Local Bifurcation Analysis of a Delayed Fractional-order Dynamic Model of Dual Congestion Control Algorithms

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***Abstract*—In this paper, we propose a delayed fractional-order congestion control model which is more accurate than the original integer-order model when depicting the dual congestion control algorithms. The presence of fractional orders requires the use of suitable criteria which usually make the analytical work so harder. Based on the stability theorems on delayed fractional-order differential equations, we study the issue of the stability and bifurcations for such a model by choosing the communication delay as the bifurcation parameter. By analyzing the associated characteristic equation, some explicit conditions for the local stability of the equilibrium are given for the delayed fractional-order model of congestion control algorithms. Moreover, the Hopf bifurcation conditions for general delayed fractional-order systems are proposed. The existence of Hopf bifurcations at the equilibrium is established. The critical values of the delay are identified, where the Hopf bifurcations occur and a family of oscillations bifurcate from the equilibrium. Same as the delay, the fractional order normally plays an important role in the dynamics of delayed fractional-order systems. It is found that the critical value of Hopf bifurcations is crucially dependent on the fractional order. Finally, numerical simulations are carried out to illustrate the main results.**

***Index Terms*—Congestion control algorithm, fractional-order congestion control algorithm model, Hopf bifurcation, stability.**

I. INTRODUCTION

FRACTIONAL calculus and its applications to physics, biology and engineering have become a subject of intense research activities. It has been found that dynamical equations using fractional derivatives are useful and more accurate in the mathematical modeling of real world phenomena arising from several interdisciplinary fields, such as diffusion and wave propagation [1], electromagnetic waves [2], viscoelastic liquids [3], dielectric polarization [4], control [5], and biology [6]. As a result of growing applications, the study of dynamics of fractional-order systems has attracted considerable interest of many researchers and numerous important results have been reported, including the stability [7], bifurcations [8], chaos [9], and synchronization [10].

With the rapid development of the Internet, the congestion control mechanism is a focus of interest to many researchers in the past few years [11]−[13] since the seminal work [14]. One of the important properties of congestion control algorithms is the stability. Sufficient conditions for stability are given for congestion control systems [15]−[18]. However, it is found in [19], [20] that some common AQM (active queue management) schemes coupled with the current congestion avoidance TCP (transmission control protocol) algorithm may lose the local stability due to an increase in delays or capacity, or a decrease in the number of connections. The loss of stability causes some nonlinear dynamical behaviors such as chaos and bifurcation. Therefore, in addition to investigation of stability, the Hopf bifurcation and control have also begun to draw much attention from researchers [21]−[26].

Unlike integer-order derivatives that are local operators, fractional-order derivatives are non-local integro-differential operators [27]. As such, they can be used to represent memory effects and long-range dispersion processes. In the last decade, fractional-order models have been an active field of research both from a theoretical and applied perspective. For instance, the resistance-capacitance-inductance (RLC) interconnect model of a transmission line is a fractional-order model [28]. Heat conduction can be more adequately modeled by fractional-order models than by their integer order counterparts [29]. In biology, it has been shown that the membranes of cells of biological organism have a fractional-order electrical conductance [30]. In economics, it is known that some financial systems can display fractional-order dynamics [31].

There have been many results on Hopf bifurcations for a variety of delayed integer-order congestion control systems recently [21]−[26]. However, to the best of our knowledge, few studies of Hopf bifurcations for delayed fractional-order congestion control systems have been, reported. It should be

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mentioned that the qualitative theory of Hopf bifurcations for the case of fractional-order dynamical systems has not completely settled yet. Thus, the Hopf bifurcation theory in fractional-order dynamical systems is still an open problem. In this paper, we will establish some bifurcation conditions for delayed fractional-order dynamical systems.

Motivated by the above discussions, this paper is devoted to investigating the stability and bifurcations for a delayed fractional-order congestion control model. The sufficient conditions for the stability of the equilibrium are given for the delayed fractional-order congestion control model. The Hopf bifurcation conditions are proposed for delayed fractional-order systems when the delay is chosen as the bifurcation parameter. Then, the critical values of the delay are identified in the delayed fractional-order congestion control model, where Hopf bifurcations occur and a family of oscillations bifurcate from the equilibrium. It is worth mentioning that the observations in this paper can help to design the Hopf bifurcation of congestion control systems with the desired bifurcation point via adjusting the delay and fractional-order.

The paper is organized as follows. In Section II, some preliminaries on delayed fractional-order systems are summarized. In Section III, a delayed fractional-order model of fair dual congestion control algorithms is proposed. In Section IV, by analyzing the associated characteristic equation, the stability condition is derived for the delayed fractional-order congestion control model. The existence of the Hopf bifurcation is established when the communication delay is chosen as the bifurcation parameter. In Section V, numerical simulations are given to illustrate the results. Finally, the conclusions are drawn in Section VI.

II. PRELIMINARIES

Generally speaking, there are three definitions of fractional derivative, i.e., the Grünwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative, and Caputo fractional derivative [27]. Due to taking on the same form as integer order differential on the initial conditions, which has well-understood physical meanings and has more applications in engineering, here we only discuss the Caputo derivative which is defined as follows:

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \tag{1} $$

where $n-1 < \alpha < n$, $n \in \mathbb{N}$, and $\Gamma(\cdot)$ is the Gamma function. The symbol α denotes the value of the fractional order that is usually chosen in the range $0 < \alpha \le 1$ in engineering.

