Output Feedback Stabilization of High-Order Nonlinear Time-Delay Systems With Low-Order and High-Order Nonlinearities

Meng-Meng Jiang, Kemei Zhang, and Xue-Jun Xie

***Abstract*—In this paper, the output feedback stabilization of a class of high-order nonlinear time-delay systems with more general low-order and high-order nonlinearities is investigated. By constructing the new Lyapunov-Krasovskii functional and reduced-order observer, based on homogeneous domination theory together with the adding a power integrator method, an output feedback controller is developed to guarantee the equilibrium of the closed system globally uniformly asymptotically stable.**

***Index Terms*—High-order nonlinear time-delay systems, low-order and high-order nonlinearities, output feedback stabilization.**

I. Introduction

IN this paper, we consider high-order nonlinear time-delay systems as follows:

$$\begin{aligned}\dot{\eta}_i(t) &= \eta_{i+1}^{p_i}(t)+\phi_i(t,\eta(t),\eta_1(t-\tau_1(t)),\cdots,\eta_i(t-\tau_i(t))),\\ &\qquad i=1,\cdots,n-1,\\ \dot{\eta}_n(t) &= u^{p_n}(t)+\phi_n(t,\eta(t),\eta_1(t-\tau_1(t)),\cdots,\eta_n(t-\tau_n(t))),\\ y(t) &= \eta_1(t),\end{aligned} \tag{1}$$

where $\eta(t)=[\eta_1(t),\cdots,\eta_n(t)]^\top\in R^n$, $u(t)\in R$ and $y(t)\in R$ are the system state, control input and output respectively, $\eta_2(t),\cdots,\eta_n(t)$ are unmeasurable. For $i=1,\cdots,n$, $\tau_i(t)$ is time-varying delay with $0\le\tau_i(t)\le\varepsilon_i$, where ε_i is a positive constant; $p_i\in R_{odd}^{\ge 1}\triangleq\{\frac{p}{q}\in R^+$: p and q are odd integers, $p\ge q\}$; $\phi_i: R^+\times R^n\times R^i\to R$ is an unknown $\mathcal{C}$ function with $\phi_i(t,0,0)=0$. The system's initial condition is $\eta(\theta)=\zeta_0(\theta),\forall\theta\in[-\varepsilon_M,0]$ with $\varepsilon_M=\max\{\varepsilon_1,\cdots,\varepsilon_n\}$ and $\zeta_0(\theta)$ being a specified $\mathcal{C}$ function. System (1) is called as high-order system if there exists at least one $i\in\{1,\cdots,n\}$ such that $p_i>1$.

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Particularly, when $\tau_i(t)=0$ for all $i=1,\cdots,n$, system (1) becomes

$$\begin{aligned}&\dot{\eta}_i(t)=\eta_{i+1}^{p_i}(t)+\phi_i(t,\eta(t)),\quad i=1,\cdots,n-1,\\ &\dot{\eta}_n(t)=u^{p_n}(t)+\phi_n(t,\eta(t)),\\ &y(t)=\eta_1(t).\end{aligned} \tag{2}$$

In the literature of output feedback stabilization of high-order nonlinear system (2), most of these results require that the nonlinearity ϕ_i satisfies certain restrictive conditions, that is, ϕ_i depends on the output y, or the states in the bounding functions are of an order equal to $\frac{1}{p_j\cdots p_{i-1}}$, or greater than $\frac{1}{p_j\cdots p_{i-1}}$, or less than $\frac{1}{p_j\cdots p_{i-1}}$, e.g., see [1]−[13] and the reference therein.

Recently, the restrictive condition was relaxed by [14]−[17], in which all the states in the bounding condition were allowed to be of both an order greater than $\frac{1}{p_j\cdots p_{i-1}}$ and an order equal to $\frac{1}{p_j\cdots p_{i-1}}$. These assumptions can be summarized as the following form:

$$|\phi_i(t,\eta(t))|\le c\sum_{j=1}^{i}(|\eta_j(t)|^{\nu_{lj}}+|\eta_j(t)|^{\nu_{uj}}), \tag{3}$$

where low-order $\nu_{lj}=\frac{1}{p_j\cdots p_{i-1}}$ and high-order $\nu_{uj}=\frac{\bar{r}_i+\bar{\omega}_2}{\bar{r}_j}$ are some ratios of odd integers in $[\frac{1}{p_j\cdots p_{i-1}},+\infty)$ with $\bar{r}_1=1,\bar{r}_{i+1}=\frac{\bar{r}_i+\bar{\omega}_2}{p_i}$ and $\bar{\omega}_2\ge 0$.

For the special case of $p_i=1$, [18], [19] weakened the growth condition (3) by allowing both low-order $0<\nu_{lj}\le 1$ and high-order $1\le\nu_{uj}<+\infty$, i.e.,

$$|\phi_i(t,\eta(t))|\le c\sum_{j=1}^{i}\left(|\eta_j(t)|^{\frac{1+i\bar{\omega}_1}{1+(j-1)\bar{\omega}_1}}+|\eta_j(t)|^{\frac{1+i\bar{\omega}_2}{1+(j-1)\bar{\omega}_2}}\right),$$
$$-\frac{1}{n}<\bar{\omega}_1\le 0,\ \bar{\omega}_2\ge 0.$$

However, for the general case of $p_i\ge 1$, few further results on the output feedback stabilization of nonlinear system (2) have been achieved to relax the condition (3) until now.

For high-order nonlinear systems (1) with $\tau_i(t)\ne 0$, since time-delay is always encountered in many practical control systems and its emergence often causes instability and serious deterioration in the systems performance, many attention has been paid on the control design of time-delay system (1) and there have been some results achieved, see [20]−[24] and the reference therein. However, [20]−[22] only considered high-order nonlinearities, [23] only had an order in $(0,+\infty)$, [24] allowed low-order to be $\frac{1}{p_j\cdots p_{i-1}}$ and high-order to take any value in $[\frac{1}{p_j\cdots p_{i-1}},+\infty)$.

Based on the above discussion, an interesting problem is immediately proposed: For high-order nonlinear time-delay system (1), under the condition

$$\begin{aligned}&|\phi_i(t,\eta(t),\eta_1(t-\tau_1(t)),\cdots,\eta_i(t-\tau_i(t)))|\\&\le c\sum_{j=1}^{i}(|\eta_j(t)|^{\nu_{lj}}+|\eta_j(t)|^{\nu_{uj}}\\&\quad+|\eta_j(t-\tau_j(t))|^{\nu_{lj}}+|\eta_j(t-\tau_j(t))|^{\nu_{uj}}),\end{aligned}\tag{4}$$

is it possible to relax condition (4) by allowing low-order ν_{lj} and high-order ν_{uj} to take any value in $(0,\frac{1}{p_j\cdots p_{i-1}}]$ and $[\frac{1}{p_j\cdots p_{i-1}},+\infty)$, respectively? Under the weaker condition, can an output feedback controller be designed for system (1)?

This paper will substantially solve this problem. By constructing the new Lyapunov-Krasovskii functional and the reduced-order observer, a global output feedback controller based on the homogeneous domination theory and the adding a power integrator method is developed to guarantee the equilibrium of the closed-loop system globally uniformly asymptotically stable. The main difficulties in the design and analysis are:

(i) In this paper, due to $p_i\ge 1$, the multiple time-varying delays and the growth condition on nonlinearity ϕ_i in (4) being largely weakened, some substantial obstacles will be inevitably produced, e.g., the L-K functionals and observers in the existing results are no longer applicable to system (1), so a difficult work is how to choose an appropriate L-K functional and construct an available observer. Besides, many more complex nonlinear terms will be inevitably produced due to multiple time-varying delays, low-order and high-order nonlinearities, how to handle them is challenging.

(ii) How to give the design and rigorous analysis of controller isn't an easy work.

This paper is organized as follows. Section II gives preliminaries. Section III gives problem statement and assumptions. Sections IV and V give the design and analysis of the output feedback controller, following a simulation example in Section VI. Section VII concludes the paper. Some proofs are given in the Appendix.

II. Mathematical Preliminaries

Some notations and lemmas are to be used throughout this paper.

Notations: R^+ stands for the set of all the nonnegative real numbers. For any vector $x=[x_1,\cdots,x_n]^\top\in R^n$, denote $x_t=x(t+\theta)$, $\|x\|=(\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$, $\|x_t\|_{\mathcal{C}}=\sup_{-\varepsilon_M\le\theta\le 0}\|x(t+\theta)\|$. For $i=1,\cdots,n$, $\bar{x}_i\triangleq[x_1,\ldots,x_i]^\top\in R^i$, $\bar{x}_{i,t}\triangleq\bar{x}_i(t+\theta)$. A function $f:R^n\to R$ is $\mathcal{C}$ if it is continuous and is $\mathcal{C}^1$ if it is continuously differential. $\mathcal{K}$ denotes the set of all functions: $R^+\to R^+$ that are continuous, strictly increasing and vanishing at zero, $\mathcal{K}_\infty$ denotes the set of all functions that are of class $\mathcal{K}$ and unbounded.

The following first five lemmas are the key tools for adding a power integrator technique. Lemma 6 is used to select observer gains.

Lemma 1 [25]: For $x,y\in R$, $p\ge 1$ is a constant, then $|x+y|^p\le 2^{p-1}|x^p+y^p|$, $(|x|+|y|)^{\frac{1}{p}}\le |x|^{\frac{1}{p}}+|y|^{\frac{1}{p}}\le 2^{1-\frac{1}{p}}(|x|+|y|)^{\frac{1}{p}}$.

