The Exp-function Method for Some Time-fractional Differential Equations

Ahmet Bekir, Ozkan Guner, and Adem Cevikel

***Abstract*—In this article, the fractional derivatives in the sense of modified Riemann-Liouville derivative and the Exp-function method are employed for constructing the exact solutions of nonlinear time fractional partial differential equations in mathematical physics. As a result, some new exact solutions for them are successfully established. It is indicated that the solutions obtained by the Exp-function method are reliable, straightforward and effective method for strongly nonlinear fractional partial equations with modified Riemann-Liouville derivative by Jumarie's. This approach can also be applied to other nonlinear time and space fractional differential equations.**

***Index Terms*—Exact solution, exp-function method, fractional differential equation.**

I. INTRODUCTION

FRACTIONAL partial differential equations (FPDEs) have gained much attention as they are widely used to describe various complex phenomena in various applications such as the fluid flow, signal processing, control theory, systems identification, finance and fractional dynamics, physics and other areas. Oldman and Spanier first considered the partial fractional differential equations arising in diffusion problems [1]. The fractional partial differential equations have been investigated by many researchers [2]−[4].

In recent decades, a large amount of literature has been provided to construct the exact solutions of fractional ordinary differential equations and fractional partial differential equations of physical interest. Many powerful and efficient methods have been proposed to obtain exact solutions of fractional partial differential equations, such as the fractional sub-equation method, the first integral method, the (G'/G)-expansion method exp-function method and so on [5]−[19].

The exp-function method [20]−[27] can be used to construct the exact solutions for some time and space fractional differential equations. The present paper investigates for the first time the applicability and effectiveness of the exp-function method on fractional nonlinear partial differential equations. The objective of this paper is to extend the application of the exp-function method to obtain exact solutions to some fractional partial differential equations in mathematical physics. These equations include Fitzhugh-Nagumo equation and KdV equation.

This Letter is organized as follows: In Section II, some basic properties of Jumarie's modified Riemann-Liouville derivative are given. The main steps of the exp-function method is given in Section III. In Sections IV and V, we construct the exact solutions of the time fractional Fitzhugh-Nagumo and KdV equations via this method. Some conclusions and discussions are shown in Section VI.

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II. MODIFIED RIEMANN-LIOUVILLE DERIVATIVE

In decades years, in order to improve the local behavior of fractional types, a few local versions of fractional derivatives have been proposed, i.e., the Caputo's fractional derivative [28], the Grünwald-Letnikov's fractional derivative [29], the Riemann-Liouville's derivative [29], the Jumarie's modified Riemann-Liouville derivative [30], [31]. The Jumarie's derivative is defined as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} \left[f(\xi) - f(0) \right] d\xi, 0 < \alpha < 1 \tag{1}$$

where f: $\mathbb{R} \to \mathbb{R}$, $t \to f(t)$ denotes a continuous (but not necessarily first-order-differentiable) function. We list some properties for the modified Riemann–Liouville derivative as follows:

Property 1:

$$D_t^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0. \tag{2}$$

Property 2:

$$D_t^{\alpha}(cf(t)) = cD_t^{\alpha} f(t), \quad c = \text{constant}. \tag{3}$$

Property 3:

$$D_t^{\alpha}\{af(t) + bg(t)\} = aD_t^{\alpha} f(t) + bD_t^{\alpha} g(t) \tag{4}$$

where a and b constants.

Property 4:

$$D_t^{\alpha} c = 0, \quad c = \text{constant}. \tag{5}$$

III. THE EXP-FUNCTION METHOD

We consider the following general nonlinear FPDE of the type

$$F(u, D_t^{\alpha} u, D_x^{\beta} u, D_y^{\psi} u, D_t^{\alpha} D_t^{\alpha} u, D_t^{\alpha} D_x^{\beta} u, D_x^{\beta} D_x^{\beta} u, D_x^{\beta} D_y^{\psi} u, D_y^{\psi} D_y^{\psi} u, \dots) = 0, \quad 0 < \alpha, \beta, \psi < 1 \tag{6}$$

where u is an unknown function, and F is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of the exp-function method.