The Laplace transform of the Caputo fractional derivative (1) at $a = 0$ is given by

$$ \mathscr{L}\{{}_0^C D_t^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). \tag{2} $$

If $f^{(k)}(0) = 0, k = 0, 1, \dots, n-1$, then $\mathscr{L}\{{}_0^C D_t^{\alpha} f(t)\} = s^{\alpha} F(s)$.

A class of n-dimensional linear fractional-order systems with multiple time delays can be represented in the following form [32]:

$$ \begin{aligned} \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} &= a_{11} x_1(t-\tau_{11}) + a_{12} x_2(t-\tau_{12}) \\ &\quad + \cdots + a_{1n} x_n(t-\tau_{1n}) \\ \frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} &= a_{21} x_1(t-\tau_{21}) + a_{22} x_2(t-\tau_{22}) \\ &\quad + \cdots + a_{2n} x_n(t-\tau_{2n}) \\ &\vdots \\ \frac{d^{\alpha_n} x_n}{dt^{\alpha_n}} &= a_{n1} x_1(t-\tau_{n1}) + a_{n2} x_2(t-\tau_{n2}) \\ &\quad + \cdots + a_{nn} x_n(t-\tau_{nn}) \end{aligned} \tag{3} $$

where $0 < \alpha_i \le 1$ for $i = 1, 2, \dots, n$, and the notation $\frac{d^{\alpha_i}}{dt^{\alpha_i}}$ is chosen as the Caputo fractional derivative (1). The initial values $x_i(t) = \phi_i(t)$ are given for $-\tau_{\max} \le t \le 0$, $i = 1, 2, \dots, n$, where $\tau_{\max} = \max_{1 \le i,j \le n}\{\tau_{ij}\}$.

Next, we introduce some stability results on the delayed fractional-order system (3). The stability of the zero solution of system (3) depends on the distribution of roots of the associated characteristic equation (4), as shown at the bottom of this page.

Theorem 1 [32]: The zero solution of system (3) is Lyapunov globally asymptotically stable if all the roots of the characteristic equation (4) have negative real parts.

Remark 1: If $\alpha_i = 1$, $i = 1, 2, \dots, n$, then the characteristic equation of (3) is reduced to the characteristic equation of delay differential equations. If $\tau_{ij} = 0$, $i, j = 1, 2, \dots, n$ and $\alpha_i = 1$, $i = 1, 2, \dots, n$, then the characteristic equation of (3) is reduced to $\det(sI - A) = 0$, where the coefficient $A = (a_{ij})_{n \times n}$. This coincides with the definition of the characteristic equation for ordinary differential equations.

Corollary 1 [32]: Suppose that $\tau_{ij} = 0, i, j = 1, 2, \dots, n$ and $\alpha_i = \alpha \in (0, 1]$, $i = 1, 2, \dots, n$. If all the roots of the characteristic equation $\det(sI - A) = 0$ satisfy $|\arg(s)| > \alpha\pi/2$, then the zero solution of system (3) is Lyapunov globally asymptotically stable.

Corollary 1 is the Matignon criterion (Theorem 2 of [33]).

Corollary 2 [32]: If $\alpha_i = \alpha \in (0, 1]$, $i = 1, 2, \dots, n$, all the eigenvalues λs of A satisfy $|\arg(s)| > \alpha\pi/2$ and the

$$ \det \begin{pmatrix} s^{\alpha_1} - a_{11} e^{-s\tau_{11}} & -a_{12} e^{-s\tau_{12}} & \cdots & -a_{1n} e^{-s\tau_{1n}} \\ -a_{21} e^{-s\tau_{21}} & s^{\alpha_2} - a_{22} e^{-s\tau_{22}} & \cdots & -a_{2n} e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} e^{-s\tau_{n1}} & -a_{n2} e^{-s\tau_{n2}} & \cdots & s^{\alpha_n} - a_{nn} e^{-s\tau_{nn}} \end{pmatrix} = 0 \tag{4} $$

characteristic equation (4) has no purely imaginary roots for any $\tau_{ij} > 0,\ i,\ j = 1, 2, \ldots, n$, then the zero solution of system (3) is Lyapunov globally asymptotically stable.

There are a number of substantial differences between integer-order dynamical systems and fraction-order dynamical systems. Therefore, most results on the delayed integer-order model of congestion control algorithms cannot be simply extended to the case of fractional order one. As is well known, limit cycles of integer-order dynamical systems are isolated periodic oscillations, whose appearance can be explained using the Hopf bifurcation theory [34]. However, to the best of our knowledge, there is no Hopf bifurcation qualitative theory developed thoroughly for the case of fractional-order dynamical systems yet, and thus, the Hopf bifurcation theory in fractional-order dynamical systems is still an open problem.

The Hopf bifurcation conditions for fractional-order dynamical systems without delays were proposed based on numerical simulations, but were not proved in [35], [36]. There are seldom reports about the Hopf bifurcation of delayed fractional-order dynamical systems.

In this paper, we are interested in the stability and bifurcation in delayed fractional-order congestion control systems.