Lemma 2 [25]: For $x,y\in R$, if $p\in R^{\ge 1}_{odd}$, then $|x-y|^p\le 2^{p-1}|x^p-y^p|$, $|x^{\frac{1}{p}}-y^{\frac{1}{p}}|\le 2^{1-\frac{1}{p}}|x-y|^{\frac{1}{p}}$.

Lemma 3 [25]: If $p\in R^{\ge 1}_{odd}$, then $|x^p-y^p|\le p|x-y|(x^{p-1}+y^{p-1})\le c|x-y||(x-y)^{p-1}+y^{p-1}|$ for a constant $c>0$ and $x,y\in R$.

Lemma 4 (Young's inequality): Let real numbers $p\ge 1$ and $q\ge 1$ satisfy $\frac{1}{p}+\frac{1}{q}=1$, then for any $x,y\in R$ and any given positive number $\gamma>0$, $xy\le\gamma|x|^p+\frac{1}{q}(p\gamma)^{-\frac{q}{p}}|y|^q$.

Lemma 5 [17]: Let $0\le\mu_1\le\cdots\le\mu_n$ and $c_j>0$, $j=1,\cdots,n$, be real numbers, then $c_1|x|^{\mu_1}+c_n|x|^{\mu_n}\le\sum_{j=1}^n c_j|x|^{\mu_j}\le(\sum_{j=1}^n c_j)(|x|^{\mu_1}+|x|^{\mu_n})$ for any $x\in R$.

Lemma 6 [26]: Let the real number $r\in(0,1)$ be a ratio of odd integers, then for $\varepsilon\in(0,1)$ and $x\ge 0$, $x^r+(1-x)^r+\varepsilon^2x^{1+r}\ge(2^r-1)\varepsilon^{1-r}$.

Lemma 7 [27]: Consider system

$$\dot{x}=f(x_t,t),\tag{5}$$

where $x(t)\in R^n$ and $f:R\times\mathcal{C}\to R^n$.

Suppose that $f:R\times\mathcal{C}\to R^n$ given in (5), maps every $R\times$ (bounded set in $\mathcal{C}$) into a bounded set in R^n, and that $u,v,w:R^+\to R^+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s>0$, and $u(0)=v(0)=0$. If there exists a continuous differentiable functional $V:R\times\mathcal{C}\to R$ such that

$$u(\|\phi(0)\|)\le V(t,\phi)\le v(\|\phi\|_{\mathcal{C}})$$

and

$$\dot{V}(t,\phi)\le -w(\|\phi(0)\|),$$

then the trivial solution of (5) is uniformly stable. If $w(s)>0$ for $s>0$, then it is uniformly asymptotically stable. In addition, if $\lim_{s\to\infty}u(s)=\infty$, then it is globally uniformly asymptotically stable.

III. Problem Statement and Assumptions

The purpose of this paper is to design an output feedback controller for system (1) such that the closed-loop system is globally uniformly asymptotically stable.

To achieve the purpose, we need the following assumptions.

Assumption 1: For each $i=1,\cdots,n$, there is a positive constant γ_i such that $\tau_i:R^+\to R$ satisfies $\dot{\tau}_i(t)\le\gamma_i<1$.

Assumption 2: For each $i=1,\cdots,n$, there are constants $c>0$, $-\frac{1}{\sum_{l=1}^n p_1\cdots p_{l-1}}<\bar{\omega}_1\le 0$ with $p_0=1$ and $\bar{\omega}_2\ge 0$ such that

$$\begin{aligned}&|\phi_i(t,\eta(t),\eta_1(t-\tau_1(t)),\cdots,\eta_i(t-\tau_i(t)))|\\&\le c\sum_{j=1}^{i}\Bigg(|\eta_j(t)|^{\frac{\bar{m}_i+\bar{\omega}_1}{\bar{m}_j}}+|\eta_j(t)|^{\frac{\bar{r}_i+\bar{\omega}_2}{\bar{r}_j}}\\&\quad+|\eta_j(t-\tau_j(t))|^{\frac{\bar{m}_i+\bar{\omega}_1}{\bar{m}_j}}+|\eta_j(t-\tau_j(t))|^{\frac{\bar{r}_i+\bar{\omega}_2}{\bar{r}_j}}\Bigg),\end{aligned}\tag{6}$$

where $\bar{m}_i$ and $\bar{r}_i$ are defined as

$$\bar{m}_1=\bar{r}_1=1,\ \bar{m}_{i+1}=\frac{\bar{m}_i+\bar{\omega}_1}{p_i},\ \bar{r}_{i+1}=\frac{\bar{r}_i+\bar{\omega}_2}{p_i}.\tag{7}$$

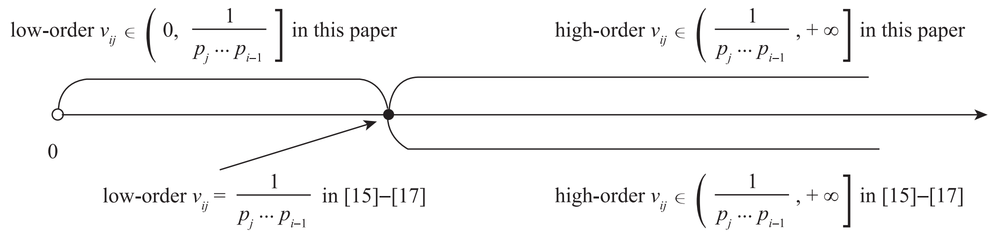

Fig. 1. The value range of low-order and high-order.

Remark 1: Assumption 2 encompasses and extends high-order and/or low-order results. We discuss this point from two cases.

Case I: $\tau_i(t) \equiv 0$ for all i. Condition (6) becomes

$$|\phi_i(t, \eta(t))| \le c \sum_{j=1}^{i} \left(|\eta_j(t)|^{\frac{\bar{m}_i + \bar{\omega}_1}{\bar{m}_j}} + |\eta_j(t)|^{\frac{\bar{r}_i + \bar{\omega}_2}{\bar{r}_j}} \right). \quad (8)$$

(8) includes the following results.

When $\bar{\omega}_1 = \bar{\omega}_2$, (8) reduces to high-order growth condition in [11] with $\bar{\omega}_2 \ge 0$,

$$|\phi_i(t, \eta(t))| \le c \sum_{j=1}^{i} |\eta_j(t)|^{\frac{\bar{r}_i + \bar{\omega}_2}{\bar{r}_j}}, \quad (9)$$

and low-order growth condition in [10] with $\bar{\omega}_1 = 0$,

$$|\phi_i(t, \eta(t))| \le c \sum_{j=1}^{i} |\eta_j(t)|^{\frac{1}{p_j \cdots p_{i-1}}}. \quad (10)$$

When $\bar{\omega}_1 = 0$ and $\bar{\omega}_2 \ge 0$, (8) is changed into the assumptions of both low-order and high-order in the latest papers [15]−[17],

$$|\phi_i(t, \eta(t))| \le c \sum_{j=1}^{i} \left(|\eta_j(t)|^{\frac{1}{p_j \cdots p_{i-1}}} + |\eta_j(t)|^{\frac{\bar{r}_i + \bar{\omega}_2}{\bar{r}_j}} \right), \quad (11)$$

which is a combination of (9) and (10).

We further discuss its significance from value range of both low-order and high-order. From $\bar{\omega}_1 \in (-\frac{1}{\sum_{l=1}^{n} p_1 \cdots p_{l-1}}, 0]$, $\bar{\omega}_2 \in [0, +\infty)$ and (7), it is easy to see that $0 < \frac{\bar{m}_i + \bar{\omega}_1}{\bar{m}_j} \le \frac{1}{p_j \cdots p_{i-1}}$, $\frac{\bar{r}_i + \bar{\omega}_2}{\bar{r}_j} \ge \frac{1}{p_j \cdots p_{i-1}}$, which implies that both low-order and high-order in Assumption 2 can take any value in $(0, \frac{1}{p_j \cdots p_{i-1}}]$, $[\frac{1}{p_j \cdots p_{i-1}}, +\infty)$, respectively. While for [15]−[17], low-order is just a fixed point $\frac{1}{p_j \cdots p_{i-1}}$, high-order $\frac{\bar{r}_i + \bar{\omega}_2}{\bar{r}_j} \ge \frac{1}{p_j \cdots p_{i-1}}$. Fig.1 clearly demonstrates the significance of Assumption 2.

Case II: $\tau_i(t) \neq 0$ for some i. Several new results [20]−[24] have been achieved on feedback stabilization of high-order nonlinear time-delay systems. The nonlinearities in [20]−[24] only have high-order terms. The nonlinearities in [23] only have an order in $(0, +\infty)$ by allowing $\bar{\omega}_2 > -\frac{1}{\sum_{l=1}^{n} p_1 \cdots p_{l-1}}$. The nonlinearities in [24] include linear and nonlinear parts, and their nonlinear parts only allow low-order $\nu_{lj} = \frac{1}{p_j \cdots p_{i-1}}$ and high-order $\nu_{uj} = \frac{\bar{r}_i + \bar{\omega}_2}{\bar{r}_j}$ with $\bar{\omega}_2 \ge 0$.

While in this paper, (6) not only includes time-delays but relaxes the intervals of low-order and high-order. □

IV. Design of Output Feedback Controller

A. Change of Coordinates

Introduce the following coordinate transformation

$$x_i(t) = \frac{\eta_i(t)}{L^{\lambda_i}},\ i = 1, \cdots, n,\ v(t) = \frac{u(t)}{L^{\lambda_{n+1}}}, \quad (12)$$

then system (1) is transformed into

$$\begin{aligned} \dot{x}_i(t) &= L x_{i+1}^{p_i}(t) + f_i(t, x(t), x_1(t - \tau_1(t)), \\ &\quad \cdots, x_i(t - \tau_i(t))),\ i = 1, \cdots, n-1, \\ \dot{x}_n(t) &= L v^{p_n}(t) + f_n(t, x(t), x_1(t - \tau_1(t)), \\ &\quad \cdots, x_n(t - \tau_n(t))), \\ y(t) &= x_1(t), \end{aligned} \quad (13)$$

where $L \ge 1$ is a constant to be determined, $\lambda_1 = 0, \lambda_i = \frac{\lambda_{i-1} + 1}{p_{i-1}},\ i = 2, \cdots, n+1,\ f_i = \frac{\phi_i}{L^{\lambda_i}}$.