Step 1: Li and He [32] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$u(x,y,t) = U(\xi)$$
$$\xi = \frac{\tau x^{\beta}}{\Gamma(1+\beta)} + \frac{\delta y^{\psi}}{\Gamma(1+\psi)} + \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \tag{7}$$

where τ, δ and λ are non zero arbitrary constants.

By using the chain rule

$$D_x^{\alpha} u = \sigma_x^{'} \frac{dU}{d\xi} D_x^{\alpha} \xi$$

$$D_y^{\alpha} u = \sigma_y^{'} \frac{dU}{d\xi} D_y^{\alpha} \xi$$

$$D_t^{\alpha} u = \sigma_t^{'} \frac{dU}{d\xi} D_t^{\alpha} \xi \tag{8}$$

where $\sigma_x^{'}$, $\sigma_y^{'}$ and $\sigma_t^{'}$ are called the sigma indexes see [33], [34], without loss of generality we can take $\sigma_x^{'} = \sigma_y^{'} = \sigma_t^{'} = l$, where l is a constant.

Substituting (7) with (2) and (8) into (6), we can rewrite (6) in the following nonlinear ODE

$$Q(U, U^{'}, U'', U''', \dots) = 0 \tag{9}$$

where the prime denotes the derivation with respect to ξ. If possible, we should integrate (9) term by term one or more times.

Step 2: According to exp-function method, which was developed by He and Wu [35], we assume that the wave solution can be expressed in the following form

$$U(\xi) = \frac{\sum\limits_{n=-c}^{d} a_n \exp\big[n\xi\big]}{\sum\limits_{m=-p}^{q} b_m \exp\big[m\xi\big]} \tag{10}$$

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite (10) in the following equivalent form.

$$U(\xi) = \frac{a_{-c} \exp\big[-c\xi\big] + \cdots + a_d \exp\big[d\xi\big]}{b_{-p} \exp\big[-p\xi\big] + \cdots + b_q \exp\big[q\xi\big]}. \tag{11}$$

Step 3: This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and p, we balance the linear term of lowest order of equation (9) with the lowest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of highest order of (9) with highest order nonlinear term [36]−[39].

In the remaining sections, we will show the exact solutions to nonlinear time fractional differential equations using exp-function method.

IV. The Time Fractional Fitzhugh-nagumo Equation

We take into account the time fractional Fitzhugh-Nagumo equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\mu), \quad t > 0;\ 0 < \alpha \leq 1; x \in \mathbb{R} \tag{12}$$

subject to the initial condition

$$u(x,0) = \frac{1}{(1 + e^{-\frac{x}{\sqrt{2}}})} \tag{13}$$

which is an important nonlinear reaction-diffusion equation, applied to model the transmission of nerve impulses [40], [41], and also used in biology and the area of population genetics in circuit theory [42]. When $\mu = -1$, the Fitzhugh-Nagumo equation reduces to the real Newell-Whitehead equation [43].

For our goal, we present the following transformation

$$u(x,t) = U(\xi), \qquad \xi = cx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \tag{14}$$

where c and $\lambda \neq 0$ are constants.

Then by use of (14) with (2) and (8) into (12), (12) can be turned into an ODE

$$\lambda U' + c^2 U'' + U(1-U)(U-\mu) = 0 \tag{15}$$

where $U' = \dfrac{dU}{d\xi}$.

