^T \sigma_l(x_k), \hat{u}_k, \hat{v}_k \right) \triangleq e_r.$$

Applying the least square method [24], the unknown weight vector of NN is updated such that it minimizes the residual error e_r. The minimization of residual error e_r is done by projecting e_r on $\frac{de_r}{dw}$, i.e., $\langle \frac{de_r}{dw}, e \rangle = 0$ where $\langle a, b \rangle = \int_\Omega ab dx$ is the Lebesgue integral. Due to the difficulty in this integration process, $\hat{w}$ is approximated using a mesh having ρ points on Ω from Riemann integration theory. The mesh point ρ is selected as $\rho \leq l$ with a mesh size Δx. Adopting Riemann approximation of integration, the $\langle \frac{de_r}{dw}, e \rangle = 0$ can be expressed as

$$X\hat{w} + Y. \tag{21}$$

This helps to derive the weight update law with least square error minimizing rule as

$$\hat{w} = -(X^T X)^{-1}(XY) \tag{22}$$

where X and Y are defined as (23) and (24), shown at the bottom of this page.

Using estimated weight (22), the cost-functional is also estimated by applying (20). The estimated cost-functional (20) is applied to derive the approximated control inputs $\hat{u}_k$ and $\hat{v}_k$. An algorithmic representation of numerical steps to achieve the suboptimal inputs $\hat{u}_k^*$ and $\hat{v}_k^*$ is given next.

Remark 3: Given admissible control inputs $u_0 \in \Omega_u$ and $v_0 \in \Omega_v$, the solution $\hat{V}_i$ of DT-GHJB (12) iteratively converges to its optimal solution V^* by updating the control inputs using (25). This claim can be proved analytically using the results reported in [17], [21].

B. Stability of Uncertain Systems Using Approximate Inputs

The derived approximated optimal inputs (26) for (8) ensure the asymptotic stability of uncertain system (3). This information is stated as a theorem in Algorithm 1.

Theorem 1: Suppose there exists a continuously differentiable positive function $\hat{V}^*(x_k)$ which satisfies (12) with the inequality

$$\begin{aligned} d_{\max}^2 \geq \phi^T \{ & R_2 + (g^+ S)^T (R_1 + g^T \nabla^2 \hat{V}^* g)(g^+ S) \\ & + M^T \nabla^2 \hat{V}^* M \} \phi. \end{aligned} \tag{23}$$

The approximated optimal control input $\hat{u}_k^*$ defined in (26) for (8) will be the robust solution of unmatched system (3) if the following condition holds

$$v_{\max} \geq \hat{v}_k^{*T} (2R_2 + M^T \nabla^2 \hat{V}^* M) \hat{v}_k^*. \tag{24}$$

$$X = \begin{bmatrix} \left\{ \nabla \hat{V}^T (f + g\hat{u} + M\hat{v} - x) + \frac{1}{2}(f + g\hat{u} + \cdots + M\hat{v} - x)^T \nabla^2 \hat{V} (f + g\hat{u} + M\hat{v} - x) \right\} |_{x=x_1} \\ \vdots \\ \left\{ \nabla \hat{V}^T (f + g\hat{u} + M\hat{v} - x) + \frac{1}{2}(f + g\hat{u} + \cdots + M\hat{v} - x)^T \nabla^2 \hat{V} (f + g\hat{u} + M\hat{v} - x) \right\} |_{x=x_\rho} \end{bmatrix} \tag{25}$$

$$Y = \begin{bmatrix} d_{\max}^2(x_k) + v_{\max}^2(x_k) + x(k)^T Q x(k) + \cdots + \hat{u}(k)^T R_1 \hat{u}(k) + \hat{v}(k)^T R_2 \hat{v}(k) |_{x=x_1} \\ \vdots \\ d_{\max}^2(x_k) + v_{\max}^2(x_k) + x(k)^T Q x(k) \cdots + \hat{u}(k)^T R_1 \hat{u}(k) + \hat{v}(k)^T R_2 \hat{v}(k) |_{x=x_\rho} \end{bmatrix} \tag{26}$$

Algorithm 1 Optimal inputs using NN based approximation

1: Initialization: $k \Leftarrow 0$, $i \Leftarrow 0$, $x \Leftarrow x_0$, $u \Leftarrow u_0$, $v \Leftarrow v_0$.