B. Output Feedback Controller of System (13)

Initial Step. Define $\xi_1 = x_1$,

$$\begin{aligned} m_1 &= r_1 = 1,\ m_{i+1} = \frac{m_i + \omega_1}{p_i}, \\ r_{i+1} &= \frac{r_i + \omega_2}{p_i},\ i = 2, \cdots, n, \end{aligned} \quad (14)$$

where ω_1 and ω_2 are both ratios of an even integer over an odd integer and satisfy $-\frac{1}{\sum_{l=1}^{n} p_1 \cdots p_l} < \omega_1 \le \bar{\omega}_1 \le 0, \omega_2 \ge \bar{\omega}_2 \ge 0$, respectively. Choose $\mu = \max_{i=1,\cdots,n+1}\{\frac{r_i}{m_i}\}$ and

$$\begin{aligned} V_1(t, \xi_{1,t}) =\ & \frac{\xi_1^{2 - m_2 p_1 + 1}}{2 - m_2 p_1 + 1} + \frac{\xi_1^{2\mu - r_2 p_1 + 1}}{2\mu - r_2 p_1 + 1} \\ & + \frac{(n+1)L}{1 - \gamma_1} \int_{t - \tau_1(t)}^{t} \left(\xi_1^2(s) \mathrm{d}s + \xi_1^{2\mu}(s) \right) \mathrm{d}s \\ & + \frac{nL}{1 - \gamma_2} \int_{t - \tau_2(t)}^{t} \left(\xi_1^2(s) \mathrm{d}s + \xi_1^{2\mu}(s) \right) \mathrm{d}s. \end{aligned} \quad (15)$$

Obviously, μ, m_i and $r_i, i = 1, \cdots, n+1$, are all ratios of odd numbers and V_1 is $\mathcal{C}^1$, positive definite and radially unbounded. From (13), (15), it follows that

$$\begin{aligned} \dot{V}_1(t, \xi_{1,t}) \ =\ & L \left(\xi_1^{2 - m_2 p_1} + \xi_1^{2\mu - r_2 p_1} \right) \left(x_2^{p_1} - x_2^{*p_1} \right) \\ & + \frac{(n+1)L}{1 - \gamma_1} \big(\xi_1^2 - \xi_1^2(t - \tau_1(t))(1 - \dot{\tau}_1(t)) \\ & + \xi_1^{2\mu} - \xi_1^{2\mu}(t - \tau_1(t))(1 - \dot{\tau}_1(t)) \big) \\ & + \frac{nL}{1 - \gamma_2} \big(\xi_1^2 - \xi_1^2(t - \tau_2(t))(1 - \dot{\tau}_2(t)) \\ & + \xi_1^{2\mu} - \xi_1^{2\mu}(t - \tau_2(t))(1 - \dot{\tau}_2(t)) \big) \\ & + \left(\xi_1^{2 - m_2 p_1} + \xi_1^{2\mu - r_2 p_1} \right) f_1 \\ & + L \left(\xi_1^{2 - m_2 p_1} + \xi_1^{2\mu - r_2 p_1} \right) x_2^{*p_1}. \end{aligned} \quad (16)$$

By (6), (7), (14), $m_2p_1 \le \bar{m}_2p_1 \le \bar{r}_2p_1 \le r_2p_1, 1 \le \frac{r_2}{m_2} \le \mu$ and Lemmas 4,5, one has

$$\left(\xi_1^{2-m_2p_1} + \xi_1^{2\mu-r_2p_1}\right) f_1 \le g_1 L\left(\xi_1^2 + \xi_1^{2\mu}\right) + L\left(\xi_1^2(t-\tau_1(t)) + \xi_1^{2\mu}(t-\tau_1(t))\right) \quad (17)$$

where $g_1 > 0$ is a constant independent on L. By choosing the first virtual controller $x_2^* = -2^{\frac{1-m_2p_1}{p_1}}(\frac{n+1}{1-\gamma_1} + \frac{n}{1-\gamma_2} + g_1 + n)^{\frac{1}{p_1}}(\xi_1 + \xi_1^{\frac{m_1r_2}{r_1m_2}})^{m_2} \triangleq -\beta_1(\xi_1 + \xi_1^{\frac{m_1r_2}{r_1m_2}})^{m_2}$, and using Assumption 1, Lemma 1 and (17), (16) becomes

$$\begin{aligned} \dot{V}_1(t,\xi_{1,t}) \le\ & -nL(\xi_1^2 + \xi_1^{2\mu} + \xi_1^2(t-\tau_1(t)) + \xi_1^{2\mu}(t-\tau_1(t)) \\ & +\xi_1^2(t-\tau_2(t)) + \xi_1^{2\mu}(t-\tau_2(t))) \\ & +L\left(\xi_1^{2-m_2p_1} + \xi_1^{2\mu-r_2p_1}\right)(x_2^{p_1} - x_2^{*p_1}). \end{aligned} \quad (18)$$

Inductive Step. Since the design is recursive, we state this step by the following proposition.

Proposition 1: Suppose at Step $k-1 (2 \le k \le n)$, there exist a $\mathcal{C}^1$ function $V_{k-1}(t, \bar{\xi}_{k-1,t})$ with

$$\pi_{k-1,1}(\|\bar{\xi}_{k-1}\|) \le V_{k-1}(t, \bar{\xi}_{k-1,t}) \le \pi_{k-1,2}(\|\bar{\xi}_{k-1,t}\|_{\mathcal{C}}), \quad (19)$$

and a series of virtual controllers $x_1^*, \cdots, x_k^*$ defined by

$$\begin{aligned} & x_1^* = 0, \quad x_i^* = -\beta_{i-1}\left(\xi_{i-1} + \xi_{i-1}^{\frac{m_{i-1}r_i}{r_{i-1}m_i}}\right)^{m_i}, \\ & \xi_{i-1} = x_{i-1}^{\frac{1}{m_{i-1}}} - x_{i-1}^{*\frac{1}{m_{i-1}}}, \quad i = 2, \cdots, k, \end{aligned} \quad (20)$$

such that

$$\begin{aligned} & \dot{V}_{k-1}(t, \bar{\xi}_{k-1,t}) \\ & \le -\sum_{j=1}^{k-1} L(n-k+2)\Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} + \xi_j^2(t-\tau_j(t)) \\ & \quad +\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t)) + \xi_j^2(t-\tau_{j+1}(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big) \\ & \quad +L\left(\xi_{k-1}^{2-m_kp_{k-1}} + \xi_{k-1}^{\frac{(2\mu-r_kp_{k-1})m_{k-1}}{r_{k-1}}}\right)\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right), \end{aligned} \quad (21)$$

where $\pi_{k-1,1}(\cdot), \pi_{k-1,2}(\cdot)$ are $\mathcal{K}_\infty$ functions, $\beta_1, \cdots, \beta_{k-1}$ are positive constants.

Then the kth function $V_k(t, \bar{\xi}_{k,t}) = V_{k-1}(t, \bar{\xi}_{k-1,t}) + W_{Hk}(\bar{\xi}_k) + W_{Dk}(\bar{\xi}_k) + L_{Hk}(t) + L_{Dk}(t)$ is a $\mathcal{C}^1$ function and satisfies

$$\pi_{k,1}(\|\bar{\xi}_k\|) \le V_k(t, \bar{\xi}_{k,t}) \le \pi_{k,2}(\|\bar{\xi}_{k,t}\|_{\mathcal{C}}), \quad (22)$$

and we can design a virtual controller $x_{k+1}^* = -\beta_k(\xi_k + \xi_k^{\frac{m_kr_{k+1}}{r_km_{k+1}}})^{m_{k+1}}$ with $\xi_k = x_k^{\frac{1}{m_k}} - x_k^{*\frac{1}{m_k}}$ and β_k being positive constant such that

$$\begin{aligned} & \dot{V}_k(t, \bar{\xi}_{k,t}) \\ & \le -L(n-k+1)\sum_{j=1}^{k}\Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} + \xi_j^2(t-\tau_j(t)) \\ & \quad +\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t)) + \xi_j^2(t-\tau_{j+1}(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big) \\ & \quad +L\left(\xi_k^{2-m_{k+1}p_k} + \xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\left(x_{k+1}^{p_k} - x_{k+1}^{*p_k}\right), \end{aligned} \quad (23)$$

where $\pi_{k,1}(\cdot), \pi_{k,2}(\cdot)$ are $\mathcal{K}_\infty$ functions, and

$$\begin{aligned} W_{Hk}(\bar{\xi}_k) &= \int_{x_k^*}^{x_k}\left(s^{\frac{1}{m_k}} - x_k^{*\frac{1}{m_k}}\right)^{2-m_{k+1}p_k} \mathrm{d}s, \\ W_{Dk}(\bar{\xi}_k) &= \int_{x_k^*}^{x_k}\left(s^{\frac{1}{m_k}} - x_k^{*\frac{1}{m_k}}\right)^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}} \mathrm{d}s, \\ L_{Hk}(t) &= \frac{L(n-k+2)}{1-\gamma_k}\int_{t-\tau_k(t)}^{t}\left(\xi_k^2(s) + \xi_k^{\frac{2\mu m_k}{r_k}}(s)\right)\mathrm{d}s, \\ L_{Dk}(t) &= \frac{L(n-k+1)}{1-\gamma_{k+1}}\int_{t-\tau_{k+1}(t)}^{t}\left(\xi_k^2(s) + \xi_k^{\frac{2\mu m_k}{r_k}}(s)\right)\mathrm{d}s. \end{aligned} \quad (24)$$