Balancing the order of U'' and U^3 in (15), we have

$$U^3 = \frac{c_1 \exp\big[(3c+p)\xi\big] + \cdots}{c_2 \exp\big[4p\xi\big] + \cdots} \tag{16}$$

and

$$U'' = \frac{c_3 \exp\big[(3p+c)\xi\big] + \cdots}{c_4 \exp\big[4p\xi\big] + \cdots} \tag{17}$$

where c_i are determined coefficients only for simplicity. Balancing lowest order of exp-function in (16) and (17) we have

$$3p + c = 3c + p \tag{18}$$

which leads to the result

$$p = c. \tag{19}$$

Similarly to determine values of d and q, we balance the linear term of highest order in (15)

$$U'' = \frac{\cdots + d_1 \exp\big[-(3q+d)\xi\big]}{\cdots + d_2 \exp\big[-4q\xi\big]} \tag{20}$$

and

$$U^3 = \frac{\cdots + d_3 \exp\big[-(3d+q)\xi\big]}{\cdots + d_4 \exp\big[-4q\xi\big]} \tag{21}$$

where d_i are determined coefficients only for simplicity. From (20) and (21), we obtain

$$-(3q+d) = -(3d+q) \tag{22}$$

and this gives

$$q = d. \tag{23}$$

To simplify, we set $p = c = 1$ and $q = d = 1$, so (11) degrades to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{24}$$

Substituting (24) into (15), and by the help of symbolic computation, we have

$$\frac{1}{A}\big[R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 + R_{-1}\exp(-\xi) + R_{-2}\exp(-2\xi) + R_{-3}\exp(-3\xi)\big] = 0 \tag{25}$$

where

$$A = (b_{-1}\exp(-\xi) + b_0 + b_1 \exp(\xi))^3$$

$$R_3 = a_1^2 b_1 + a_1^2 k b_1 - a_1 b_1^2 k - a_1^3$$

$$\begin{aligned}R_2 = {} & a_1^2 b_0 - 3a_1^2 a_0 + c^2 a_0 b_1^2 + 2a_1 b_1 a_0 - \lambda a_0 b_1^2 + k a_1^2 b_0 \\ & - a_0 b_1^2 k + 2a_1 a_0 k b_1 - 2a_1 b_1 k b_0 - c^2 a_1 b_1 b_0 + \lambda a_1 b_1 b_0\end{aligned}$$

$$\begin{aligned}R_1 = {} & a_1^2 b_{-1} - 3a_1^2 a_{-1} - 3a_1 a_0^2 + a_0^2 b_1 + c^2 a_1 b_0^2 \\ & - 2\lambda a_{-1} b_1^2 - a_1 b_0^2 k + 4c^2 a_{-1} b_1^2 + 2a_1 b_1 a_{-1} \\ & - a_{-1} b_1^2 k + \lambda a_1 b_0^2 + a_1^2 b_{-1} k + a_0^2 k b_1 + 2a_1 b_0 a_0 \\ & - c^2 a_0 b_1 b_0 + 2a_1 b_0 a_0 k - 4c^2 a_1 b_1 b_{-1} - 2a_1 b_1 b_{-1} k \\ & - \lambda a_0 b_1 b_0 - 2a_0 b_1 b_0 k + 2\lambda a_1 b_1 b_{-1} + 2a_1 b_1 a_{-1} k\end{aligned}$$

$$\begin{aligned}R_0 = {} & -a_0^3 - a_0 b_0^2 k + a_0^2 k b_0 - 2a_1 b_0 k b_{-1} \\ & + 2a_1 a_0 k b_{-1} - 2a_{-1} b_1 k b_0 + 2a_1 b_0 a_{-1} + 2a_1 b_{-1} a_0 \\ & + a_0^2 b_0 + 3\lambda a_1 b_0 b_{-1} - 3\lambda a_{-1} b_1 b_0 + 3c^2 a_1 b_0 b_{-1} \\ & + 3c^2 a_{-1} b_0 b_1 - 6c^2 a_0 b_1 b_{-1} - 6a_1 a_0 a_{-1} \\ & + 2a_0 b_1 a_{-1} + 2a_1 a_{-1} k b_0 - 2a_0 b_1 k b_{-1} + 2a_0 a_{-1} k b_1\end{aligned}$$