2: Select any value of a scalar $\epsilon > 0$ and number of mesh points ρ.

3: Initial inputs u_0 and v_0 are admissible control inputs.

4: Create an NN as $\hat{V}(x)$ using (20).

5: Compute $\hat{V}_i$ using (20), (22)$-$(24). Here $i = 0, 1, 2, \ldots$, denotes the number of iteration.

6: Compute the approximate control inputs using following equation

$$\begin{bmatrix}\hat{u}_{i+1}\\ \hat{v}_{i+1}\end{bmatrix} = -\begin{bmatrix}(2R_1 + g^T\nabla^2\hat{V}_i g) & g^T\nabla^2\hat{V}_i M\\ M^T\nabla^2\hat{V}_i g & (2R_2 + M^T\nabla^2\hat{V}_i M)\end{bmatrix}^{-1} \times \begin{bmatrix}g^T(\nabla\hat{V}_i + \nabla^2\hat{V}_i(f-x))\\ M^T(\nabla\hat{V}_i + \nabla^2\hat{V}_i(f-x))\end{bmatrix}. \tag{27}$$

7: Update the control inputs (25).

8: **if** $\hat{V}_i - \hat{V}_{i+1} \geq \epsilon$

9: Go to line 5

10: **else**

11: Optimal cost-function $\hat{V}^* = \hat{V}_i$

12: Using $\hat{V}^*$, the approximate optimal inputs $\hat{u}_k^*$ and $\hat{v}_k^*$ are computed as

$$\begin{bmatrix}\hat{u}_k^*\\ \hat{v}_k^*\end{bmatrix} = -\begin{bmatrix}(2R_1 + g^T\nabla^2\hat{V}^* g) & g^T\nabla^2\hat{V}^* M\\ M^T\nabla^2\hat{V}^* g & (2R_2 + M^T\nabla^2\hat{V}^* M)\end{bmatrix}^{-1} \times \begin{bmatrix}g^T(\nabla\hat{V}^* + \nabla^2\hat{V}^*(f-x))\\ M^T(\nabla\hat{V}^* + \nabla^2\hat{V}^*(f-x))\end{bmatrix}. \tag{28}$$

13: **end if**

Proof of Theorem 1: Let $V(x_k)$ is the solution of (12) and it is approximated as $\hat{V}^*(x_k)$ using the estimated inputs (26). The approximated solution $\hat{V}^*(x_k)$ and inputs (26) also satisfy the following equation

$$\begin{aligned}&d_{\max}^2(x_k) + v_{\max}^2(x_k) + x_k^TQx_k + \hat{u}_k^{*T}R_1\hat{u}_k^* + \hat{v}_k^{*T}R_2\hat{v}_k^*\\ &+\nabla\hat{V}_k^{*T}(x_{k+1}-x_k) + \frac{1}{2}(x_{k+1}-x_k)^T\nabla^2\hat{V}_k^*(x_{k+1}-x_k) = 0.\end{aligned} \tag{29}$$

Now, with the control inputs (26), the difference of $\hat{V}^*(x_k)$ $\big[\Delta\hat{V}^* = \hat{V}^*(x_{k+1})) - \hat{V}^*(x_k)\big]$ along the solution of (3) is

$$\begin{aligned}\Delta\hat{V}^* =&\nabla\hat{V}^{*T}(f + g\hat{u}_k^* + M\hat{v}_k^* - x)\\ &+ \frac{1}{2}(f+g\hat{u}_k^*+M\hat{v}_k^*-x)^T\nabla^2\hat{V}^*(f+g\hat{u}_k^*+M\hat{v}_k^*-x)\\ &+ \nabla\hat{V}^{*T}(d - M\hat{v}_k^*)\\ &+ (f + g\hat{u}_k^* + M\hat{v}_k^* - x)^T\nabla^2\hat{V}^*(d - M\hat{v}_k^*)\\ &+ \frac{1}{2}(d - M\hat{v}_k^*)^T\nabla^2\hat{V}^*(d - M\hat{v}_k^*).\end{aligned} \tag{30}$$