Proof: See the Appendix. □

Step n. By choosing $V_n(t, \xi_t) = V_{n-1}(t, \bar{\xi}_{n-1,t}) + W_{Hn}(\xi) + W_{Dn}(\xi) + L_{Hn}(t) + L_{Dn}(t)$, one can find a series of virtual controllers $x_1^*, \cdots, x_{n+1}^*$ defined by

$$\begin{aligned} & x_1^* = 0, \ x_i^* = -\beta_{i-1}\left(\xi_{i-1} + \xi_{i-1}^{\frac{m_{i-1}r_i}{r_{i-1}m_i}}\right)^{m_i}, \\ & \xi_{i-1} = x_{i-1}^{\frac{1}{m_{i-1}}} - x_{i-1}^{*\frac{1}{m_{i-1}}}, \ i = 2, \cdots, n+1, \end{aligned} \quad (25)$$

such that

$$\begin{aligned} & \dot{V}_n(t, \xi_t) \\ & \le -L\sum_{j=1}^{n}\Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} + \xi_j^2(t-\tau_j(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t)) \\ & \quad +\xi_j^2(t-\tau_{j+1}(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big) \\ & \quad +L\left(\xi_n^{2-m_{n+1}p_n} + \xi_n^{\frac{(2\mu-r_{n+1}p_n)m_n}{r_n}}\right)\left(v^{p_n} - x_{n+1}^{*p_n}\right), \end{aligned}$$

where $\tau_{n+1}(t) = 0$.

Introduce variables $z_2, \cdots, z_n$ as

$$x_i^{p_{i-1}} = (z_i + l_{i-1}x_{i-1})^{\frac{r_ip_{i-1}}{r_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}}, \quad (26)$$

where the gains $l_1 \ge 1, \cdots, l_{n-1} \ge 1$ are constants to be determined. By (26), one deduces that

$$\begin{aligned} \dot{z}_i = & -l_{i-1}Lx_i^{p_{i-1}} - l_{i-1}f_{i-1} \\ & +\mathcal{I}_{1,i}^{-1}p_{i-1}x_i^{p_{i-1}-1}(Lx_{i+1}^{p_i} + f_i)(z_i + l_{i-1}x_{i-1})^{-\frac{\omega_1}{m_{i-1}}} \end{aligned} \quad (27)$$

where $x_{n+1} = v, \mathcal{I}_{1,i} = \frac{r_ip_{i-1}}{r_{i-1}}(z_i + l_{i-1}x_{i-1})^{\frac{\omega_2}{r_{i-1}} - \frac{\omega_1}{m_{i-1}}} + \frac{m_ip_{i-1}}{m_{i-1}}$. Based on (26) and (27), the reduced-order observer is constructed

$$\begin{aligned} & \dot{\hat{z}}_i = -l_{i-1}L\hat{x}_i^{p_{i-1}}, \\ & \hat{x}_i^{p_{i-1}} = (\hat{z}_i + l_{i-1}\hat{x}_{i-1})^{\frac{r_ip_{i-1}}{r_{i-1}}} + (\hat{z}_i + l_{i-1}\hat{x}_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}} \end{aligned} \quad (28)$$

where $\hat{x}_i$ is the estimate of x_i, $i = 2, \cdots, n$, $\hat{x}_1 = x_1$. Using the certainty equivalence principle and (25), we obtain output feedback controller of system (13):

$$\begin{aligned} & v(t) = \hat{x}_{n+1}^* = -\beta_n\left(\hat{\xi}_n + \hat{\xi}_n^{\frac{m_nr_{n+1}}{r_nm_{n+1}}}\right)^{m_{n+1}}, \\ & \hat{\xi}_i = \hat{x}_i^{\frac{1}{m_i}} - \hat{x}_i^{*\frac{1}{m_i}}, \quad \hat{x}_1^* = 0, \\ & \hat{x}_{i+1}^* = -\beta_i\left(\hat{\xi}_i + \hat{\xi}_i^{\frac{m_ir_{i+1}}{r_im_{i+1}}}\right)^{m_{i+1}}, \quad i = 1, \cdots, n. \end{aligned} \quad (29)$$

Defining the estimate error as $e_i = z_i - \hat{z}_i, \quad i = 2, \cdots, n$, and using (27), (28) yield

$$\begin{aligned} \dot{e}_i = & -l_{i-1}L(x_i^{p_{i-1}} - \hat{x}_i^{p_{i-1}}) - l_{i-1}f_{i-1} \\ & +\mathcal{I}_{1,i}^{-1}p_{i-1}x_i^{p_{i-1}-1}(Lx_{i+1}^{p_i} + f_i)(z_i + l_{i-1}x_{i-1})^{-\frac{\omega_1}{m_{i-1}}}. \end{aligned} \quad (30)$$

C. Output Feedback Controller of System (1)

We next give the output feedback controller for system (1) by determining $l_1, \cdots, l_{n-1}$ and L. Introduce a set of $\mathcal{C}^1$ functions for the error dynamics (30), which consist of a low-order part and a high-order part,

$$U_i(e_i) = \frac{m_{i-1}}{2-\omega_1} e_i^{\frac{2-\omega_1}{m_{i-1}}} + \int_{z_i+l_{i-1}x_{i-1}}^{\hat{z}_i+l_{i-1}x_{i-1}} \Big(s^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} - (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \Big) \mathrm{d}s,\ \ i=2,\cdots,n. \quad (31)$$

It is easy to see that U_i is $\mathcal{C}^1$. When $\hat{z}_i > z_i$, it follows from Lemma 2 that

$$\begin{aligned} &\int_{z_i+l_{i-1}x_{i-1}}^{\hat{z}_i+l_{i-1}x_{i-1}} \Big(s^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} - (z_i+l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \Big) \mathrm{d}s \\ &\geq 2^{1-\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \int_{z_i+l_{i-1}x_{i-1}}^{\hat{z}_i+l_{i-1}x_{i-1}} (s - (z_i + l_{i-1}x_{i-1}))^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \mathrm{d}s \\ &= \frac{2^{1-\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} r_{i-1}}{2\mu - \omega_2} e_i^{\frac{2\mu - \omega_2}{r_{i-1}}}. \end{aligned} \quad (32)$$

When $\hat{z}_i \leq z_i$, it can be shown that (32) also holds in a similar way.

It follows from (31) and (32) that $U_i(e_i) \geq \frac{m_{i-1}}{2-\omega_1} e_i^{\frac{2-\omega_1}{m_{i-1}}} + \frac{2^{1-\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} r_{i-1}}{2\mu - \omega_2} e_i^{\frac{2\mu - \omega_2}{r_{i-1}}}$. It is easy to see that $U_i(e_i) \geq 0$ and $U_i(e_i) = 0$ if and only of $e_i = 0$. Hence U_i is positive definite.

From (13), (26), (27), (30), (31), it follows that

$$\begin{aligned} \dot{U}_i \ = \ & -l_{i-1}\Big(e_i^{\frac{2m_i p_{i-1}}{m_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \\ & - (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \Big) f_{i-1} + \Big(e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}} \\ & + \frac{2\mu - r_i p_{i-1}}{r_{i-1}} e_i (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}} - 1} \Big) \\ & \cdot \mathcal{I}_{1,i}^{-1} p_{i-1} x_i^{p_{i-1}-1} (L x_{i+1}^{p_i} + f_i)(z_i + l_{i-1}x_{i-1})^{-\frac{\omega_1}{m_{i-1}}} \\ & - l_{i-1} L \Big(e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \\ & - (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \Big) \mathcal{I}_{2,i} \\ & - l_{i-1} L \Big(e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \\ & - (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \Big) \mathcal{I}_{3,i}, \end{aligned} \quad (33)$$

where $\mathcal{I}_{2,i} = (z_i + l_{i-1}x_{i-1})^{\frac{r_i p_{i-1}}{r_{i-1}}} - (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{r_i p_{i-1}}{r_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{m_i p_{i-1}}{m_{i-1}}} - (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{m_i p_{i-1}}{m_{i-1}}}$, $\mathcal{I}_{3,i} = (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{r_i p_{i-1}}{r_{i-1}}} - (\hat{z}_i + l_{i-1}\hat{x}_{i-1})^{\frac{r_i p_{i-1}}{r_{i-1}}} + (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{m_i p_{i-1}}{m_{i-1}}} - (\hat{z}_i + l_{i-1}\hat{x}_{i-1})^{\frac{m_i p_{i-1}}{m_{i-1}}}$.

We next give the upper bounds of all the terms on the right-hand side of (33) by Proposition 2-Proposition 6, whose proofs are placed in the Appendix. In the following deduction, $h_{i,j} \geq 1$ is used to represent a generic constant related to $l_1, \cdots, l_i$.

Proposition 2: For $i = 2, \cdots, n$,

$$\begin{aligned} & -l_{i-1}\Big(e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \\ & - (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \Big) f_{i-1} \\ & \leq c_1 h_{i-1,i-1} L^{1-\nu} \Big(e_i^{\frac{2}{m_{i-1}}} + e_i^{\frac{2\mu}{r_{i-1}}} + \sum_{j=1}^{i} \Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} \Big) \Big) \\ & + \frac{L}{3n} \sum_{j=1}^{i} \Big(\xi_j^2(t - \tau_j(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t - \tau_j(t)) \\ & + \xi_j^2(t - \tau_{j+1}(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t - \tau_{j+1}(t)) \Big), \end{aligned}$$

where $c_1 > 0$ is a constant independent on $l_1, \cdots, l_{i-1}$.