$$\begin{aligned}R_{-1} = {} & a_0^2 b_{-1} - 3a_{-1} a_0^2 - 3a_1 a_{-1}^2 + b_1 a_{-1}^2 - 2\lambda a_{-1} b_1 b_{-1} \\ & + 2a_1 a_{-1} k b_{-1} + \lambda a_0 b_{-1} b_0 + 2a_0 a_{-1} k b_0 - c^2 a_0 b_0 b_{-1} \\ & - 2a_{-1} b_1 k b_{-1} - 2a_0 b_0 k b_{-1} - 4c^2 a_{-1} b_1 b_{-1} + 2\lambda a_1 b_{-1}^2 \\ & - \lambda a_{-1} b_0^2 + a_{-1}^2 k b_1 + 2a_1 b_{-1} a_{-1} - a_1 b_{-1}^2 k + 4c^2 a_1 b_{-1}^2 \\ & + a_0^2 k b_{-1} + 2a_0 b_0 a_{-1} + c^2 a_{-1} b_0^2 - a_{-1} b_0^2 k\end{aligned}$$

$$\begin{aligned}R_{-2} = {} & -3a_0 a_{-1}^2 + b_0 a_{-1}^2 - 2a_{-1} b_0 k b_{-1} + 2a_0 a_{-1} k b_{-1} \\ & - c^2 a_{-1} b_0 b_{-1} - \lambda a_{-1} b_0 b_{-1} + 2a_0 b_{-1} a_{-1} \\ & - a_0 b_{-1}^2 k + \lambda a_0 b_{-1}^2 + a_{-1}^2 k b_0 + c^2 a_0 b_{-1}^2\end{aligned}$$

$$R_{-3} = a_{-1}^2 b_{-1} + a_{-1}^2 k b_{-1} - a_{-1} b_{-1}^2 k - a_{-1}^3.$$

Solving this system of algebraic equations by using Maple, we obtain the following results

Case 1:

$$a_0 = 0, \quad b_{-1} = \frac{1}{5}a_{-1}, \quad b_0 = 0, \quad b_1 = a_1$$
$$\mu = 5, \quad \lambda = 6, \quad c = \pm\sqrt{2} \tag{26}$$

where a_{-1} and a_1 are free parameters. Substituting these results into (24), we obtain the exact solution (27), shown at the bottom of the page.

The evolution of exact solution for (27) with $\alpha = 0.5$ and $\alpha = 1.0$ is shown in Fig. 1.

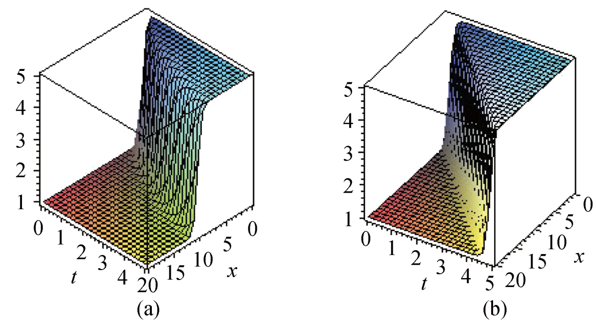


Fig. 1. The exact solution for (27) with (a) $\alpha = 0.5$ and (b) $\alpha = 1$, respectively, when $a_1 = 1, a_{-1} = -1$.

Case 2 :

$$a_0 = 0, \quad b_{-1} = a_{-1}, \quad b_0 = 0, \quad b_1 = \frac{1}{5}a_1$$
$$\mu = 5, \quad \lambda = -6, \quad c = \pm\sqrt{2} \tag{28}$$

where a_{-1} and a_1 are free parameters. Substituting these results into (24), we obtain the exact solution (29), shown at the bottom of the page.