Using (7) in (30), the following is obtained

$$\begin{aligned}\Delta\hat{V}^* =&\nabla\hat{V}^{*T}(f + g\hat{u}_k^* + M\hat{v}_k^* - x)\\ &+ \frac{1}{2}(f+g\hat{u}_k^*+M\hat{v}_k^*-x)^T\nabla^2\hat{V}^*(f+g\hat{u}_k^*+M\hat{v}_k^*-x)\\ &+ \nabla\hat{V}^{*T}(gN\phi + M\phi - Mv_k^*)\\ &+ (f + g\hat{u}_k^* + M\hat{v}_k^* - x)\nabla^2\hat{V}^*(gN\phi + M\phi - M\hat{v}_k^*)\\ &+ \frac{1}{2}(gN\phi+M\phi-M\hat{v}_k^*)^T\nabla^2\hat{V}^*(gN\phi+M\phi-M\hat{v}_k^*)\end{aligned} \tag{31}$$

where matrix $N = g^+S$. After further simplification, equation (26) can be rewritten as

$$g^T\nabla^2\hat{V}^*(f + g\hat{u}_k^* + M\hat{v}_k^* - x) = -(2R_1\hat{u}_k^* + g^T\nabla\hat{V}^*) \tag{32}$$

$$M^T\nabla^2\hat{V}^*(f + g\hat{u}_k^* + M\hat{v}_k^* - x) = -(2R_2\hat{v}_k^* + M^T\nabla\hat{V}^*). \tag{33}$$

Applying (29), (32) and (33) in (31), $\Delta\hat{V}^*$ is simplified as

$$\begin{aligned}\Delta\hat{V}^* =&- d_{\max}^2 - v_{\max}^2 - x_k^TQx_k - \hat{u}_k^{*T}R_1\hat{u}_k^* - \hat{v}_k^{*T}R_2\hat{v}_k^*\\ &+ \nabla\hat{V}^{*T}(gN\phi + M\phi - M\hat{v}^*)\\ &- (2R_1\hat{u}^* + g^T\nabla\hat{V}^*)^Tg^+S\phi\\ &- (2\hat{v}^{*T}R_2 + \nabla^T\hat{V}^*M)(\phi - \hat{v}^*)\\ &+ \frac{1}{2}(gN\phi+M\phi-M\hat{v}^*)^T\nabla^2\hat{V}^*(gN\phi+M\phi-M\hat{v}^*).\end{aligned}$$

After further simplification $\Delta\hat{V}^*$ reduces to

$$\begin{aligned}\Delta\hat{V}^* \leq&- x_k^TQx_k - \big\{v_{\max}^2 - \hat{v}_k^{*T}(2R_2 + M^T\nabla^2\hat{V}^*M)\hat{v}_k^*\big\}\\ &- \big\{d_{\max}^2 - \phi^T(R_2 + N^T(R_1 + g^T\nabla^2\hat{V}^*g)N\\ &+ M^T\nabla^2\hat{V}^*M)\phi\big\}.\end{aligned} \tag{34}$$

From the above inequality, $\Delta\hat{V}^*$ is negative definite if the conditions (27) and (28) hold. This proves the asymptotic convergence of (3) under periodic feedback of control input $u_k,\ \forall k$. ■

The proposed robust control framework considers the general system uncertainty, which includes both matched and mismatched component. Without mismatched part, system (3) reduces to matched system (defined in (5)), i.e.

$$x_{k+1} = f(x_k) + g(x_k)(u_k + d(x_k)). \tag{35}$$

Moreover, due to the absence of mismatched part, the virtual control input v_k is not necessary in (8) and (9). Therefore the nominal system and cost-functional for (35) reduce to

$$x_{k+1} = f(x_k) + g(x_k)u_k(x_k) \tag{36}$$

$$J_k = \frac{1}{2}\sum_{k=0}^{\infty}\left\{d_{\max}^2(x_k) + x_k^TQx_k + u_k^TR_1u_k\right\} \tag{37}$$

where $\| d(x_k) \| \leq d_m(x_k),\ \forall k$.

As a special case of Theorem 1, Corollary 1 is introduced for matched system.

