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Letter

Distributed Minimum-Energy Containment Control of Continuous-Time Multi-Agent Systems by Inverse Optimal Control

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Dear Editor,

This letter focuses on the distributed optimal containment control of continuous-time multi-agent systems (CTMASs) with respect to the minimum-energy performance index over fixed topology. To achieve this, we firstly investigate the optimal containment control problem using the inverse optimal control method, where all states of followers asymptotically converge to the convex hull spanned by the leaders while some quadratic performance indexes get minimized. A sufficient condition for existence of the distributed optimal containment control protocol is derived. By introducing the parametric algebraic Riccati equation (PARE), it is strictly proved that the global performance index can be used to approximate the standard minimum-energy performance index as the parameters tends to infinity. In consequence, the standard minimum-energy cooperative containment control can be solved by local steady state feedback protocols. Finally, numerical examples are given to demonstrate the effectiveness of theoretical results.

For decades, cooperative control of MASs has sparked a surge in interest for its extensive applications in formation control, flocking and aggregation problem, see in [1]–[5]. The MASs with multiple leaders are more typical. Then the containment control problem arises, where the followers are to be impelled into a given geometric space spanned by the leaders. Distributed dynamic containment protocol is established for the continuous-time/discrete-time MASs with general linear dynamics under fixed directed topologies in [6]. The distributed containment control problem is investigated over directed communication networks in [7].

Researchers have attached great importance to the distributed optimal cooperative control problem for MASs [8]–[10], whose goal is to find distributed protocols to achieve consensus while optimizing quadratic performance indexes. For the purpose of addressing the globally optimal cooperative control problem, the linear quadratic regulator (LQR) method is employed in [8]. The inverse optimal method is employed to establish the necessary and sufficient condition for the designed performance index on global optimality in [9]. Based on the network approximation (NA) technique, Chen and Chen [10] proposes a distributed optimal consensus algorithm to minimize the global energy cost. However, MASs may require a restricted energy supply in mobility and vehicle coordination. Thus, the existence of distributed optimal solution is still worth discussing. The unavailability of the global information for the whole MASs makes the design of a distributed protocol contradictory. Therefore, it's quite challenging to design the distributed containment protocol to optimize the global performance index, which motivates our present work.

1) For CTMASs over fixed topology, the sufficient condition of the distributed optimal containment control protocol with respect to some global performance indexes is proposed by employing the inverse optimal control method.

2) The standard minimum-energy distributed containment control problem is solved by local steady state feedback protocols. It's shown that, for marginally stable MASs, the energy cost can be arbitrarily small; for MASs with strictly unstable dynamics, the minimum-energy type performance index exists a strict lower bound since energy expenditure to stabilize the system is essential.

Notations: The Kronnecker product is denoted by $\otimes$. The set of all symmetric positive definite (SPD) matrices is denoted by $\mathbb{SR}^{n\times n}$. The spectral radius of matrix A is defined as $\rho(A)$.

Problem formulation: Consider the CTMAS with N nodes,

$$\dot{x}_i = Ax_i + Bu_i, \quad \forall i \in \mathcal{N} \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are, respectively, state vector and control vector, And the system matrix A and the input matrix B are with compatible dimensions. The compact form of MAS (1) is

$$\dot{x} = (I_N \otimes A)x + (I_M \otimes B)u \tag{2}$$

where $x = (x_1^T, x_2^T, \ldots, x_N^T)^T \in \mathbb{R}^{nN}$ are global state vector and $u = (u_1^T, u_2^T, \ldots, u_M^T)^T \in \mathbb{R}^{mM}$ are global control input vector respectively. Assume that there are $M(M<N)$ followers, described by undirected graph, while $(N-M)$ leaders do not receive information from any other agent. Attaching $1,\ldots,M$ to M followers and $M+1,\ldots,N$ to $(N-M)$ leaders, then the leader set and the follower set, which are, respectively, $\alpha \triangleq \{M+1,\ldots,N\}$ and $\beta \triangleq \{1,\ldots,M\}$.

Assumption 1: For each follower, there exists at least one leader who has a directed path to that follower.