III. MODEL DESCRIPTIONS

The dual algorithms are a subset of a larger class of congestion control mechanisms. In these algorithms the resource determines its congestion measure or price, by an averaging process at the link, which is then communicated back to the end-systems. To facilitate a control theoretic study, caricatures of rate control or window-based algorithms are often converted into delayed integer-order differential equations [19], [37]−[39]. It has been found from the study of such integer-order equations that some congestion control mechanisms may lose the local stability with an increase in delays or capacity, or a decrease in the number of connections, which is triggered by the Hopf bifurcation.

Raina [19] introduced the following dynamical representation of a fair dual congestion control algorithm:

$$\frac{d}{dt}p(t) = \kappa p(t)(x(t-\tau) - C) \tag{5}$$

where the variable p is the price at the link, τ is the communication delay, $\kappa > 0$ is the gain parameter, and the scalar $C > 0$ is the capacity. In addition, $x(t) = \mathcal{D}(p(t))$ with $\mathcal{D}(p)$, $p \geq 0$, a non-negative, continuous and strictly decreasing demand function, and $\mathcal{D}(p)$ can be expressed by $(w/p)^{1/\gamma}$, where $w > 0$ may be viewed as a willingness to pay parameter of the user, and $\gamma > 0$ is the fair allocation parameter [40].

The integer-order dual algorithm model (5) has been extensively studied regarding its bifurcation and control by many researchers in the past years [19], [23], [41]. The local Hopf bifurcation was studied for model (5) by choosing the non-dimensional parameter κ as the bifurcation parameter [19]. Explicit conditions were derived to ensure the onset of stable limit cycles as model (5) just loses its local stability, and the direction of Hopf bifurcations was also determined by applying the normal form theory and center manifold theorem. On the other hand, unlike the work in [19] where the gain parameter κ was considered as the bifurcation parameter, the authors used the communication delay τ as the bifurcation parameter [23]. It was demonstrated that model (5) loses its stability and a Hopf bifurcation occurs when the delay τ passes through a critical value. Moreover, the bifurcating periodic solution was calculated by means of the perturbation method. A hybrid control strategy using both the state feedback and parameter perturbation was applied to control the undesirable Hopf bifurcation of model (5) [41]. It was shown that this proposed method can delay the onset of bifurcations effectively, and thus extend the stable range in the parameter space and improve the performance of congestion control systems.

Compared with the classical integer-order models, fractional-order models are characterized by infinite memory. Congestion control systems include round trip propagation delays. Therefore, the incorporation of a memory term into a congestion control model is an extremely important improvement. Moreover, the fractional-order congestion control models are more accurate than the original integer-order models when modeling some congestion control algorithms. Thus, studying fractional-order congestion control models is of great significance.

In this paper, we replace the usual integer-order derivative by the fractional-order Caputo derivative (1) in the fair dual congestion control algorithm model (5). The new model is then described by the following delayed fractional-order differential equation:

$$\frac{d^\alpha p}{dt^\alpha} = \kappa p(t)(x(t-\tau) - C) \tag{6}$$

where $\alpha \in (0, 1]$.

Suppose that p^* is a non-zero equilibrium of (6). Then it satisfies the following equation:

$$\mathcal{D}(p^*) = C. \tag{7}$$

It should be underlined that p^* is an equilibrium of model (6) with the fractional order α if and only if it is an equilibrium of the integer-order model (5).

IV. STABILITY AND BIFURCATION ANALYSIS

In this section, we investigate the stability and bifurcation of the delayed fractional-order model (6) of fair dual congestion control algorithms.

A. Stability Analysis

Let $u(t) = p(t) - p^*$ and the equilibrium p^* is shifted to the origin. The linearized model of (6) is

$$\frac{d^\alpha u}{dt^\alpha} = \kappa p^* \mathcal{D}'(p^*) u(t-\tau) \tag{8}$$

with the characteristic equation

$$s^\alpha - \kappa p^* \mathcal{D}'(p^*) e^{-s\tau} = 0. \tag{9}$$

Theorem 2: If $[-\kappa p^* \mathcal{D}'(p^*)]^{1/\alpha} \neq [(2k+1)\pi - \alpha\pi/2]/\tau$, where $k \in \mathbb{Z}$, then the equilibrium p^* of model (6) is Lyapunov globally asymptotically stable.

Proof: Let $s = i\omega = \omega(\cos \pi/2 + i \sin \pi/2)(\omega > 0)$ be a root of (9). Then

$$\omega^\alpha(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}) - \kappa p^* \mathcal{D}'(p^*)(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts gives

$$\begin{aligned}&\omega^\alpha\cos\frac{\alpha\pi}{2}-\kappa p^*\mathcal{D}'(p^*)\cos\omega\tau=0\\&\omega^\alpha\sin\frac{\alpha\pi}{2}+\kappa p^*\mathcal{D}'(p^*)\sin\omega\tau=0.\end{aligned}\tag{10}$$

Taking square on the both sides of (10) and summing them up give

$$\begin{aligned}&(\omega^\alpha)^2+[\kappa p^*\mathcal{D}'(p^*)]^2\\&\quad-2\omega^\alpha\kappa p^*\mathcal{D}'(p^*)\cos(\tfrac{\alpha\pi}{2}+\omega\tau)=0.\end{aligned}\tag{11}$$

Notice that $\kappa>0, p^*>0$, and $\mathcal{D}'(p^*)<0$. It is straightforward to obtain that

$$\begin{aligned}&(\omega^\alpha)^2+[\kappa p^*\mathcal{D}'(p^*)]^2-2\omega^\alpha\kappa p^*\mathcal{D}'(p^*)\cos(\frac{\alpha\pi}{2}+\omega\tau)\\&\geq(\omega^\alpha)^2+[\kappa p^*\mathcal{D}'(p^*)]^2+2\omega^\alpha\kappa p^*\mathcal{D}'(p^*)\\&=[\omega^\alpha+\kappa p^*\mathcal{D}'(p^*)]^2.\end{aligned}$$

Obviously, if $[-\kappa p^*\mathcal{D}'(p^*)]^{1/\alpha}\neq[(2k+1)\pi-\alpha\pi/2]/\tau$, then (11) has no positive real roots, meaning that (9) has no purely imaginary roots with positive imaginary parts.