Case 3:

$$a_0 = 0, \quad b_{-1} = b_{-1}, \quad b_0 = 0, \quad b_1 = a_1$$
$$\mu = \frac{a_{-1}}{b_{-1}}, \quad \lambda = \frac{a_{-1}^2 - b_{-1}^2}{4b_{-1}^2}, \quad c = \pm\frac{\sqrt{2}}{4b_{-1}}(a_{-1} - b_{-1}) \tag{30}$$

where a_{-1} and b_{-1} are free parameters which exist provided that $b_{-1} \neq 0$ and $a_{-1}^2 \neq b_{-1}^2$. Substituting these results into (24), we obtain the exact solution (31), shown at the bottom of the page.

$$u(x,t) = \frac{a_1 \exp(\pm\sqrt{2}x + \frac{6t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(\pm\sqrt{2}x + \frac{6t^\alpha}{\Gamma(1+\alpha)}))}{\frac{a_1}{5}\exp(\pm\sqrt{2}x + \frac{6t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(\pm\sqrt{2}x + \frac{6t^\alpha}{\Gamma(1+\alpha)}))} \tag{27}$$

$$u(x,t) = \frac{a_1 \exp(\pm\sqrt{2}x + \frac{6t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(\pm\sqrt{2}x + \frac{6t^\alpha}{\Gamma(1+\alpha)}))}{\frac{a_1}{5}\exp(\pm\sqrt{2}x + \frac{6t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(\pm\sqrt{2}x + \frac{6t^\alpha}{\Gamma(1+\alpha)}))} \tag{29}$$

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))}{a_1 \exp(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}) + b_{-1}\exp(-(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))} \tag{31}$$

Case 4 :

$$a_0 = 0, \quad b_{-1} = a_{-1}, \quad b_0 = 0, \quad b_1 = b_1$$
$$\mu = \frac{a_1}{b_1}, \quad \lambda = -\frac{a_1^2 - b_1^2}{4b_1^2}, \quad c = \pm\frac{\sqrt{2}}{4b_1}(a_1 - b_1) \tag{32}$$

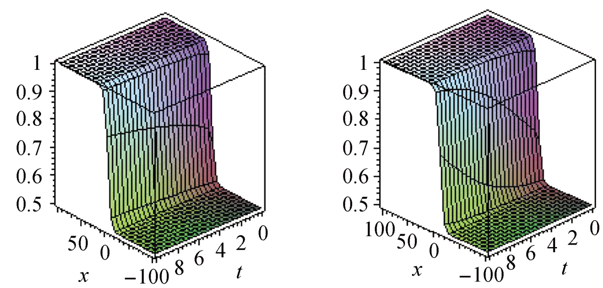
where a_1 and b_1 are free parameters which exist provided that $b_1 \neq 0$ and $a_1^2 \neq b_1^2$. Substituting these results into (24), we obtain the exact solution (33), shown at the bottom of the page.

Case 5 :

$$a_0 = 0, \quad b_{-1} = 2a_{-1}, \quad b_0 = \sqrt{-a_1 a_{-1}}, \quad b_1 = a_1$$
$$\mu = \frac{1}{2}, \quad \lambda = -\frac{3}{8}, \quad c = \pm\frac{\sqrt{2}}{4} \tag{27}$$

where a_1 and a_{-1} are free parameters. Substituting these results into (24), we obtain the exact solution (35), shown at the bottom of the page.

Obtained exact solution is described in Fig. 2.



(a) (b)

Fig. 2. The exact solution for (35) with (a) $\alpha = 0.5$ and (b) $\alpha = 1$, respectively, when $a_1 = 1, a_{-1} = -1$.

Case 6 :

$$a_0 = 0, \quad \lambda = \frac{a_{-1}^2 - b_{-1}^2}{2b_{-1}^2}$$
$$b_{-1} = b_{-1}, \quad b_0 = \pm\sqrt{-\frac{a_1}{a_{-1}}}(a_{-1} - b_{-1}), \quad b_1 = a_1$$
$$\mu = \frac{a_{-1}}{b_{-1}}, \quad c = \pm\frac{\sqrt{2}}{2b_{-1}}(a_{-1} - b_{-1}) \tag{36}$$

where a_1 and a_{-1} are free parameters which exist provided that $b_{-1} \neq 0$ and $a_{-1}^2 \neq b_{-1}^2$. Substituting these results into (24), we obtain the exact solution (37), shown at the bottom of the page.