Corollary 1: Suppose there exists a continuously differentiable positive function $\hat{V}^*(x_k)$ which satisfies

$$\begin{aligned}d_{\max}^2(x_k) &+x_k^TQx_k + \hat{u}_k^{*T}R_1\hat{u}_k^* + \nabla\hat{V}^{*T}(x_{k+1} - x_k)\\ &+\frac{1}{2}(x_{k+1} - x_k)^T\nabla^2\hat{V}^*(x_{k+1} - x_k) = 0.\end{aligned} \tag{38}$$

Then the designed optimal control input

$$\hat{u}_k^* = -\{g(x_k)^T \nabla^2 \hat{V}^* g(x_k) + 2R_1\}^{-1} g(x_k)^T \times\{\nabla \hat{V}^* + \nabla^2 \hat{V}^*(f - x_k)\} \quad (39)$$

for (36) which minimizes (37) is also a robust solution of (35) if the uncertainty $d(x_k)$ satisfies the following bound

$$d_{\max}^2(x_k) \geq \phi^T(N^T(R_1 + g^T \nabla^2 \hat{V}^* g)N)\phi. \quad (40)$$

Proof: Due to space limitation, the proof of this corollary is omitted. ■

C. Robustness With Input Uncertainty

The proposed framework can be extended in the presence of input uncertainty. A system with mismatched input uncertainty is described as

$$x_{k+1} = f(x_k) + \{g(x_k) + d(x_k)\}u_k(x_k) \quad (41)$$

where function $d(x_k)$ is the bounded uncertainty affecting the input function $g(x_k)$. To design the robust control input the virtual nominal system (8) and cost-functional (9) are considered. To tackle the mismatched uncertainty in input function, the optimal control problem is solved for (8) and (9) with the control inputs (26). The following theorem states the robust problem under presence of input uncertainty.

Theorem 2: Suppose there exists a continuously differentiable positive function $\hat{V}^*(x_k)$ which satisfies (29) with the inequality

$$d_{\max}^2 \geq (\phi u_k^*)^T\{R_2 + (g^+S)^T R_1(g^+S) + M^T \nabla^2 \hat{V}^* M + (gg^+S + M)^T \nabla^2 \hat{V}^*(gg^+S + M)\}(\phi u_k^*). \quad (42)$$

The approximate optimal control input $\hat{u}_k^*$ defined in (26) for (8) which minimizes (9) will be the robust solution of (41) if the following condition holds

$$v_{\max} \geq \hat{v}_k^{*T}(2R_2 + M^T \nabla^2 \hat{V}^* M)\hat{v}_k^*. \quad (43)$$

Proof: The proof of this theorem is similar to the proof of Theorem 1. ■

Remark 4: It is observed that the DT-HJB (10) is approximated using Taylor series expansion and it reduces to DT-GHJB (12). Due to this approximation, the optimal input (11) is converted to near optimal input (16). The approximated virtual input $\hat{v}_k^*$ is not used to stabilize system (3) but it is used to design the $\hat{u}_k^*$. The input $\hat{v}_k^*$ is used to verify the condition (28).

D. Comparison With Existing Results

This subsection compares the main results of this paper with the existing work reported in [8]. In 2016, D. Wang *et al.* have proposed an approximate optimal control based robust control technique for discrete-time nonlinear system. To realize the robust control law, they have considered that the system is affected by matched uncertainty. For the purpose of comparison with the results described in [8], the mismatched component of the uncertainty is neglected. It is observed that without mismatched component, the virtual input v_k is not necessary. Therefore without virtual input v_k, the nominal dynamics and cost-functional defined in this paper are in a form similar to that as mentioned in [8].

So the results reported in [8] can be recovered as a special case of the proposed work. To solve the nonlinear optimal control problem, a NN based approximation technique is adopted from [8], [21]. But, the presence of control input v_k in nominal system (8) modifies the DT-HJB equation reported in [8], [21]. To tackle the mismatched uncertainty, the cost-functional (9) consists of two extra terms as $v_{\max}^2(x_k)$ and $v_k^T R_2 v_k$. These two extra terms directly affect the computation of matrices X and Y as mentioned in (23) and (24) respectively. Moreover, the computation of approximated cost-functional $\hat{V}^*$ also depends on both the control inputs $\hat{u}_k^*$ and $\hat{v}_k^*$. The absence of virtual input v_k in [8], [21], makes it easy to compute $\hat{V}^*$.