Proposition 3: For $i = 2, \cdots, n-1$,

$$\begin{aligned} & \Big(e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}} + \frac{2\mu - r_i p_{i-1}}{r_{i-1}} e_i (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}} - 1} \Big) \\ & \cdot \mathcal{I}_{1,i}^{-1} p_{i-1} x_i^{p_{i-1}-1} (L x_{i+1}^{p_i} + f_i)(z_i + l_{i-1}x_{i-1})^{-\frac{\omega_1}{m_{i-1}}} \\ & \leq L c_2 \Big(e_i^{\frac{2}{m_{i-1}}} + e_i^{\frac{2\mu}{r_{i-1}}} \Big) + \frac{L}{8n} \sum_{j=1}^{i+1} \Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} \Big) \\ & + \frac{L}{3n} \sum_{j=1}^{i+1} \Big(\xi_j^2(t - \tau_j(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t - \tau_j(t)) \\ & + \xi_j^2(t - \tau_{j+1}(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t - \tau_{j+1}(t)) \Big), \end{aligned}$$

where $c_2 > 0$ is a constant independent on $l_1, \cdots, l_{i-1}$.

Proposition 4: For $i = 2, \cdots, n$,

$$\begin{aligned} & -l_{i-1} L \Big(e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \\ & - (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \Big) \mathcal{I}_{2,i} \\ & \leq -L \Big(2^{2-\frac{2\mu}{r_{i-1}}} \Big(2^{\frac{m_i p_{i-1}}{m_{i-1}}} - 1 \Big) l_{i-1}^{\frac{2m_i p_{i-1}}{m_{i-1}+m_i p_{i-1}}} - c_3 \Big) \\ & \cdot \Big(e_i^{\frac{2}{m_{i-1}}} + e_i^{\frac{2\mu}{r_{i-1}}} \Big) + \frac{L}{8n} \sum_{j=i-1}^{i} \Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} \Big), \end{aligned}$$

where $c_3 > 0$ is a constant independent on $l_1, \cdots, l_{i-1}$.

Proposition 5: For $i = 3, \cdots, n$,

$$\begin{aligned} & -l_{i-1} L \Big(e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \\ & - (\hat{z}_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} \Big) \mathcal{I}_{3,i} \\ & \leq L c_4 \Big(e_i^{\frac{2}{m_{i-1}}} + e_i^{\frac{2\mu}{r_{i-1}}} \Big) + \frac{L}{8n} \sum_{j=1}^{i} \Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} \Big) \\ & + L \sum_{j=2}^{i-1} h_{i-1,j} \Big(e_j^{\frac{2}{m_{j-1}}} + e_j^{\frac{2\mu}{r_{j-1}}} \Big), \end{aligned}$$

where $c_4 > 0$ is a constant independent on $l_1, \cdots, l_{i-1}$.

Proposition 6: There holds

$$\begin{aligned} & \Big(e_n^{\frac{2-m_n p_{n-1}}{m_{n-1}}} + \frac{2\mu - r_n p_{n-1}}{r_{n-1}} e_n (z_n + l_{n-1}x_{n-1})^{\frac{2\mu - r_{n-1}}{r_{n-1}} - 1} \Big) \\ & \cdot \mathcal{I}_{1,n}^{-1} p_{n-1} x_n^{p_{n-1}-1} (L v^{p_n}(\hat{x}) + f_n)(z_n + l_{n-1}x_{n-1})^{-\frac{\omega_1}{m_{n-1}}} \\ & \leq L c_5 \Big(e_n^{\frac{2}{m_{n-1}}} + e_n^{\frac{2\mu}{r_{n-1}}} \Big) + \frac{L}{8} \sum_{j=1}^{n} \Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} \Big) \\ & + \frac{L}{3} \sum_{j=1}^{n} \Big(\xi_j^2(t - \tau_j(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t - \tau_j(t)) \end{aligned}$$

$$+\xi_j^2(t-\tau_{j+1}(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big)$$
$$+L\sum_{j=2}^{n-1}h_{n-1,j}\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right),$$

where $c_5>0$ is a constant independent on $l_1,\cdots,l_{n-1}$.

Using Propositions 2-6, it is easy to prove that the derivative of $U=\sum_{i=2}^n U_i$ satisfies

$$\begin{aligned}\dot U\ \leq\ & -L\sum_{j=2}^{n-1}\bigg(2^{2-\frac{2\mu}{r_{j-1}}}\left(2^{\frac{m_jp_{j-1}}{m_{j-1}}}-1\right)l_{j-1}^{\frac{2m_jp_{j-1}}{m_{j-1}+m_jp_{j-1}}}\\ & -h_{n-1,j}-c_2-c_3-c_4\bigg)\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right)\\ & -L\bigg(2^{2-\frac{2\mu}{r_{n-1}}}\left(2^{\frac{m_np_{n-1}}{m_{n-1}}}-1\right)l_{n-1}^{\frac{2m_np_{n-1}}{m_{n-1}+m_np_{n-1}}}-c_3\\ & -c_4-c_5\bigg)\left(e_n^{\frac{2}{m_{n-1}}}+e_n^{\frac{2\mu}{r_{n-1}}}\right)+\frac{L}{2}\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)\\ & +L\sum_{j=1}^{n}\bigg(\xi_j^2(t-\tau_j(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t))\\ & +\xi_j^2(t-\tau_{j+1}(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\bigg)\\ & +h_{n-1,1}L^{1-\nu}\bigg(\sum_{j=2}^{n}\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right)\\ & +\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)\bigg).\end{aligned}\tag{34}$$

Choose $T=U+V_n$, then

$$\begin{aligned}\dot T\ \leq\ & -\frac{L}{2}\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)+L\bigg(\xi_n^{2-m_{n+1}p_n}\\ & +\xi_n^{\frac{(2\mu-r_{n+1}p_n)m_n}{r_n}}\bigg)(v^{p_n}-x_{n+1}^{*p_n})\\ & -L\bigg(2^{2-\frac{2\mu}{r_{n-1}}}\left(2^{\frac{m_np_{n-1}}{m_{n-1}}}-1\right)l_{n-1}^{\frac{2m_np_{n-1}}{m_{n-1}+m_np_{n-1}}}\\ & -c_3-c_4-c_5\bigg)\left(e_n^{\frac{2}{m_{n-1}}}+e_n^{\frac{2\mu}{r_{n-1}}}\right)\\ & -L\sum_{j=2}^{n-1}\bigg(2^{2-\frac{2\mu}{r_{j-1}}}\left(2^{\frac{m_jp_{j-1}}{m_{j-1}}}-1\right)l_{j-1}^{\frac{2m_jp_{j-1}}{m_{j-1}+m_jp_{j-1}}}\\ & -h_{n-1,j}-c_2-c_3-c_4\bigg)\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right)\\ & +h_{n-1,1}L^{1-\nu}\bigg(\sum_{j=2}^{n}\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right)\\ & +\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)\bigg).\end{aligned}\tag{35}$$

$L(\xi_n^{2-m_{n+1}p_n}+\xi_n^{\frac{(2\mu-r_{n+1}p_n)m_n}{r_n}})(v^{p_n}-x_{n+1}^{*p_n})$ in (35) can be estimated by the following proposition whose proof is in the Appendix.

Proposition 7: There holds

$$L\left(\xi_n^{2-m_{n+1}p_n}+\xi_n^{\frac{(2\mu-r_{n+1}p_n)m_n}{r_n}}\right)\left(v^{p_n}-x_{n+1}^{*p_n}\right)$$
$$\leq Lc_6\left(e_n^{\frac{2}{m_{n-1}}}+e_n^{\frac{2\mu}{r_{n-1}}}\right)$$
$$+L\sum_{j=2}^{n-1}h_{n-1,j}\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right)+\frac{L}{4}\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right),$$

where $c_6>0$ is a constant independent on $l_1,\cdots,l_{n-1}$.

With the aid of Proposition 7, (35) is simplified as

$$\begin{aligned}\dot T\ \leq\ & -\frac{L}{4}\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)-L\bigg(2^{2-\frac{2\mu}{r_{n-1}}}\left(2^{\frac{m_np_{n-1}}{m_{n-1}}}-1\right)\\ & \cdot l_{n-1}^{\frac{2m_np_{n-1}}{m_{n-1}+m_np_{n-1}}}-c_3-c_4-c_5-c_6\bigg)\left(e_n^{\frac{2}{m_{n-1}}}+e_n^{\frac{2\mu}{r_{n-1}}}\right)\\ & -L\sum_{j=2}^{n-1}\bigg(2^{2-\frac{2\mu}{r_{j-1}}}\left(2^{\frac{m_jp_{j-1}}{m_{j-1}}}-1\right)l_{j-1}^{\frac{2m_jp_{j-1}}{m_{j-1}+m_jp_{j-1}}}\\ & -h_{n-1,j}-c_2-c_3-c_4\bigg)\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right)\\ & +h_{n-1,1}L^{1-\nu}\bigg(\sum_{j=2}^{n}\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right)\\ & +\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)\bigg).\end{aligned}\tag{36}$$

By determining $l_1,\cdots,l_{n-1}$ and L as

$$l_{n-1}=\max\left\{\left((c_3+c_4+c_5+c_6+1)\left(2^{\frac{m_np_{n-1}}{m_{n-1}}}-1\right)^{-1}\cdot 2^{\frac{2\mu}{r_{n-1}}-2}\right)^{\frac{m_{n-1}+m_np_{n-1}}{2m_np_{n-1}}},1\right\},$$
$$l_i=\max\left\{\left((c_2+c_3+c_4+1+h_{n-1,i+1})\left(2^{\frac{m_{i+1}p_i}{m_i}}-1\right)^{-1}\cdot 2^{\frac{2\mu}{r_i}-2}\right)^{\frac{m_i+m_{i+1}p_i}{2m_{i+1}p_i}},1\right\},\ i=n-2,\cdots,1,$$
$$L>\max\left\{(8h_{n-1,1})^{\frac{1}{\nu}},1\right\},$$