Case 7 :

$$a_0 = 0, \quad b_{-1} = a_{-1}, \quad b_0 = \frac{\sqrt{-a_1 a_{-1}}}{2}, \quad b_1 = \frac{a_1}{2}$$
$$\mu = 2, \quad \lambda = -\frac{3}{2}, \quad c = \pm\frac{\sqrt{2}}{2} \tag{38}$$

where a_1 and a_{-1} are free parameters. Substituting these results into (24), we obtain the exact solution (39), shown at the bottom of the page.

Case 8 :

$$a_0 = 0, \quad \lambda = \frac{a_1^2 - b_1^2}{2b_1^2}$$
$$b_{-1} = a_{-1}, \quad b_0 = \pm\sqrt{-\frac{a_{-1}}{a_1}}(a_1 - b_1), \quad b_1 = b_1$$
$$\mu = \frac{a_1}{b_1}, \quad c = \pm\frac{\sqrt{2}}{2b_1}(a_1 - b_1) \tag{40}$$

where a_1 and a_{-1} are free parameters which exist provided that $b_1 \neq 0$ and $a_1^2 \neq b_1^2$. Substituting these results into (24), we obtain the exact solution (41), shown at the bottom of the page.

V. The Time Fractional KdV Equation

We consider the time fractional KdV equation

$$\frac{\partial^\alpha}{\partial t^\alpha}u + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0;\ 0 < \alpha \leq 1; x \in \mathbb{R} \tag{42}$$

subject to the initial condition:

$$u(x,0) = \frac{1}{2}\operatorname{sech}^2\left(\frac{1}{2}x\right) \tag{43}$$

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))}{b_1 \exp(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))} \tag{33}$$

$$u(x,t) = \frac{a_1 \exp(\pm\frac{\sqrt{2}}{4}x + \frac{3t^\alpha}{8\Gamma(1+\alpha)}) + a_{-1}\exp(-(\pm\frac{\sqrt{2}}{4}x + \frac{3t^\alpha}{8\Gamma(1+\alpha)}))}{a_1 \exp(\pm\frac{\sqrt{2}}{4}x + \frac{3t^\alpha}{8\Gamma(1+\alpha)}) + \sqrt{-a_1 a_{-1}} + 2a_{-1}\exp(-(\pm\frac{\sqrt{2}}{4}x + \frac{3t^\alpha}{8\Gamma(1+\alpha)}))} \tag{35}$$

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))}{a_1 \exp(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}) + \sqrt{-a_1 a_{-1}} + 2a_{-1}\exp(-(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))} \tag{37}$$

$$u(x,t) = \frac{a_1 \exp(\pm\frac{\sqrt{2}}{2}x + \frac{3t^\alpha}{2\Gamma(1+\alpha)}) + a_{-1}\exp(-(\pm\frac{\sqrt{2}}{2}x + \frac{3t^\alpha}{2\Gamma(1+\alpha)}))}{\frac{a_1}{2}\exp(\pm\frac{\sqrt{2}}{2}x + \frac{3t^\alpha}{2\Gamma(1+\alpha)}) + \frac{\sqrt{-a_1 a_{-1}}}{2} + a_{-1}\exp(-(\pm\frac{\sqrt{2}}{2}x + \frac{3t^\alpha}{2\Gamma(1+\alpha)}))} \tag{39}$$

$$u(x,t) = \frac{a_1 \exp(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}) + a_{-1}\exp(-(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))}{b_1 \exp(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}) \pm \sqrt{-\frac{a_{-1}}{a_1}}(a_1 - b_1) + a_{-1}\exp(-(cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}))} \tag{41}$$

where α is a parameter describing the order of the fractional time-derivative. The function $u(x,t)$ is assumed to be a causal function of time.