III. Additional Results

A discrete-time linear system with state uncertainty is described as

$$x_{k+1} = (A + \Delta(p))x_k + Bu_k \quad (44)$$

where A and B are state and input matrices. The uncertain matrix $\Delta(p) = BB^+S\phi(p) + (I - BB^+)S\phi(p)$ affects the system due to bounded variation of the uncertain parameter p. The uncertainty is bounded by a known matrix F and it is defined as

$$\phi_A^T(p)S^T PS\phi_A(p) \leq F \quad (45)$$

where matrix P is a positive-definite matrix. To solve the robust control problem, the virtual nominal system and cost-functional are selected as

$$x_{k+1} = Ax_k + Bu_k + Mv_k \quad (46)$$

$$J_k = \frac{1}{2}\sum_{k=0}^{\infty}\left\{x_k^T(Q + F)x_k + \begin{bmatrix}u_k^T & v_k^T\end{bmatrix}\begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix}\begin{bmatrix}u_k\\ v_k\end{bmatrix}\right\} \quad (47)$$

where matrix $M = (I - BB^+)S$. With a quadratic Lyapunov function $V^*(x) = x_k^T Px_k$, the gradient vector and Hessian matrix can be expressed as $\nabla V = 2Px_k$ and $\nabla^2 V = 2P$. For the system (46), with a cost-functional (47), the DT-GHJB is

$$x_k^T(Q + F)x_k + \begin{bmatrix}u_k^* & v_k^*\end{bmatrix}\begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix}\begin{bmatrix}u_k^*\\ v_k^*\end{bmatrix} + (Ax_k + Bu_k^* + Mv_k^* - x_k)^T P(Ax_k + Bu_k^* + Mv_k^* - x_k) + 2x_k^T P(Ax_k + Bu_k^* + Mv_k^* - x_k). \quad (48)$$

After further simplification, equation (48) reduces to

$$A^T\left\{P^{-1} + \begin{bmatrix}B & (I - BB^+)S\end{bmatrix}\begin{bmatrix}R_1^{-1} & 0\\ 0 & R_2^{-1}\end{bmatrix} \times \begin{bmatrix}B^T\\ S^T(I - BB^+)^T\end{bmatrix}\right\}^{-1} A - P + Q + F = 0. \quad (49)$$

The optimal control inputs $u_k^* = Kx_k$ and $v_k^* = Lx_k$ are

$$u_k^* = -\left(R_1^{-1}B^T\left\{P^{-1} + \begin{bmatrix}B & (I - BB^+)S\end{bmatrix} \times \begin{bmatrix}R_1^{-1} & 0\\ 0 & R_2^{-1}\end{bmatrix}\begin{bmatrix}B^T\\ S^T(I - BB^+)^T\end{bmatrix}\right\}^{-1} A\right)x_k \quad (50)$$

$$v_k^* = -\left(R_2^{-1}S^T(I-BB^+)^T\left\{P^{-1} + \begin{bmatrix} B & (I-BB^+)S \end{bmatrix}\right.\right.$$
$$\left.\left.\times \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix} \begin{bmatrix} B^T \\ S^T(I-BB^+)^T \end{bmatrix}\right\}^{-1} A\right)x_k \qquad (51)$$

where matrices K and L are the controller gains. It is observed that the equation (49) is a discrete-time Algebraic Riccati equation (DT-ARE). Therefore the DT-ARE related with the optimal control problem for a linear system can be recovered from the proposed DT-GHJB (12). To address the robust control problem for the linear system (44), following lemma is included.

Lemma 2: Suppose their exists a positive definite solution P of Riccati equation (49). The optimal input (50) ensures the asymptotic convergence of uncertain closed-loop system (44) for all bounded variation of uncertain parameter p, if it satisfies the inequalities (45) and $A_c^T P A_c - K^T R_1 K - L^T R_2 L \geq 0$ where $A_c = A + BK$.