There are $(N-M)$ leader agents in system (1), behaving autonomously, whose dynamic is $\dot{x}_l = Ax_l$, where $l \in \alpha$. In consequence, MAS (1) can be written as

$$\dot{x}_f = Ax_f + Bu, \quad \forall f \in \beta \tag{3a}$$

$$\dot{x}_l = Ax_l \quad \forall l \in \alpha. \tag{3b}$$

The topology of MAS (1) could be represented by graph $\mathcal{G}$, whose characteristics are described by Laplacian matrix $\mathcal{L}$, divided into

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & \mathcal{L}_l \\ \mathbf{0}_{(N-M)\times M} & \mathbf{0}_{(N-M)\times(N-M)} \end{bmatrix} \tag{4}$$

where $\mathcal{L}_f \in \mathbb{SR}^{M\times M}$ is formed by the followers and $\mathcal{L}_l \in \mathbb{R}^{M\times(N-M)}$ is associated with $(N-M)$ leaders. For subtopology $\mathcal{G}_f$, It is connected. Then, the Laplacian matrix $\mathcal{L}_f \in \mathbb{SR}^{M\times M}$. Consequently, there is an unitary matrix $U \in \mathbb{R}^{M\times M}$ that makes $U^T \mathcal{L}_f U = \Pi = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_M\}$ true.

Consider linear static distributed containment controller for optimal containment control problem.

$$u_i = -cF\varepsilon_i = -cF \sum_{j\in\alpha\bigcup\beta} a_{ij}\left(x_i - x_j\right), \quad i \in \mathcal{N} \tag{5}$$

where $c > 0 \in \mathbb{R}$, $F \in \mathbb{R}^{m\times n}$ and a_{ij} is the (i,j)-th entry of the adjacency matrix $\mathcal{A}$. Then the compact form of (5) is $u = -c(\mathcal{L}_f \otimes F)\{x_f + [(\mathcal{L}_f^{-1}\mathcal{L}_l)\otimes I_n]x_l\}$. The compact form of MAS (3) in the sense of containment control can be written as

$$\begin{aligned} \dot{x}_f &= \left[(I_M \otimes A) - c\mathcal{L}_f \otimes (BF)\right]x_f - c\mathcal{L}_l \otimes (BF)x_l \\ \dot{x}_l &= (I_{N-M} \otimes A)\,x_l, \quad \forall l \in \alpha, \ \forall f \in \beta. \end{aligned} \tag{6}$$

The convex hull spanned by leaders can be defined as $\varpi = -\left[(\mathcal{L}_f^{-1}\mathcal{L}_l)\otimes I_n\right]x_l$, then the error between followers and convex hull in MAS (1) is $\delta = x_f - \varpi$.

Problem 1 (globally optimal containment control): It is to design a

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distributed optimal containment protocol (5) for (1) such that the state of all followers can asymptotically converge to the convex hull while optimizing some global quadratic performance index:

$$J=\int_0^{+\infty}\left[Q(\delta)+u^T(I_m\otimes R)u\right]dt$$

where $Q(\delta)\geq 0$.

Problem 2 (distributed minimum-energy control): It is to design a distributed optimal containment protocol (5) for (1) such that the state of all followers can asymptotically converge to the convex hull while optimizing standard minimum-energy performance index:

$$J=\int_0^{+\infty}u^Tudt. \tag{7}$$

Main results: A distributed global optimal containment protocol is proposed for Problem 1.

Theorem 1: The distributed protocol (5), where $F=R^{-1}B^TP$ by solving $A^TP+PA-PBR^{-1}B^TP+Q=0$ (ARE) and the coupling gain $c\geq c_{\min}$ with $c_{\min}=1/\min_{i\in\beta}\{\lambda_i\}$, solves Problem 1 and the minimized global quadratic performance index follows:

$$\begin{aligned}\widetilde{J}=\int_0^{+\infty}&\left\{u^T(I_M\otimes R)u+c^2\delta^T[\mathcal{L}_f^2\otimes(PBR^{-1}B^TP)]\delta\right.\\&\left.-c\delta^T[(I_M\otimes A^T)(\mathcal{L}_f\otimes P)+(\mathcal{L}_f\otimes P)(I_M\otimes A)]\delta\right\}dt.\end{aligned} \tag{8}$$

Proof: To generate a function associated with δ, we derive that

$$\dot{x}_f+\left[\left(\mathcal{L}_f^{-1}\mathcal{L}_l\right)\otimes I_n\right]\dot{x}_l=\frac{d}{dt}\left[x_f+\left[\left(\mathcal{L}_f^{-1}\mathcal{L}_l\right)\otimes I_n\right]x_l\right].$$

The controller (5) follows:

$$u=-\left\{(I_N\otimes R)^{-1}(I_N\otimes B)^T[c(\mathcal{L}_f\otimes P)]\right\}\delta. \tag{9}$$