(36) becomes

$$\dot T\leq-\frac{L}{8}\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)-\frac{7L}{8}\sum_{j=2}^{n}\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right).\tag{37}$$

Then, we get the output feedback controller of system (1) with the form

$$\begin{aligned}& u(t)=-L^{\lambda_{n+1}}v(t)=-\beta_nL^{\lambda_{n+1}}\left(\hat\xi_n+\hat\xi_n^{\frac{m_nr_{n+1}}{r_nm_{n+1}}}\right)^{m_{n+1}},\\ & \hat\xi_i=\hat x_i^{\frac{1}{m_i}}-\hat x_i^{*\frac{1}{m_i}},\ x_0^*=0,\\ & \hat x_i^*=-\beta_{i-1}\left(\hat\xi_{i-1}+\hat\xi_{i-1}^{\frac{m_{i-1}r_i}{r_{i-1}m_i}}\right)^{m_i},\ i=1,\cdots,n,\end{aligned}\tag{38}$$

where $\hat x_2,\cdots,\hat x_n$ are observed by

$$\begin{aligned}& \dot{\hat z}_i=-l_{i-1}L\hat x_i^{p_{i-1}},\\ & \hat x_i^{p_{i-1}}=(\hat z_i+l_{i-1}\hat x_{i-1})^{\frac{r_ip_{i-1}}{r_{i-1}}}+(\hat z_i+l_{i-1}\hat x_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}}.\end{aligned}\tag{39}$$

V. Stability Analysis

We state the main result in this paper.

Theorem 1: If Assumptions 1-2 hold for system (1), the output feedback controller (38) and (39) guarantees that

(1) All the solutions of the closed-loop system (1), (38) and (39) are well defined on $[-\varepsilon_M,+\infty)$.

(2) The equilibrium $\eta=0$ of the closed-loop system is globally uniformly asymptotically stable.

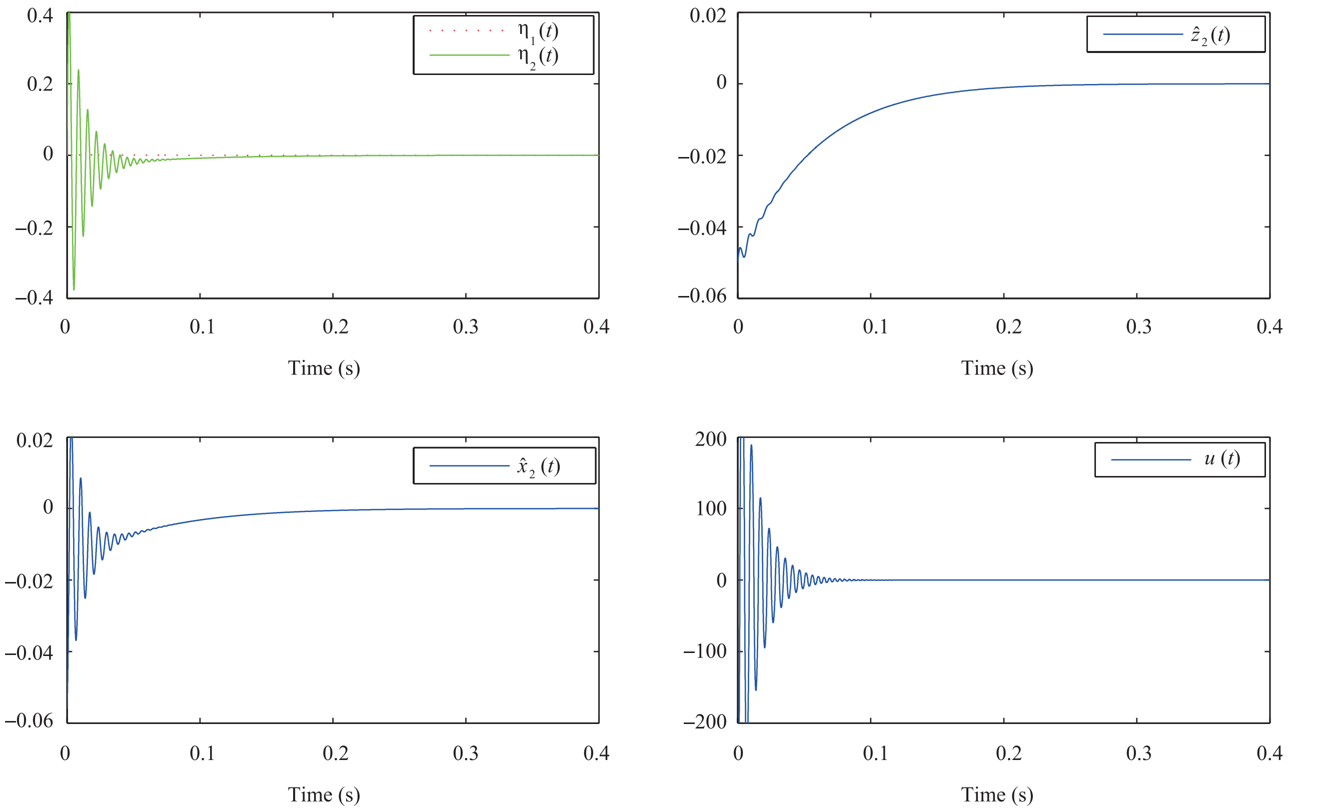

Fig. 2. The response of the closed-loop system (43), (44).

Proof: (1) Under (38), system (1) can be equivalently transformed into a ξ-system

$$\dot{\xi}_i(t) = \varphi_i(t, \xi(t), \xi_1(t-\tau_1(t)), \cdots, \xi_i(t-\tau_i(t)), \xi_1(t-\tau_2(t)), \cdots, \xi_{i-1}(t-\tau_i(t)), u(t)), \quad (40)$$

where $\varphi_i(\cdot) : R^+ \times R^n \times R^i \times R^{i-1} \times R \to R$ is a $\mathcal{C}^1$ function with $\varphi_i(t,0,0,0,0) = 0$. Define $\mathcal{Z} = [\xi_1, \cdots, \xi_n, e_2, \cdots, e_n]^\top$, by the existence and continuation of the solution, the solution $\mathcal{Z}(t)$ of $\mathcal{Z}$-system (30), (40) is defined on $[-\varepsilon_M, t_M)$ with t_M being infinite or not.

Firstly, we prove that $T(\mathcal{Z})$ is $\mathcal{C}^1$, positive definite and radially unbounded.

Proposition 1 and $\frac{2-\omega_1}{m_{i-1}} \geq 1, \frac{2\mu - r_i p_{i-1}}{r_{i-1}} \geq 1$ imply that $T(\mathcal{Z})$ is $\mathcal{C}^1$.

We show that $T(\mathcal{Z})$ is positive definite. On one hand, Proposition 1 yields that V_n is positive definite, then when $T(\mathcal{Z}) = 0$, $\xi = 0$ and $U_i = 0, i = 2, \cdots, n$, then $\mathcal{Z} = 0$. On the other hand, if $\mathcal{Z} = 0$, $T(\mathcal{Z}) = 0$. So $T(\mathcal{Z})$ is positive definite.

We know from Proposition 1 that V_n is positive definite and radially unbounded, so there is a $\mathcal{K}_\infty$ function $\pi_{n,1}(\cdot)$ such that $V_n \geq \pi_{n,1}(\|\xi\|)$. From (31) and (32), one has

$$T(\mathcal{Z}) \geq \pi_{n,1}(\|\xi\|) + \sum_{i=2}^{n} \left(\frac{m_{i-1}}{2-\omega_1} e_i^{\frac{2-\omega_1}{m_{i-1}}} + \frac{2^{1-\frac{2\mu - r_i p_{i-1}}{r_{i-1}}} r_{i-1}}{2\mu - \omega_2} e_i^{\frac{2\mu-\omega_2}{r_{i-1}}} \right) \triangleq \pi_1(\|\mathcal{Z}\|). \quad (41)$$

Obviously, $T(\mathcal{Z})$ is radially unbounded since $\pi_{n,1}(\cdot)$ is a $\mathcal{K}_\infty$ function.

Secondly, we show that $\eta(t)$ is well defined on $[-\varepsilon_M, +\infty)$. Noticing that $T(\mathcal{Z})$ and the term $-\frac{L}{8}\sum_{j=1}^{n}(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}}) - \frac{7L}{8}\sum_{j=2}^{n}(e_j^{\frac{2}{m_{j-1}}} + e_j^{\frac{2\mu}{r_{j-1}}})$ on the right-hand side of (37) are positive definite and radially unbounded, one can find $\mathcal{K}_\infty$ functions $\pi_2(\cdot)$ and $\pi_3(\cdot)$ such that

$$T(\mathcal{Z}) \leq \pi_2(\|\mathcal{Z}\|), \quad \dot{T}(\mathcal{Z}) \leq -\pi_3(\|\mathcal{Z}\|). \quad (42)$$

Since $\pi_1(\|\mathcal{Z}\|)$ is a $\mathcal{K}_\infty$ function, for any $\delta > 0$, one can always find a $\beta = \beta(\delta)$ with $\beta > \delta > 0$ such that $\pi_2(\delta) \leq \pi_1(\beta)$. If $\|\mathcal{Z}_0(\theta)\|_{\mathcal{C}} < \delta$, $\theta \in [-\varepsilon_M, 0]$, (41) and (42) yield $\pi_1(\|\mathcal{Z}\|) \leq T(\mathcal{Z}_t(\theta)) \leq T(\mathcal{Z}_0(\theta)) \leq \pi_2(\|\mathcal{Z}_0(\theta)\|_{\mathcal{C}}) \leq \pi_1(\beta)$, $\forall t \in [0, t_M)$, which means that $\|\mathcal{Z}(t)\| \leq \beta$ for any $t \in [-\varepsilon_M, t_M)$. Hence, t_M is not an escape time, i.e., $\mathcal{Z}(t)$ is well defined on $[-\varepsilon_M, +\infty)$, so is $\eta(t)$.