For our purpose, we introduce the following transformations;

$$u(x,t)=U(\xi),\qquad \xi=cx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \tag{44}$$

where c and λ are non-zero constants.

Substituting (44) with (2) and (8) into (42), we can show that (42) reduced into following ODE

$$-\lambda U'+6cUU'+c^3U'''=0 \tag{45}$$

where $U'=\dfrac{dU}{d\xi}$.

Integrating (45) with respect to ξ yields

$$-\lambda U+3cU^2+c^3U''+\xi_0=0 \tag{46}$$

where ξ_0 is a constant of integration.

By the same procedure as illustrated in Section III, we can determine values of c and p by balancing terms U^2 and U'' in (46). By symbolic computation, we have

$$U''=\frac{c_1\exp\big[(3p+c)\xi\big]+\cdots}{c_2\exp\big[4p\xi\big]+\cdots} \tag{47}$$

and

$$U^2=\frac{\cdots+c_3\exp\big[2c\xi\big]}{\cdots+c_4\exp\big[2p\xi\big]} \tag{48}$$

where c_i are determined coefficients only for simplicity. According to exp-function method, balancing lowest order of (47) and (48), we have

$$3p+c=2c+2p \tag{49}$$

that gives

$$p=c. \tag{50}$$

In the same way, we balance the linear term of highest order in (46)

$$U''=\frac{\cdots+d_1\exp\big[-(3q+d)\xi\big]}{\cdots+d_2\exp[-4q\xi]} \tag{51}$$

and

$$U^2=\frac{d_3\exp\big[-2d\xi\big]+\cdots}{d_4\exp\big[-2q\xi\big]+\cdots} \tag{52}$$

where d_i are determined coefficients only for simplicity. From

(51) and (52), we get

$$-(3q+d)=-(2d+2q) \tag{53}$$

and this gives

$$q=d. \tag{54}$$

For simplicity, we set $p=c=1$ and $q=d=1$, so (11) reduces to

$$U(\xi)=\frac{a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)}{b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)}. \tag{55}$$

Substituting (55) into (46), and by the help of Maple, we have

$$\frac{1}{A}\Big[R_3\exp(3\xi)+R_2\exp(2\xi)+R_1\exp(\xi)+R_0 +R_{-1}\exp(-\xi)+R_{-2}\exp(-2\xi)+R_{-3}\exp(-3\xi)\Big]=0 \tag{56}$$

where

$$A=(b_{-1}\exp(-\xi)+b_0+b_1\exp(\xi))^3$$

$$R_3=-\lambda a_1b_1^2+kb_1^3+3ca_1^2b_1$$

$$R_2=c^3a_0b_1^2+3kb_1^2b_0-\lambda a_0b_1^2+3ca_1^2b_0-2\lambda a_1b_1b_0 +6ca_1a_0b_1-c^3a_1b_1b_0$$

$$R_1=-2\lambda a_0b_1b_0+6ca_1b_0a_0-c^3a_0b_1b_0+3kb_1b_0^2-\lambda a_1b_0^2 +3ca_0^2b_1+c^3a_1b_0^2+4c^3a_{-1}b_1^2+3kb_1^2b_{-1}+3ca_1^2b_{-1} -\lambda a_{-1}b_1^2-2\lambda a_1b_1b_{-1}+6ca_1a_{-1}b_1-4c^3a_1b_1b_{-1}$$

$$R_0=3ca_0^2b_0+6ca_0a_{-1}b_1+kb_0^3-\lambda a_0b_0^2+6kb_{-1}b_1b_0 +3c^3a_1b_0b_{-1}+3c^3a_{-1}b_1b_0-6c^3a_0b_1b_{-1}-2\lambda a_1b_0b_{-1} -2\lambda a_0b_1b_{-1}-2\lambda a_{-1}b_1b_0+6ca_1a_0b_{-1}+6ca_1a_{-1}b_0$$