It is known that $P\in\mathbb{SR}^{n\times n}$, indicating that $\mathcal{L}_f\otimes P\in\mathbb{SR}^{Mn\times Mn}$ as well. So we establish the following Lyapunov function:

$$V(\delta)=\delta^T(c\mathcal{L}_f\otimes P)\delta \tag{10}$$

so as to create the sufficient condition $\dot{V}(\delta)|_{\frac{1}{2}u}$ in [9], take the derivative of $V(\delta)$ with respect to time along error system $\dot{\delta}=[I_M\otimes A-(c/2)\mathcal{L}_f\otimes(BF)]\delta$. The result is

$$\begin{aligned}\dot{V}(\delta)|_{\frac{1}{2}u}=c\delta^T&(U\otimes I_n)\times\mathrm{diag}\{\Omega_1,\Omega_2,\ldots,\Omega_M\}\\&\times(U\otimes I_n)^T\delta\end{aligned} \tag{11}$$

where $\Omega_i=\lambda_i(A^TP-(c\lambda_i/2)F^TB^TP+PA-(c\lambda_i/2)PBF),\ i\in\beta$.

Substituting the solution F for Ω_i, we obtain that $\Omega_i=\lambda_i(A^TP+PA-c\lambda_iPBR^{-1}B^TP),\ i\in\beta$. In the light of structure of ARE and $Q>0$, when $c\lambda_i>1,\Omega_i<0$. In further terms, it is obvious that $\dot{V}(\delta)|_{\frac{1}{2}u}<0$. By the inverse optimal control in [9], the optimal containment control in Problem 1 is solved. ■

As an application of the inverse optimal control, we focus on the distributed minimum-energy containment control in Problem 2.

Consider the following parametric ARE (PARE):

$$A^TP+PA-\frac{1}{\tau}PBB^TP+Q=0 \tag{12}$$

where $Q\in\mathbb{SR}^{n\times n}$, $\tau>0$ is a scalar.

Lemma 1: If A is marginally stable, then for the solution P of PARE (12), the spectral radius $\rho(P/\tau)$ is monotonically decreasing as τ increases. Furthermore, $\rho(P/\tau)\to 0$ as $\tau\to+\infty$.

Proof: Taking the following LQR problem into account:

$$\dot{x}=Ax+Bu,x(0)=x_0$$

$$J(\tau)=\int_0^{+\infty}\left(x^TQx+\tau u^Tu\right)dt \tag{13}$$

which is equivalent to

$$\frac{J(\tau)}{\tau}=\int_0^{+\infty}(x^T\frac{Q}{\tau}x+u^Tu)\mathrm{d}t \tag{14a}$$

$$A^T\frac{P}{\tau}+\frac{P}{\tau}A-\frac{P}{\tau}BB^T\frac{P}{\tau}+\frac{Q}{\tau}=0. \tag{14b}$$

Then, we have $\frac{d}{d\tau}\left[\frac{J(\tau)}{\tau}\right]=-\frac{1}{\tau^2}\int_0^{+\infty}x^TQx\mathrm{d}t<0$. Apparently, $\frac{J(\tau)}{\tau}$ is monotonically decreasing as τ increases. Based the inverse optimal control, the optimal controller u^* corresponding to scalar τ satisfies

$$u^*=-F(\tau)x \tag{15}$$

where $F(\tau)=(1/\tau)B^TP(\tau)$. Meanwhile, the quadratic performance index in (13) has the optimal value $J^*(\tau)=x_0^TP(\tau)x_0$.

For arbitrary $\tau_2>\tau_1>0$, consider the performance index

$$\frac{J(\tau_i)}{\tau_i}=\int_0^{+\infty}(x^T\frac{Q}{\tau_i}x+u_i^Tu_i)\mathrm{d}t,\ i=1,2. \tag{16}$$

The optimal value is bound to follow $\frac{J^*(\tau_i)}{\tau_i}=x_0^T\frac{P_i}{\tau_i}x_0,i=1,2$. As a result, it can be deduced that

$$\begin{aligned}\left.\frac{J^*(\tau_i)}{\tau_i}\right|_{u_i^*=-F_ix}&=\int_0^{+\infty}\left(x^T\frac{Q}{\tau_i}x+u_i^Tu_i\right)dt\\&=x_0^T\frac{P_i}{\tau_i}x_0,\ i=1,2.\end{aligned}$$

Set $\tau=\tau_2$, under the control of the optimal controller u_1 corresponding to the fixed value $\tau=\tau_1$, $\frac{J(\tau)}{\tau}$ can be transformed into $\left.\frac{J(\tau_2)}{\tau_2}\right|_{u_1=-F_1x}=\int_0^{+\infty}(x^T\frac{Q}{\tau_2}x+u_1^Tu_1)\mathrm{d}t$.