(2) Since $t_M = +\infty$, according to (41), (42) and Lemma 7, the equilibrium $\mathcal{Z} = 0$ of system (30) and (40) is globally uniformly asymptotically stable. Since $x_i^*(\xi_{i-1})$ is continuous on ξ_{i-1} and $x_i^*(0) = 0$, by (39) and the globally uniformly asymptotic stability of system (30) and (40), it is easy to recursively prove that the equilibrium $\eta = 0$ of the closed-loop system (1), (38) and (39) is globally uniformly asymptotically stable. □

VI. A Simulation Example

Consider a simple system

$$\dot{\eta}_1 = \eta_2^{\frac{21}{19}},$$

$$\dot{\eta}_2 = u + \frac{\eta_1^{\frac{17}{23}} \sin \eta_1}{3} + \frac{\ln(1 + |\eta_1|^{\frac{21}{19}})}{2(1+\eta_2^2)} + \frac{\eta_1^{\frac{17}{23}}(t - \frac{1}{2}\sin^2 t)}{3}$$

$$
\begin{aligned}
&+\frac{\eta_1^{\frac{21}{19}}(t-\frac{1}{2}\sin^2 t)}{2(1+\eta_2^2(t-\frac{1}{2}\cos^2 t))},\\
&y=\eta_1, \qquad (43)
\end{aligned}
$$

where $p_1=\frac{21}{19}$, $p_2=1, \tau_1(t)=\frac{1}{2}\sin^2 t, \tau_2(t)=\frac{1}{2}\cos^2 t$. Choose $\bar{\omega}_1=-\frac{2}{23}\in(-\frac{19}{40},0]$, $\bar{\omega}_2=\frac{2}{19}\in[0,+\infty)$, then $\bar{m}_1=1,\bar{m}_2=\frac{\bar{m}_1+\bar{\omega}_1}{p_1}=\frac{19}{23},\bar{m}_3=\frac{\bar{m}_2+\bar{\omega}_1}{p_2}=\frac{17}{23}$, $\bar{r}_1=1,\bar{r}_2=\frac{\bar{r}_1+\bar{\omega}_2}{p_1}=1,\bar{r}_3=\frac{\bar{r}_2+\bar{\omega}_2}{p_2}=\frac{21}{19}$. Obviously, $\varepsilon_M=\frac{1}{2}$, Assumption 1 holds with $\gamma_1=\frac{1}{2},\gamma_2=\frac{1}{2}$, Assumption 2 holds with $\phi_1=0,|\phi_2|\le\frac{1}{2}(|\eta_1|^{\frac{17}{23}}+|\eta_1|^{\frac{21}{19}}+|\eta_1(t-\frac{1}{2}\sin^2 t)|^{\frac{17}{23}}+|\eta_1(t-\frac{1}{2}\sin^2 t)|^{\frac{21}{19}})$.

Introduce a change of coordinates $x_1=\eta_1,x_2=\frac{\eta_2}{L^{\frac{19}{21}}},v=\frac{u}{L^{\frac{40}{21}}}$. Following the design procedure in Section IV, by choosing $\omega_1=\bar{\omega}_1=-\frac{2}{23},\omega_2=\bar{\omega}_2=\frac{2}{19},\mu=\frac{483}{323}$, a direct but redundant computation leads to an output feedback controller of system (43) with the form

$$
\begin{aligned}
&u=-L^{\frac{40}{21}}\beta_2\left(\hat{\xi}_2+\hat{\xi}_2^{\frac{21}{17}}\right)^{\frac{17}{23}},\\
&\hat{\xi}_2=\hat{x}_2^{\frac{23}{19}}+\beta_1^{\frac{23}{19}}\left(x_1+x_1^{\frac{23}{19}}\right),\\
&\hat{x}_2^{\frac{21}{19}}=(\hat{z}_2+l_1x_1)^{\frac{21}{23}}+(\hat{z}_2+l_1x_1)^{\frac{21}{19}},\\
&\dot{\hat{z}}_2=-l_1L\hat{x}_2^{\frac{21}{19}}, \qquad (44)
\end{aligned}
$$

where $\beta_1=2.68,\beta_2=600,l_1=30,L=3$.

From Theorem 1, output feedback controller (44) can guarantee the globally uniformly asymptotical stability of system (43). By choosing the initial values $\eta_1(\theta)=0.001$, $\eta_2(\theta)=-0.002,\hat{z}_2(\theta)=-0.05,\theta\in[-\frac{1}{2},0]$, Fig.2 verifies the effectiveness of the control scheme.

VII. Conclusion

This paper addresses the global output feedback stabilization of high-order nonlinear time-delay systems with more general low-order and high-order nonlinearities.

Some problems are still remained, e.g., 1) Can we design an output feedback controller when there are time delays in input? 2) Recently, many results including our papers [28]−[31] on stochastic high-order nonlinear systems have been achieved, another problem is whether the result in this paper can be extended to stochastic nonlinear time-delay systems.

Appendix A
Proof of Proposition 1

Step 1: We first prove that $V_k(\bar{\xi}_k)$ is $\mathcal{C}^1$. It's easy to see that L_{Hk},L_{Dk} is $\mathcal{C}^1$. By (14) and the definition of μ, it's deduced that $2-m_{k+1}p_k\ge 1$ and

$$
\begin{aligned}
&\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}\\
&=\frac{2m_k\max_{i=1,\cdots,n+1}\{r_ip_{i-1}\}}{r_k\min_{i=1,\cdots,n+1}\{m_ip_{i-1}\}}-\frac{r_{k+1}m_kp_k}{r_k}\\
&\ge\frac{m_k\max_{i=1,\cdots,n+1}\{r_ip_{i-1}\}}{r_k\min_{i=1,\cdots,n+1}\{m_ip_{i-1}\}}\ge 1,
\end{aligned}
$$

which means that W_{Hk}, W_{Dk} and then V_k are $\mathcal{C}^1$.

Step 2: By (20), (24), and Lemma 2, we have

$$
W_{Hk}\le|\xi_k|^{2-m_{k+1}p_k}|x_k-x_k^*|\le 2^{1-m_k}|\xi_k|^{2-\omega_1},
$$

$$
W_{Dk}\le|\xi_k|^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}|x_k-x_k^*|\le 2^{1-m_k}|\xi_k|^{\frac{(2\mu-\omega_2)m_k}{r_k}},
$$

which and (19) yield

$$
\begin{aligned}
&V_k(t,\bar{\xi}_{k,t})\\
&\le\pi_{k-1,2}(\|\bar{\xi}_{k-1,t}\|_{\mathcal{C}})+2^{1-m_k}\left(|\xi_k|^{2-\omega_1}+|\xi_k|^{\frac{(2\mu-\omega_2)m_k}{r_k}}\right)\\
&\quad+(2n-2k+3)\int_{t-\varepsilon_M}^{t}\left(\xi_k^2(s)+\xi_k^{\frac{2\mu m_k}{r_k}}(s)\right)\mathrm{d}s\\
&\le\pi_{k-1,2}(\|\bar{\xi}_{k-1,t}\|_{\mathcal{C}})+2^{1-m_k}\left(|\xi_k|^{2-\omega_1}+|\xi_k|^{\frac{(2\mu-\omega_2)m_k}{r_k}}\right)\\
&\quad+(2n-2k+3)\varepsilon_M\left(\left(\sup_{-\varepsilon_M\le\theta\le 0}|\xi_k(t+\theta)|\right)^2\right.\\
&\quad\left.+\left(\sup_{-\varepsilon_M\le\theta\le 0}|\xi_k(t+\theta)|\right)^{\frac{2\mu m_k}{r_k}}\right)\\
&\le\pi_{k-1,2}(\|\bar{\xi}_{k,t}\|_{\mathcal{C}})+2^{1-m_k}\left(\|\bar{\xi}_{k,t}\|_{\mathcal{C}}^{2-\omega_1}+\|\bar{\xi}_{k,t}\|_{\mathcal{C}}^{\frac{(2\mu-\omega_2)m_k}{r_k}}\right)\\
&\quad+(2n-2k+3)\varepsilon_M\left(\|\bar{\xi}_{k,t}\|_{\mathcal{C}}^2+\|\bar{\xi}_{k,t}\|_{\mathcal{C}}^{\frac{2\mu m_k}{r_k}}\right)\\
&\triangleq\pi_{k,2}(\|\bar{\xi}_{k,t}\|_{\mathcal{C}}). \qquad (45)
\end{aligned}
$$

Obviously, $\pi_{k,2}(\cdot)$ is a $\mathcal{K}_\infty$ function.

When $x_k>x_k^*$, it follows from Lemma 2 that

$$
\begin{aligned}
&W_{Hk}\ge\frac{2^{\frac{(2-m_{k+1}p_k)(m_k-1)}{m_k}}m_k}{2-\omega_1}|x_k-x_k^*|^{\frac{2-\omega_1}{m_k}},\\
&W_{Dk}\ge\frac{2^{\frac{(2\mu-r_{k+1}p_k)(m_k-1)}{r_k}}r_k}{2\mu-\omega_2}|x_k-x_k^*|^{\frac{2\mu-\omega_2}{r_k}}. \qquad (46)
\end{aligned}
$$

When $x_k\le x_k^*$, it can be shown that (46) also holds in a similar way.