Proof: Proof of this Lemma is omitted. ■

IV. RESULTS

The section uses a numerical example to validate the proposed control algorithm. Consider a state space form of uncertain discrete-time nonlinear system as (3) where functions $f(x_k)$, $g(x_k)$ and $\phi(x_k)$ are defined as

$$f(x_k) = \begin{bmatrix} -0.8x_{2k} \\ \sin(0.8x_{1k}) - x_{2k} + 1.8x_{2k} \end{bmatrix}$$
$$g(x_k) = \begin{bmatrix} 0 \\ -x_{2k} \end{bmatrix}$$
$$\phi(x_k) = p\ \sin(0.8k)x_{1k}$$

where p is the uncertain parameter. This system has mismatched uncertainty and hence the results of [8] are not applicable. To solve the optimal control problem for virtual nominal system, the design parameters $Q = I,\ R_1 = 0.5I$ and $R_2 = 0.5I$ are selected. The scaling matrix S is selected as $S = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T$. The upper bound of uncertainty $d(x_k)$, defined in (4) is considered as $d_{\max} = \|x_k\|^2$. The parameter p can vary within -0.5 to 0.5. To estimate the optimal cost function through the NN realization, the NN is constructed as

$$\hat{V}(x) = \hat{w}_1 x_1^2 + \hat{w}_2 x_2^2 + \hat{w}_3 x_1 x_2. \qquad (52)$$

The mesh point $\rho = 6$ and mesh size $\Delta x = 0.01$ are selected. For simulation, the initial admissible control inputs $u_0 = x_1 + 1.5x_2$ and $v_0 = 0.049x_1$ are used. The simulation is carried out in MATLAB simulation platform for 10 iterations with the initial states $[0.5, -0.5]^T$. After 5 iterations, the NN weight w converges to $w = \begin{bmatrix} 6.97 & 8.35 & 6.72 \end{bmatrix}^T$.

Analysis of Simulation Results: Fig. 2(a) shows that the system has converged to its equilibrium point through the admissible control inputs u_0. Figs. 3(a) and 3(b) show the convergence of NN weight and approximated value function. In Fig. 2(b), the systems state trajectories reach their equilibrium point in-spite-of uncertainty. The simulation results show that the proposed robust suboptimal control technique guarantees the closed-loop stability in presence of mismatched uncertainty. The variation of stabilizing input $\hat{u}_k^*$ and virtual input $\hat{v}_k^*$ is shown in Figs. 4(a) and 4(b).

Now, for a selection of the scaling matrix $S = g(x_k)$, the same example is solved numerically. This selection converts the mismatched system (3) to a matched system as defined in (35). The closed loop behavior of (35), is shown in Fig. 5a and 5b which replicates the results of matched system as stated in [8].

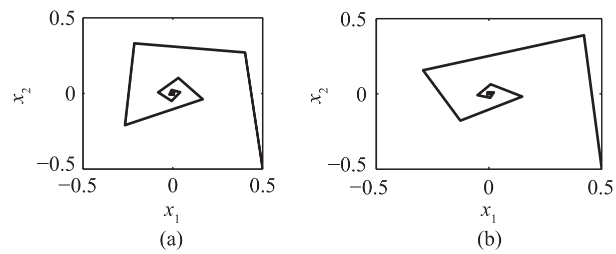

Fig. 2. Results of proposed robust control technique. (a) Convergence of state trajectory (x_1, x_2) with the initial admissible control inputs u_0 and v_0 for $p = 0$. (b) Convergence of state trajectory (x_1, x_2) with the designed robust control input $\hat{u}_k^*$ for $p = 0.5$.

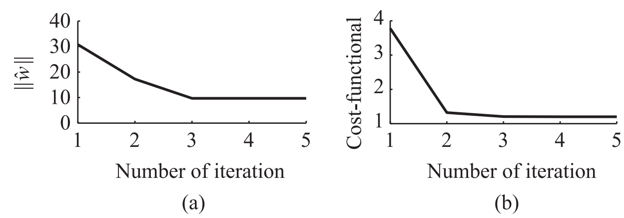

Fig. 3. Results of NN based approximation. (a) Convergence of norm of weight vector ($\|\hat{w}\|$). (b) Convergence of approximated cost-functional.

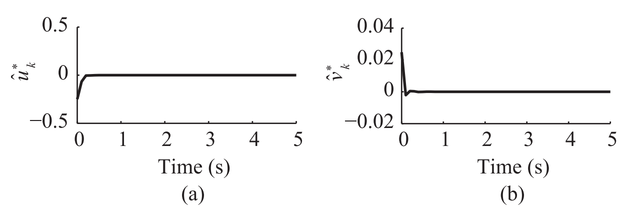

Fig. 4. Convergences of control inputs. (a) Convergence of approximated control input $\hat{u}_k^*$. (b) Convergence of approximated virtual input $\hat{v}_k^*$.