Let $s=i\omega=-\omega[\cos\pi/2+i\sin(-\pi/2)]$ $(\omega<0)$ be a root of (9). It is similar to prove that (9) has no purely imaginary roots with negative imaginary parts under the assumption $[-\kappa p^*\mathcal{D}'(p^*)]^{1/\alpha}\neq[(2k+1)\pi-\alpha\pi/2]/\tau$.

Thus, if $[-\kappa p^*\mathcal{D}'(p^*)]^{1/\alpha}\neq[(2k+1)\pi-\alpha\pi/2]/\tau$, then the characteristic equation (9) has no purely imaginary roots.

On the other hand, it is easy to see that the coefficient A of the linearized model (8) has one eigenvalue $s=\kappa p^*\mathcal{D}'(p^*)<0$ satisfying $|\arg(s)|>\alpha\pi/2$.

Applying Corollary 2, the equilibrium p^* of model (6) is Lyapunov globally asymptotically stable. ∎

Remark 2: Although nonlinear dynamics of integer-order congestion control systems were investigated in [19], [23], [37]−[41], to date, the theoretical results on the stability with respect to the system parameters and order have not been reported yet for fractional-order congestion control systems.

For illustration of Theorem 2, we consider the fractional-order model (6) with $\kappa=0.02, C=40, \tau=1, \alpha=0.9$, and the proportional fairness [19] with $\gamma=1, w=1$. The equilibrium can be found by solving (7), yielding $p^*=0.025$. It is easy to verify that the condition $[-\kappa p^*\mathcal{D}'(p^*)]^{1/\alpha}\neq[(2k+1)\pi-\alpha\pi/2]/\tau$ holds. Fig. 1 shows that the state $p(t)$ of model (6) is globally asymptotically decreasing toward the equilibrium p^*.

B. Hopf Bifurcation

It is well known that the Hopf bifurcation is the birth of a limit cycle from an equilibrium in integer-order dynamical systems, when the equilibrium changes the stability via a pair of purely imaginary eigenvalues. However, the qualitative theory of Hopf bifurcations for fractional-order dynamical systems has not been constructed yet. In this Subsection, we study the local bifurcation of the delayed fractional-order model (6) by regarding the delay τ as the bifurcation parameter.

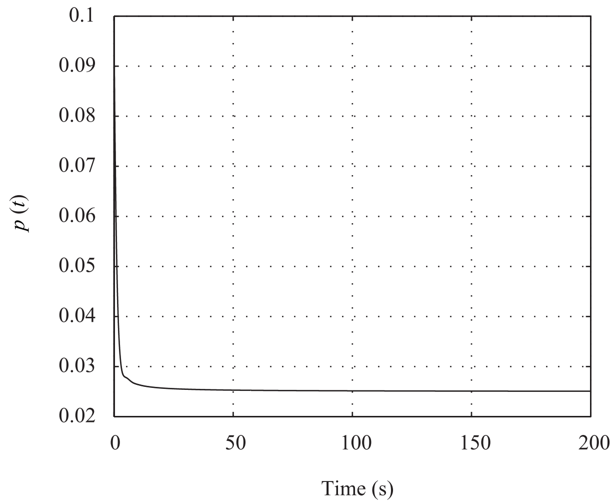

Fig. 1. Equilibrium $p^*=0.025$ of model (6) is Lyapunov globally asymptotically stable when $\kappa=0.02, C=40, \tau=1, \gamma=1, w=1, \alpha=0.9$, and the initial condition $p^0=0.1$.

First, we put forward the Hopf bifurcation conditions for general delayed fractional-order systems. Consider the following n-dimensional fractional-order system with delay:

$$\frac{d^\alpha x_i}{dt^\alpha}=f_i(x_1,x_2,\ldots,x_n;\tau),\ i=1,2,\ldots,n\tag{12}$$

where $0<\alpha\leq1$ and the time delay $\tau\geq0$. According to Corollary 2, we propose the conditions of (12) to undergo a Hopf bifurcation at the equilibrium $x^*=(x_1^*,x_2^*,\ldots,x_n^*)$ when $\tau=\tau_0$ as follows:

1) All the eigenvalues of the coefficient matrix of the linearized system of (12) satisfy $|\arg(s)|>\alpha\pi/2$.

2) The characteristic equation of (12) has a purely imaginary roots $\pm i\omega_0$ when $\tau=\tau_0$.

3) $\frac{d\mathrm{Re}[s(\tau)]}{d\tau}\big|_{\tau=\tau_0}>0$, where $\mathrm{Re}\{\cdot\}$ denotes the real part of the complex eigenvalue.