Because $\frac{J(\tau)}{\tau}$ is monotonically decreasing as τ increases with a fixed controller u, then the following inequalities are obtained:

$$\left.\frac{J^*(\tau_1)}{\tau_1}\right|_{u_1^*=-F_1x}>\left.\frac{J(\tau_2)}{\tau_2}\right|_{u_1=-F_1x}>\left.\frac{J^*(\tau_2)}{\tau_2}\right|_{u_2^*=-F_2x}.$$

Apparently, $\frac{J^*(\tau)}{\tau}$ is monotonically decreasing as τ increases with a corresponding optimal controller u^*, and $\rho(\frac{P}{\tau})$ is monotonically decreasing as well. Moreover, because $\rho(\frac{Q}{\tau})\to 0$ as $\tau\to\infty$, then it's obvious that $\frac{P}{\tau}=\mathbf{0}$ is a solution of ARE (14b) when $\frac{Q}{\tau}\equiv\mathbf{0}$. Due to the continuity of the solution, the spectral radius $\rho(\frac{P}{\tau})$ decreases monotonically with the increase of τ, and $\rho(\frac{P}{\tau})\to 0$ as $\tau\to\infty$. ■

Theorem 2: For the marginally stable MAS (1), the distributed global optimal containment control protocol (15) minimizes the global performance index (8), and the global performance index tends to the standard minimum-energy performance index (7) as the parameter $\tau\to+\infty$.

Proof: The global performance index for PARE (12) is

$$\widetilde{J}(\tau)=\int_0^{+\infty}\left(\delta^T\widetilde{Q}\delta+\tau u^Tu\right)dt \tag{17}$$

where $\widetilde{Q}=c[\frac{c}{\tau}\mathcal{L}_f^2\otimes(PBB^TP)-\mathcal{L}_f\otimes(A^TP+PA)]$. Naturally, $\frac{\widetilde{Q}}{\tau}=c[c\mathcal{L}_f^2\otimes(\frac{P}{\tau}BB^T\frac{P}{\tau})-\mathcal{L}_f\otimes(A^T\frac{P}{\tau}+\frac{P}{\tau}A)]$.

It is explicable that the positive semi-definite global state penalty term $\delta^T\widetilde{Q}\delta$ is added to drive the states of all followers to achieve a synchronization with the convex hull. By Lemma 1, $\rho(\frac{\widetilde{Q}}{\tau})$ is monotonically decreasing as τ increases because of the decreasing monotonicity of $\rho(\frac{P}{\tau})$. Moreover, the value of $\widetilde{Q}$ converges to zero as $\tau\to\infty$. Therefore, the performance index (17) can be used to approximate the performance index (7) as $\frac{\widetilde{J}(\tau)}{\tau}\approx J$.

By Theorem 1, the distributed containment controller (5) is optimal to the performance index (17), hence approximately optimal to the minimum-energy performance index (7). ■

Theorem 2 indicates that the energy cost can be arbitrarily small. When the matrix A is strictly unstable, because $\frac{P}{\tau}$ is the unique positive definite solution of PARE (14b), a similar conclusion holds that the spectral radius $\rho(\frac{P}{\tau})$ is monotonically decreasing as τ increases and $\rho(\frac{P}{\tau})$ asymptotically converges to a strict lower bound as $\tau\to\infty$, so does $\frac{\widetilde{Q}}{\tau}$. Eventually, the minimum-energy type performance index in (7) asymptotically converges to a strict lower bound as $\tau\to\infty$.

Algorithm 1: Problem 1 is solved by protocol (5), where $F=R^{-1}B^TP$ and the coupling gain $c\geq c_{\min}$, with $c_{\min}=\frac{1}{\min_{i\in\beta}\{\lambda_i\}}$.

Algorithm 2: Based on Algorithm 1, Problem 2 is solved by protocol (5) where $\tau\geq\tau_0$ in PARE (12) and τ_0 a positive number that is large enough.