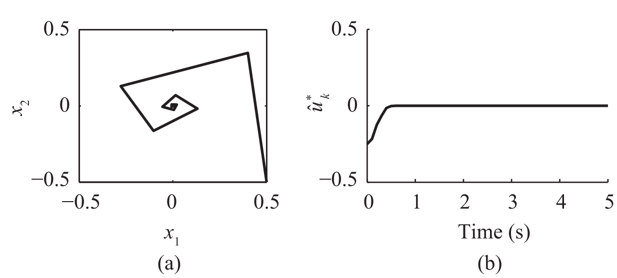

Fig. 5. Results for matched uncertain system. (a) Convergence of system states with matched uncertainty for $p = 0.5$. (b) Convergence of approximated control input $\hat{u}_k^*$ for matched system.

V. CONCLUSION

A discrete-time robust control technique for an uncertain nonlinear system is proposed in this paper. It is considered that the system is primarily affected by mismatched uncertainty. The control law is designed by formulating an optimal control problem for a virtual nominal system with a modified cost-functional. The virtual input is defined to design the stabilizing controller gain along with the stability condition. An analytical proof for ensuring asymptotic convergence of closed-loop uncertain system is also given. A comparative study between existing and proposed results is also reported. This paper has several promising future research directions. Few of them are discussed below.

The proposed control algorithm can be applied in several application and can also be extended to networked control system where subsystems are interconnected by a digital network [25]. To address this problem, coupled DT-HJB equation can be formulated [26]. The proposed control framework can also be treated as a differential game problem by considering control inputs u_k and v_k as maximizing and minimizing inputs [27].

REFERENCES

[1] F. Lin, "An optimal control approach to robust control design", *Int. J. Control* vol. 73, no.3, pp. 177−186, 2000.

[2] F. Lin and R. D. Brandt, "An optimal control approach to robust control of robot manipulators", *IEEE Trans. on Robot. and Automat.*, vol. 14, no. 1, pp. 69−77, 1998.

[3] F. Lin, W. Zhang and R. D. Brandt,"Robust hovering control of a PVTOL aircraft", *IEEE Trans. on Control Syst. Technol.*, vol. 7, no. 3, pp. 343−351, 1999.

[4] D. Wang, D. Liu, Q. Zhang, and D. Zhao, "Data-based adaptive critic designs for nonlinear robust optimal control with uncertain dynamics", *IEEE Trans. on Syst. Man and Cybern.: Syst.*, pp.1−12, 2016.

[5] D. Wang, D. Liu, and H. Li, "Policy iteration algorithm for online design of robust control for a class of continuous-time nonlinear systems." *IEEE Trans. Autom. Sci. Eng.*, vol. 11, no. 2, pp. 627−632, 2014.

[6] D. M. Adhyaru, I.N. Kar and M. Gopal, "Fixed final time optimal control approach for bounded robust controller design using Hamilton Jacobi Bellman solution", *IET Control Theory and Appl.*. vol. 3, no. 9, pp. 1183−1195, 2009.

[7] D. M. Adhyaru, I. N. Kar and M. Gopal, "Bounded robust control of systems using neural network based HJB solution", *Neural Comput and Applic*, vol. 20, no. 1, pp. 91−103, 2011.

[8] D. Wang, D. Liu, H. Li, B. Luo and H. Ma, "An approximate optimal control approach for robust stabilization of a class of discrete-time nonlinear systems with uncertainties", *IEEE Trans. on Syst. Man and Cybern.: Syst.*, vol. 46, no. 5, pp.1−5, 2016.

[9] I R Petersen, "Structural stabilization of uncertain systems: necessity of the matching condition", *SIAM J. Control Optim.*, vol. 23, no.2, pp. 286−296,1985.

[10] I. N. Kar, "Quadratic stabilization of a collection of linear systems", *Int. J. Syst. Sci.*, vol. 33, no. 2, pp. 153−160, 2002.

[11] Y. A. R. I. Mohamed, "Design and implementation of a robust current-control scheme for a PMSM vector drive with a simple adaptive disturbance observer", *IEEE Trans. Ind. Electron.*, vol. 54, no. 4, pp. 1981−1988, 2007.