Remark 3: The condition 1) guarantees the stability of the equilibrium x^* of the delayed fractional-order system (12) when $\tau=0$. It is well known that the Routh-Hurwitz criterion is the necessary and sufficient condition for the stability of the equilibrium of integer-order dynamical systems. It should be noted that this criterion can also ensure the stability of the equilibrium of fractional-order dynamical systems.

Remark 4: The condition 3) is the transversality condition of Hopf bifurcations of the delayed fractional-order system (12).

Remark 5: The Hopf bifurcation conditions for fractional-order dynamical systems without time delays by the observations from numerical simulations were proposed in [35], [36]. We formulate the conditions of Hopf bifurcation of fractional-order dynamical systems with time delays in this paper.

Lemma 1: If $\tau=\tau_k, k=0,1,\ldots$, then (9) has a purely imaginary roots $\pm i\omega_0(\omega_0>0)$, where

$$\begin{aligned}\tau_k&=\frac{(2k+1)\pi-\frac{\alpha\pi}{2}}{[-\kappa p^*\mathcal{D}'(p^*)]^{1/\alpha}}\\\omega_0&=[-\kappa p^*\mathcal{D}'(p^*)]^{1/\alpha}.\end{aligned}\tag{13}$$

Proof: From the proof of Theorem 2, we can see that (9) has a pair of purely imaginary roots when $[-\kappa p^*\mathcal{D}'(p^*)]^{1/\alpha} = [(2k+1)\pi - \alpha\pi/2]/\tau$. Therefore, the conclusion follows immediately. ■

Remark 6: Lemma 1 illustrates that the proposed condition 2) of Hopf bifurcation is reached for the delayed fractional-order model (6).

Lemma 2: Let $s(\tau) = \rho(\tau) + i\omega(\tau)$ be the root of (9) satisfying $\rho(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0 > 0, k = 0, 1, \ldots$. Then,

$$\frac{d\mathrm{Re}[s(\tau)]}{d\tau}\Big|_{\tau=\tau_k} > 0.$$

Proof: Substituting $s(\tau)$ into (9) and differentiating both sides of the resulting equation with respect to τ, we obtain

$$\alpha s^{\alpha-1}\frac{ds}{d\tau} + \kappa p^*\mathcal{D}'(p^*)e^{-s\tau}[\tau\frac{ds}{d\tau} + s] = 0.$$

Thus

$$\frac{ds}{d\tau} = \frac{-\kappa p^*\mathcal{D}'(p^*)se^{-s\tau}}{\alpha s^{\alpha-1} + \kappa p^*\mathcal{D}'(p^*)\tau e^{-s\tau}}.$$

Note that $s(\tau) = \rho(\tau) + i\omega(\tau) = r(\cos\theta + i\sin\theta)$ is the root of (9). Then we have

$$\frac{ds}{d\tau} = \frac{-\kappa p^*\mathcal{D}'(p^*)[\rho + \mathrm{i}\omega]e^{-\rho\tau}[\cos(\omega\tau) - \mathrm{i}\sin(\omega\tau)]}{\alpha[\rho + \mathrm{i}\omega]^{\alpha-1} + \kappa p^*\mathcal{D}'(p^*)\tau e^{-\rho\tau}[\cos(\omega\tau) - \mathrm{i}\sin(\omega\tau)]}.$$

From this we obtain

$$\frac{d\mathrm{Re}[s(\tau)]}{d\tau} = -\kappa p^*\mathcal{D}'(p^*)e^{-\rho\tau}\frac{P(\tau)M(\tau) + Q(\tau)N(\tau)}{M^2(\tau) + N^2(\tau)}$$

in which

$$P(\tau) = \rho\cos\omega\tau + \omega\sin\omega\tau$$
$$Q(\tau) = \omega\cos\omega\tau - \rho\sin\omega\tau$$
$$M(\tau) = \alpha r^{\alpha-1}\cos(\alpha-1)\theta + \kappa p^*\mathcal{D}'(p^*)\tau e^{-\rho\tau}\cos\omega\tau$$
$$N(\tau) = \alpha r^{\alpha-1}\sin(\alpha-1)\theta - \kappa p^*\mathcal{D}'(p^*)\tau e^{-\rho\tau}\sin\omega\tau.$$

Replacing τ by τ_k, it follows that:

$$\begin{aligned}\frac{d\mathrm{Re}[s(\tau)]}{d\tau}\Big|_{\tau=\tau_k} &= -\kappa p^*\mathcal{D}'(p^*)\frac{P(\tau_k)M(\tau_k) + Q(\tau_k)N(\tau_k)}{M^2(\tau_k) + N^2(\tau_k)}\\ &= -\kappa p^*\mathcal{D}'(p^*)\frac{\alpha(\omega_0)^\alpha\sin[\omega_0\tau_k + (\alpha-1)\frac{\pi}{2}]}{M^2(\tau_k) + N^2(\tau_k)}\end{aligned}$$

where

$$P(\tau_k) = \omega_0\sin(\omega_0\tau_k)$$
$$Q(\tau_k) = \omega_0\cos(\omega_0\tau_k)$$
$$M(\tau_k) = \alpha(\omega_0)^{\alpha-1}\cos(\alpha-1)\frac{\pi}{2} + \kappa p^*\mathcal{D}'(p^*)\tau_k\cos(\omega_0\tau_k)$$
$$N(\tau_k) = \alpha(\omega_0)^{\alpha-1}\sin(\alpha-1)\frac{\pi}{2} - \kappa p^*\mathcal{D}'(p^*)\tau_k\sin(\omega_0\tau_k).$$

It can be seen from (13) that $\omega_0\tau_k = (2k+1)\pi - \alpha\pi/2$, implying that $\sin\ [\omega_0\tau_k + (\alpha-1)\pi/2] = 1$. Moreover, note that $-\kappa p^*\mathcal{D}'(p^*) > 0$. Therefore

$$\frac{d\mathrm{Re}[s(\tau)]}{d\tau}\Big|_{\tau=\tau_k^+} > 0.$$

The conclusion follows. ■

Remark 7: Lemma 2 implies that the transversality condition 3) of Hopf bifurcations is satisfied for the delayed fractional-order model (6).