$$R_{-1}=-2\lambda a_0b_{-1}b_0+6ca_0a_{-1}b_0-c^3a_0b_0b_{-1}+3kb_0^2b_{-1} -\lambda a_{-1}b_0^2+3ca_0^2b_{-1}+c^3a_{-1}b_0^2+3ca_{-1}^2b_1+4c^3a_1b_{-1}^2 +3kb_1b_{-1}^2-\lambda a_1b_{-1}^2-2\lambda a_{-1}b_1b_{-1}+6ca_1a_{-1}b_{-1} -4c^3a_{-1}b_1b_{-1}$$

$$R_{-2}=c^3a_0b_{-1}^2+3kb_0b_{-1}^2-\lambda a_0b_{-1}^2+3cb_0a_{-1}^2-2\lambda a_2b_0b_{-1} +6ca_0a_{-1}b_{-1}-c^3a_{-1}b_0b_{-1}$$

$$R_{-3}=-\lambda a_{-1}b_{-1}^2+3ca_{-1}^2b_{-1}+kb_{-1}^3. \tag{57}$$

Solving this system of algebraic equations by using Maple, we obtain the following results

$$\begin{aligned} &a_1=a_1, && a_0=\frac{b_0}{b_1}(c^2b_1+a_1), && a_{-1}=\frac{a_1b_0^2}{4b_1^2}\\ &b_1=b_1, && b_0=b_0, && b_{-1}=\frac{b_0^2}{4b_1}\\ &\xi_0=\frac{a_1c(c^2b_1+3a_1)}{b_1^2}, && \lambda=\frac{c(c^2b_1+6a_1)}{b_1} \end{aligned} \tag{58}$$

where a_1 , b_0 and b_1 are free parameters which exist provided that $b_1\neq 0$ and $c^2b_1+6a_1\neq 0$. Substituting these results into (56), we obtain the exact solution (59), shown at the bottom of the page.

Also, $u(x,t)$ in (59) is represented in Fig. 3.

$$u(x,t)=\frac{a_1\exp(cx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+\frac{b_0}{b_1}(c^2b_1+a_1)+\frac{a_1b_0^2}{4b_1^2}\exp(-(cx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))}{b_1\exp(cx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+b_0+\frac{b_0^2}{4b_1}\exp(-(cx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}))} \tag{59}$$

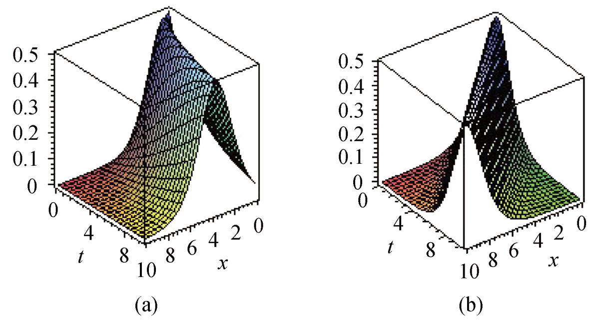


Fig. 3. The exact solution for (59) with (a) $\alpha = 0.5$ and (b) $\alpha = 1$ respectively, when $c = 1, a_1 = 0, b_1 = 1, b_0 = 2$.

Comparing our results with the results [45], it can be seen that our results are new.

VI. CONCLUSION

exp-function method known as very powerful and an effective method for solving nonlinear problems and ordinary, partial, difference, fractional equations and so many other equations. The basic idea described in this paper is expected to be further employed to solve other similar nonlinear equations in fractional calculus. To our knowledge, these new solutions have not been reported in former literature, they may be of significant importance for the explanation of some special physical phenomena. As a result, many new and more rational solitary wave solutions are obtained, from which hyperbolic function and trigonometric function solutions.

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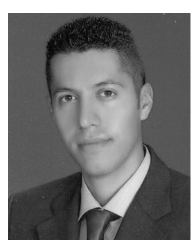

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