[12] H. W. Lee, K. C. Kim, and J. Lee, "Review of maglev train technologies", *IEEE Trans. Magn.*, vol. 42, no. 7, pp. 1917−1925, 2006.

[13] J. Yang, S. Li and X. Yu, "Sliding-mode control for systems with mismatched uncertainties via a disturbance observer", *IEEE Trans. Ind. Electron.*, vol. 60, no. 1, pp. 160−169, 2013.

[14] L. Ma, Z. Wang, Y. Bo and Z. Gua, "Robust H_∞ sliding mode control for nonlinear stochastic systems with multiple data packet losses", *Int. J. Robust & Nonlin. Control*, vol. 22, no. 5, pp. 474−491, 2012.

[15] Y. Zheng, G. M. Dimirovski, Y. jing and M. Yang, "Discrete-time sliding mode control of nonlinear systems", *American Control Conf.*, New York City, USA. pp. 3825−3830, 2007.

[16] H. K. Khalil, Nonlinear Systems, *Prentice Hall*, 3rd Edition, New Jersey, 2002.

[17] J. Sarangapani,"Neural network control of nonlinear discrete-time systems", *CRC press*, Florida, USA, 2006.

[18] R. W. Beard, "Improving the closed-loop performance of nonlinear systems." Ph.D. diss., Rensselaer Polytechnic Institute, 1995.

[19] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK. 1990.

[20] A. Al-Tamini, F. L. Lewis and M. Abu-Khalaf, "Discrete-time nonlinear HJB solution using approximate dynamic programming: Convergence Proof", *IEEE Trans. on Syst., Man and Cybern. B: Cybern.*, vol. 38, no. 4, pp. 943−949, 2008.

[21] Z. Chen and S. Jagannathan, "Generalized Hamilton-Jacobi-Bellman formulation-based neural network control of affine nonlinear discrete-time systems", *IEEE Trans. on Neural Netw.*, vol. 19, no. 1, pp. 90−106. 2008.

[22] D. S. Naidu, Optimal control systems, *CRC press*, India, 2009.

[23] I. N. Imam , "The Schur complement and the inverse M-matrix problem", *Linear Algebra and its Appl.*, vol. 62, pp. 235−240, 1984.

[24] B. A. Finlayson, "The method of weighted residuals and variational principles", *Academic Press*, New York, USA, 1972.

[25] N. S. Tripathy, I. N. Kar, and K. Paul, "Stabilization of uncertain discrete-time linear system with limited communication", *IEEE Trans. Autom. Control*, vol. 62, no. 9, pp. 4727−4733, 2017.

[26] Z. Gajic and M. T. J. Qureshi, "Lyapunov matrix equation in system stability and control", Dover Publication, New York, USA. 2008.

[27] I. R. Petersen, "Linear quadratic differential games with cheap control", *Syst. & Control Lett.*, vol. 8, pp. 181−188, 1986.

Niladri Sekhar Tripathy completed the Bachelor Degree in electronics and communication engineering and Master Degree in mechatronics engineering. in 2009 and 2011, respectively. After completion of master degree he worked as a Senior Research Fellow (SRF) in IIT Delhi from 2011 to 2012, Indial. Presently he is pursuing his Ph.D. (electrical Engg.) degree from IIT Delhi.

His research interests include control techniques for cyber-physical system, mechatonics, application of control theory in computing.

Indra Narayan Kar (M'04-SM'07) received the Ph.D. (1997) degree from Indian Institute of Technology, Kanpur, India, in electrical engineering. He worked as a research student at Nihon University, Tokyo from 1996 to 1998. He joined as a Lecturer in the department of Electrical Engineering, Indian Institute of Technology Delhi in 1998 where, at present, he is working as a Professor.

His research interests mainly include nonlinear control, contraction theory, time-delay system and cyber-physical system.

Kolin Paul (M'05) received the Ph.D. in computer science from Bengal Engineering College, Kolkata, India in 2002. He joined the Department of Computer Science and Engineering, Indian Institute of Technology Delhi, in 2004, where he is an Associate Professor.

His research interests include understanding high performance architectures and compilation systems. In particular he works in the area of adaptive/reconfigurable computing trying to understand its use and implications in embedded systems.