Theorem 3: For model (6), the following results hold.

1) The equilibrium p^* of model (6) is asymptotically stable for $\tau \in [0, \tau_0)$, and unstable when $\tau > \tau_0$.

2) Model (6) undergoes a Hopf bifurcation at the equilibrium p^* when $\tau = \tau_0$.

Proof: Note that the coefficient matrix of the linearized (8) has the eigenvalue $\lambda = \kappa p^*\mathcal{D}'(p^*) < 0$ satisfying the inequality $|\arg(s)| > \alpha\pi/2$. Thus, the condition 1) of Hopf bifurcations is satisfied for model (6).

1) It is easy to see that all the roots of (9) with $\tau = 0$ have negative real parts. From Lemma 1, the definition of τ_0 implies that all the roots of (9) have negative real parts for $\tau \in [0, \tau_0)$. The conclusion in Lemma 2 indicates that (9) has at least one root with positive real part when $\tau > \tau_0$. Thus, the conclusion follows.

2) From Remarks 6 and 7, we know that the conditions 2) and 3) of Hopf bifurcations are satisfied for model (6). Hence, a Hopf bifurcation occurs at the equilibrium p^* when $\tau = \tau_0$.■

Remark 8: The Hopf bifurcation theory in fractional-order dynamical systems is still an open problem. The Hopf bifurcation conditions for fractional-order systems without delays are proposed based on the observations from numerical simulations [35], [36]. However, there are few results on the Hopf bifurcation of delayed fractional-order systems.

Remark 9: The integer-order congestion control model (5) may display a Hopf bifurcation when the delay τ passes through the critical values [23]. However, the corresponding fractional-order model (6) will not produce the bifurcation at the same values, which will be confirmed by numerical simulations later.

Remark 10: The order and system parameter were chosen as the bifurcation parameters in fractional-order neural network models in [8]. In this paper, we use the delay τ as the bifurcation parameter in fractional-order congestion control models.

V. NUMERICAL SIMULATIONS

In this section, we present some numerical results to illustrate the analytical results obtained in the previous section, displaying the Hopf bifurcation phenomenon of the delayed fractional-order model (6) of fair dual congestion control algorithms. Simulations are performed using the method introduced in [42] to find the solution of delayed fractional-order differential equations. This method is the improved version of the Adams-Bashforth-Moulton algorithm and is proposed based on the predictor correctors scheme.

For a consistent comparison, we discuss model (6) with the same system parameters used in [23]: $\kappa = 0.01, C = 50$, and the proportional fairness [19] with $\mathcal{D}(p) = 1/p$. From (7), model (6) has a unique non-zero equilibrium $p^* = 0.02$. For model (6) with $\alpha = 1$ (integer-order model (5)), it follows from Theorem 1 in [23] that

$$\tau_0 = 3.1416, \quad \omega_0 = 0.5.$$

The dynamical behavior of the integer-order model (5) is illustrated in Figs. 2–4. From Theorem 1 in [23], it is shown

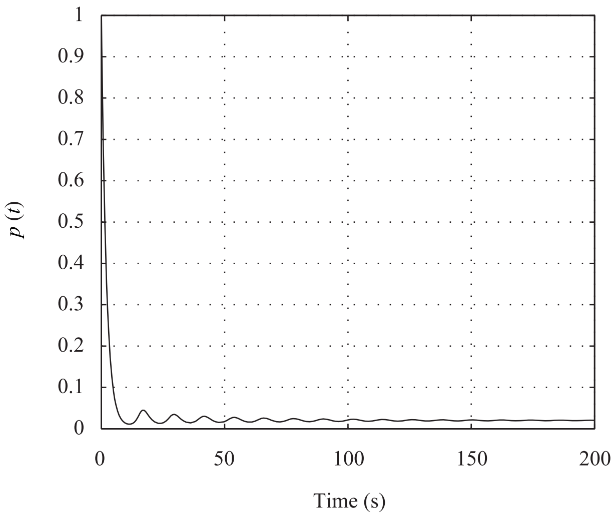

Fig. 2. The equilibrium $p^* = 0.02$ of the integer-order model (5) is asymptotically stable, where $\kappa = 0.01, C = 50, \mathcal{D}(p) = 1/p$, the initial condition $p^0 = 1$, and $\tau = 2.95 < \tau_0 = 3.1416$.