Numerical examples: The MAS in this section is presented by the communication graph described in Fig. 1. Note that $\mathcal{L}_f$ in (4) has the

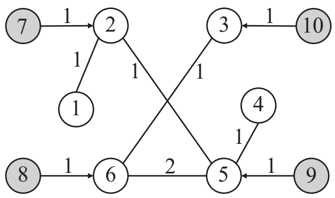

Fig. 1. The topology of MAS.

minimum eigenvalue $\lambda_{\min} = 0.4491$ and the maximum eigenvalue $\lambda_{\max} = 6.9127$.

Case 1 (Marginally stable MAS): Consider a CTMAS with marginally stable dynamics where

$$A = \begin{bmatrix} -0.020 & 0.020 \\ 0.015 & -0.015 \end{bmatrix}, \quad B = \begin{bmatrix} 0.9 \\ 7.2 \end{bmatrix}.$$

According to Theorem 1 and Algorithm 1, the coupling gain c is 2.2267. Let $Q = I_2$, by solving the ARE, the optimal feedback gain matrix is $F|_{\tau=1} = \begin{bmatrix} 0.4652 & 0.9490 \end{bmatrix}$ in Fig. 2(a). In consequence, the global optimal containment control protocol (5) is obtained, the states of the agents are shown in Fig. 3. Meanwhile, we simulate the graph of performance index with parameter τ, shown in Fig. 4(a).

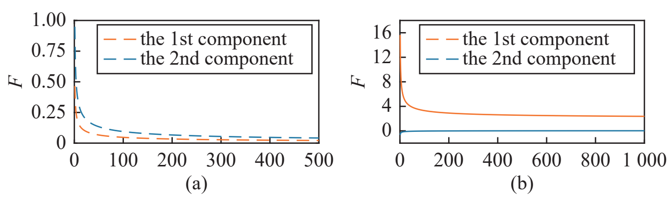

Fig. 2. Feedback gain F with scalar τ. (a) Case 1; (b) Case 2.

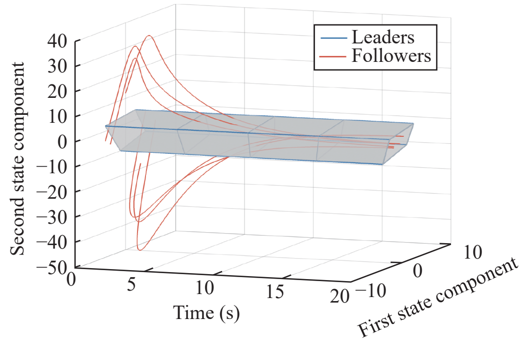

Fig. 3. The state transition curve of the agents.

Case 2 (Strictly unstable MAS): Consider another CTMAS with strictly unstable dynamics where

$$A = \begin{bmatrix} 1.020 & 0.020 \\ 0.015 & -0.015 \end{bmatrix}, \quad B = \begin{bmatrix} 0.9 \\ 7.2 \end{bmatrix}.$$

According to Theorem 2 and Algorithm 2, the coupling gain c is still 2.2267. Let $Q = I_2$, $F(\tau)$ in (15) changes continuously. Set $\tau = 1000$, we obtain $F|_{\tau=1000} = \begin{bmatrix} 2.3675 & 0.0170 \end{bmatrix}$ in Fig. 2(b). The performance index evolution curve is shown in Fig. 4(b).

It is clearly seen in Fig. 1 that the approximate performance index $\frac{\widetilde{J}(\tau)}{\tau}$ decreases monotonically as τ increases. Compared with Case 1,

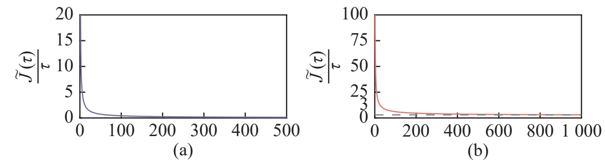

Fig. 4. Performance index $\frac{\widetilde{J}(\tau)}{\tau}$ with scalar τ. (a) Case 1; (b) Case 2.

Case 2 exists a strict lower bound.

Conclusion: In this letter, the distributed optimal containment control problem for CTMASs has been investigated with respect to the minimum-energy performance index over fixed topology. The sufficient condition of the distributed global optimal containment control protocol has been given based on the inverse optimal control method. Besides, the minimum-energy distributed containment control problem has been addressed by local state steady feedback protocols. Finally, numerical examples have been given to demonstrate the effectiveness of theoretical results.

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