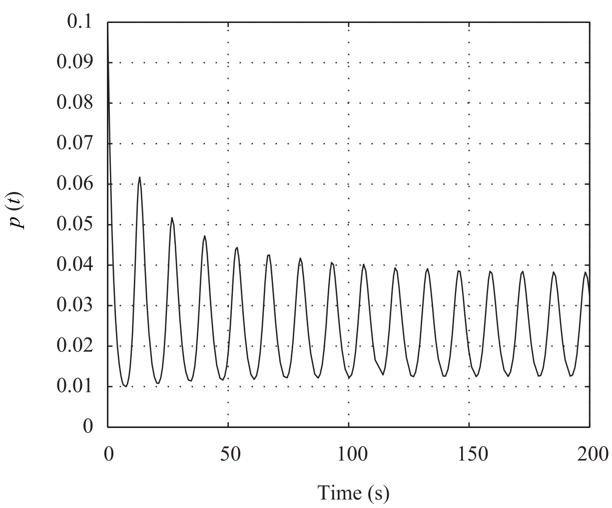

Fig. 3. A periodic oscillation bifurcates from the equilibrium $p^* = 0.02$ of the integer-order model (5), where $\kappa = 0.01, C = 50, \mathcal{D}(p) = 1/p$, the initial condition $p^0 = 0.1$, and $\tau = 3.25 > \tau_0 = 3.1416$.

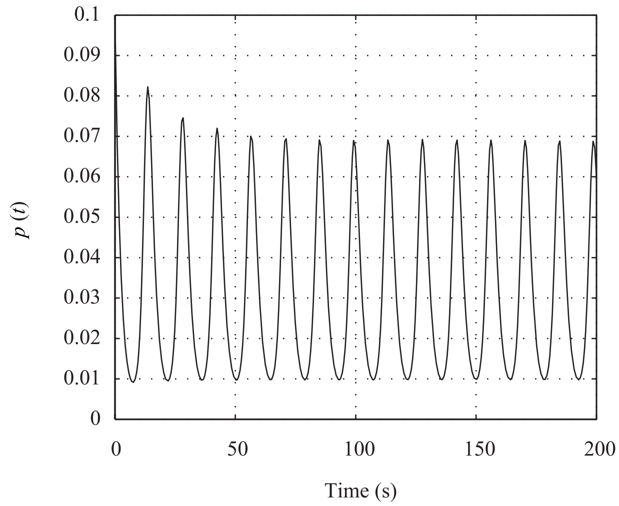

Fig. 4. A periodic oscillation bifurcates from the equilibrium $p^* = 0.02$ of the integer-order model (5), where $\kappa = 0.01, C = 50, \mathcal{D}(p) = 1/p$, the initial condition $p^0 = 0.1$, and $\tau = 3.45 > \tau_0 = 3.1416$.

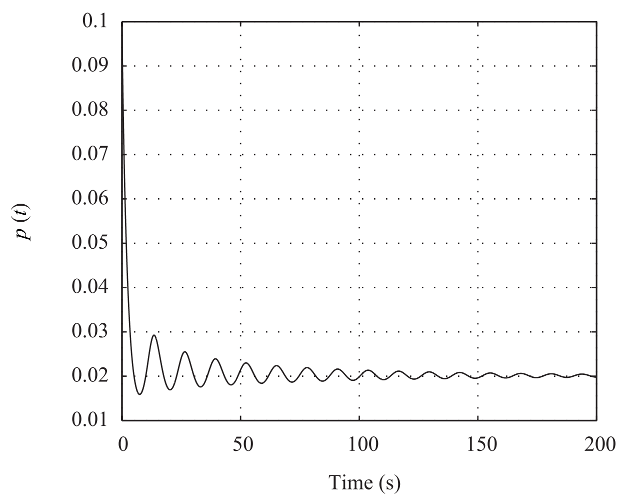

Fig. 5. The equilibrium $p^* = 0.02$ of model (6) with $\alpha = 0.92$ is asymptotically stable, where $\kappa = 0.01, C = 50, \mathcal{D}(p) = 1/p$, the initial condition $p^0 = 0.1$, and $\tau = 3.45 < \tau_0 = 3.6037$.

that when $\tau < \tau_0$, the trajectory converges to the equilibrium p^* (see Fig. 2), while as τ is increased to pass through τ_0, p^* loses its stability and a Hopf bifurcation occurs (see Figs. 3 and 4).

Next, using our Theorem 3, we display the Hopf bifurcation for the fractional-order model (6) with $\alpha \in (0, 1)$. For example, by choosing $\alpha = 0.92$, we can apply (13) in Lemma 1 to obtain

$$\tau_0 = 3.6037, \quad \omega_0 = 0.4708.$$

Note that the fractional-order model (6) with $\alpha = 0.92$ has the same equilibrium as that of the integer-order model (5), but the critical value τ_0 increases from 3.1416 to 3.6037, implying that the onset of Hopf bifurcations is delayed.

When $\alpha = 0.92$, we choose $\tau = 3.45 < \tau_0 = 3.6037$, which is the same value as that used in Fig. 4. According to Theorem 3, we conclude that instead of having a Hopf bifurcation, the fractional-order model (6) with $\alpha = 0.92$ converges to the equilibrium $p^* = 0.02$, as shown in Fig. 5.

When $\alpha = 0.92$, we choose $\tau = 3.8 > \tau_0 = 3.6037$. From Theorem 3, the equilibrium $p^* = 0.02$ is unstable, as shown in Fig. 6. It can be seen that when τ passes through the critical value $\tau_0 = 3.6037$, a Hopf bifurcation occurs (see Figs. 5 and 6).

When $\alpha = 0.92$, we choose $\tau = 3.45 < \tau_0 = 3.6037$, which is the same value as that used in Fig. 4. According to Theorem 3, we conclude that instead of having a Hopf bifurcation, the fractional-order model (6) with $\alpha = 0.92$ converges to the equilibrium $p^* = 0.02$, as shown in Fig. 5.

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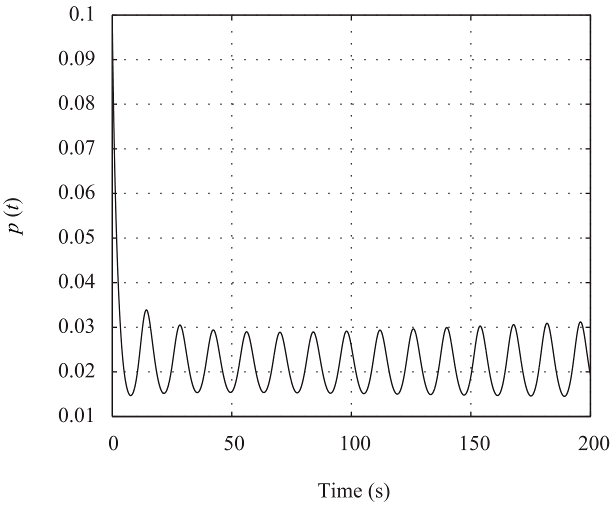

Fig. 6. A periodic oscillation bifurcates from the equilibrium $p^* = 0.02$ of model (6) with $\alpha = 0.92$, where $\kappa = 0.01, C = 50, \mathcal{D}(p) = 1/p$, the initial condition $p^0 = 0.1$, and $\tau = 3.8 > \tau_0 = 3.6037$.

It can be shown that if we choose a smaller value of α, then the fractional-order model (6) may not have a Hopf bifurcation even for the larger values of τ. This indicates that the order α can delay the onset of Hopf bifurcations, thus guaranteeing a stationary sending rate for the larger values of τ. For example, when choosing $\alpha = 0.86$, the fractional-order model (6) converges to the equilibrium $p^* = 0.02$ if $\tau < \tau_0 = 4.0092$, as shown in Fig. 7.

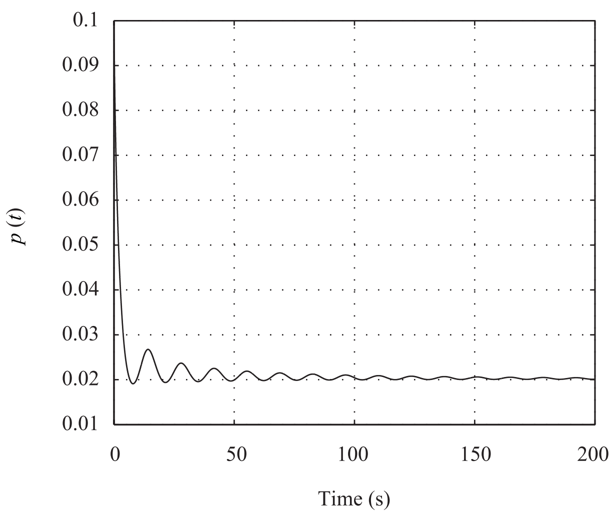

Fig. 7. Equilibrium $p^* = 0.02$ of model (6) with $\alpha = 0.86$ is asymptotically stable, where $\kappa = 0.01, C = 50, \mathcal{D}(p) = 1/p$, the initial condition $p^0 = 0.1$, and $\tau = 3.9 < \tau_0 = 4.0092$.

The effect of the order α on the values of τ_0 and ω_0 is shown in Table I. The critical value τ_0 decreases clearly with the order α, which means that the value of τ_0 is sensitive to the change of the order α.

VI. Conclusion

In this paper, we have extended a delayed integer-order model of dual congestion control algorithms to a fractional-order counterpart. We have considered the stability and bifurcations of network congestion control in the presence of communication delays and fractional order. A stability criterion for the delayed fractional-order congestion control model has been established. We have also proposed some conditions of Hopf-type bifurcations for delayed fractional-order systems. The delayed fractional-order congestion control model can exhibit a Hopf bifurcation (i.e., periodic oscillations appear) as the delay achieves a critical value which can be determined exactly. It is observed that an increase in the order may lead to a decrease of the critical value. The observations allow us to design Hopf bifurcations of congestion control systems with the desired bifurcation points by adjusting the delays and order.

TABLE I
Values of ω_0 and τ_0 for (6) With $\kappa = 0.01$, $C = 50$, $D(P) = 1/P$, and Different Values of α : α = 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, and 0.1

Fractional order of model (6)	ω_0	τ_0
α = 1	0.5	3.1416
α = 0.9	0.4629	3.7324
α = 0.8	0.4204	4.4832
α = 0.7	0.3715	5.4968
α = 0.6	0.3150	6.9818
α = 0.5	0.2500	9.4248
α = 0.4	0.1768	14.2172
α = 0.3	0.0992	26.9155
α = 0.2	0.0313	90.4779
α = 0.1	9.7656E−004	3.0561E+003

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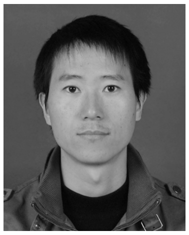

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