From (19), (24), (46), it follows that

$$
\begin{aligned}
&V_k(t,\bar{\xi}_{k,t})\\
&\ge\pi_{k-1,1}(\|\bar{\xi}_{k-1}\|)+2^{\frac{(2-m_{k+1}p_k)(m_k-1)}{m_k}}\frac{m_k}{2-\omega_1}|x_k-x_k^*|^{\frac{2-\omega_1}{m_k}}\\
&=\pi_{k-1,1}(\|\bar{\xi}_{k-1}\|)+2^{\frac{(2-m_{k+1}p_k)(m_k-1)}{m_k}}\frac{m_k}{2-\omega_1}\\
&\quad\cdot\left|\left(\xi_k+x_k^{*\frac{1}{m_k}}\right)^{m_k}-x_k^*\right|^{\frac{2-\omega_1}{m_k}}\triangleq W_k(\bar{\xi}_k). \qquad (47)
\end{aligned}
$$

We know that $W_k(\bar{\xi}_k)\ge 0$ and $W_k(\bar{\xi}_k)=0$ if and only if $\bar{\xi}_k=0$, i.e., $W_k(\bar{\xi}_k)$ is positive definite.

We next show that $W_k(\bar{\xi}_k)$ is radially unbounded. $\|\bar{\xi}_k\|\to+\infty$ includes three cases.

Case 1): $\|\bar{\xi}_{k-1}\|\to+\infty$ and $|\xi_k|\to+\infty$, then $\lim_{\|\bar{\xi}_k\|\to+\infty}W_k(\bar{\xi}_k)\ge\lim_{\|\bar{\xi}_k\|\to+\infty}\pi_{k-1,1}(\|\bar{\xi}_{k-1}\|)=+\infty$.

Case 2): $\|\bar{\xi}_{k-1}\|\to+\infty$ and $|\xi_k|\le M_1$, we have the same analysis as case 1).

Case 3): $\|\bar{\xi}_{k-1}\|\le M_2$ for a positive constant M_2 and $\|\xi_k|\to+\infty$, since $x_k^*(\xi_{k-1})$ is continuous in ξ_{k-1}, and $|\xi_{k-1}|\le\|\bar{\xi}_{k-1}\|\le M_2$, one has

$$
\begin{aligned}
\lim_{\|\bar{\xi}_k\|\to+\infty}W_k(\bar{\xi}_k)\ &\ge\ \frac{2^{\frac{(2-m_{k+1}p_k)(m_k-1)}{m_k}}m_k}{2-\omega_1}\\
&\quad\lim_{\|\bar{\xi}_k\|\to+\infty}\left|\left(\xi_k+x_k^{*\frac{1}{m_k}}\right)^{m_k}-x_k^*\right|^{\frac{2-\omega_1}{m_k}}\\
&=\ +\infty. \qquad (48)
\end{aligned}
$$

Since $W_k(\bar{\xi}_k)$ is radially unbounded, there is a $\mathcal{K}_\infty$ function $\pi_{k,1}(\cdot)$ such that

$$\pi_{k,1}(\|\bar{\xi}_k\|) \le W_k(\bar{\xi}_k) \le V_k(t,\bar{\xi}_{k,t}). \tag{49}$$

Combining (45) with (49) leads to (22).

Step 3: We prove (23). From Assumption 1 and (21), (24), we can get

$$\begin{aligned}
&\dot{V}_k(t,\bar{\xi}_{k,t})\\
&\le -L(n-k+2)\sum_{j=1}^{k-1}\bigg(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}+\xi_j^2(t-\tau_j(t))\\
&\quad+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t))+\xi_j^2(t-\tau_{j+1}(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\bigg)\\
&\quad+L\bigg(\xi_k^{2-m_{k+1}p_k}+\xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\bigg)\left(x_{k+1}^{p_k}-x_{k+1}^{*p_k}\right)\\
&\quad+L\bigg(\xi_k^{2-m_{k+1}p_k}+\xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\bigg)x_{k+1}^{*p_k}\\
&\quad-L(n-k+2)\left(\xi_k^2(t-\tau_k(t)+\xi_k^{\frac{2\mu m_k}{r_k}}(t-\tau_k(t))\right)\\
&\quad-L(n-k+1)\left(\xi_k^2(t-\tau_{k+1}(t))+\xi_k^{\frac{2\mu m_k}{r_k}}(t-\tau_{k+1}(t))\right)\\
&\quad+L\left(\frac{n+k-2}{1+\gamma_k}+\frac{n+k-1}{1+\gamma_{k+1}}\right)\bigg(\xi_k^{2-m_{k+1}p_k}+\xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\bigg)\\
&\quad+L\bigg(\xi_{k-1}^{2-m_kp_{k-1}}+\xi_{k-1}^{\frac{(2\mu-r_kp_{k-1})m_{k-1}}{r_{k-1}}}\bigg)(x_k^{p_{k-1}}-x_k^{*p_{k-1}})\\
&\quad+\bigg(\xi_k^{2-m_{k+1}p_k}+\xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\bigg)f_k\\
&\quad+\sum_{l=1}^{k-1}\left(\frac{\partial W_{Hk}}{\partial x_l}+\frac{\partial W_{Dk}}{\partial x_l}\right)\dot{x}_l.
\end{aligned}\tag{50}$$

For proceed further, we next estimate the last three terms on the right-hand side of (50).

Firstly, by Lemmas 2,4,5, there is a constant $g_{k,1}>0$ independent on L such that

$$\begin{aligned}
&L\bigg(\xi_{k-1}^{2-m_kp_{k-1}}+\xi_{k-1}^{\frac{(2\mu-r_kp_{k-1})m_{k-1}}{r_{k-1}}}\bigg)\left(x_k^{p_{k-1}}-x_k^{*p_{k-1}}\right)\\
&\le 2^{1-m_kp_{k-1}}L\bigg(|\xi_{k-1}|^{2-m_kp_{k-1}}+|\xi_{k-1}|^{\frac{(2\mu-r_kp_{k-1})m_{k-1}}{r_{k-1}}}\bigg)\\
&\quad\cdot|\xi_k|^{m_kp_{k-1}}\\
&\le\frac{L}{3}\bigg(\xi_{k-1}^2+\xi_{k-1}^{\frac{2\mu m_{k-1}}{r_{k-1}}}\bigg)+Lg_{k,1}\bigg(\xi_k^2+\xi_k^{\frac{2\mu m_k}{r_k}}\bigg).
\end{aligned}\tag{51}$$

In view of (14), we can deduce that

$$\begin{aligned}
\bar{m}_i&=\frac{1}{p_1\cdots p_{i-1}}+\bar{\omega}_1\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}},\\
\bar{r}_i&=\frac{1}{p_1\cdots p_{i-1}}+\bar{\omega}_2\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}},\\
m_i&=\frac{1}{p_1\cdots p_{i-1}}+\omega_1\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}},\\
r_i&=\frac{1}{p_1\cdots p_{i-1}}+\omega_2\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}}.
\end{aligned}\tag{52}$$

Define a function $\psi(s)=\frac{1+s\sum_{l=1}^{i}p_1\cdots p_{l-1}}{p_j\cdots p_{i-1}(1+s\sum_{l=1}^{j-1}p_1\cdots p_{l-1})}$, which is monotonically increasing on s obviously. By (52) and $\omega_1\le\bar{\omega}_1\le\bar{\omega}_2\le\omega_2$, we have

$$\begin{aligned}
\frac{m_i+\omega_1}{m_j}&=\psi(\omega_1)\le\frac{\bar{m}_i+\bar{\omega}_1}{\bar{m}_j}=\psi(\bar{\omega}_1)\\
&\le\frac{\bar{r}_i+\bar{\omega}_2}{\bar{r}_j}=\psi(\bar{\omega}_2)\le\frac{r_i+\omega_2}{r_j}=\psi(\omega_2).
\end{aligned}\tag{53}$$

Recall $-\frac{1}{\sum_{l=1}^{n}p_1\cdots p_{l-1}}<\omega_1\le 0$ and $\omega_2\ge 0$, (52) implies that for $i=1,\cdots,n$, $j=1,\cdots,i-1$,

$$\begin{aligned}
&r_im_j-m_ir_j\\
&=\left(\frac{1}{p_1\cdots p_{i-1}}+\omega_2\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}}\right)\\
&\quad\cdot\left(\frac{1}{p_1\cdots p_{j-1}}+\omega_1\sum_{l=1}^{j-1}\frac{1}{p_l\cdots p_{j-1}}\right)\\
&\quad-\left(\frac{1}{p_1\cdots p_{i-1}}+\omega_1\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}}\right)\\
&\quad\cdot\left(\frac{1}{p_1\cdots p_{j-1}}+\omega_2\sum_{l=1}^{j-1}\frac{1}{p_l\cdots p_{j-1}}\right)\\
&=\frac{\omega_2-\omega_1}{p_1\cdots p_{j-1}}\sum_{l=j}^{i-1}\frac{1}{p_l\cdots p_{i-1}}\ge 0.
\end{aligned}\tag{54}$$

Using (20), (52)-(54), $f_i=\frac{\phi_i}{L^{\lambda_i}}$, $L\ge 1$, Lemmas 1,5, one has

$$\begin{aligned}
&|f_k(t,x(t),x_1(t-\tau_1(t)),\cdots,x_k(t-\tau_k(t)))|\\
&\le c\sum_{j=1}^{k}\bigg(L^{-\lambda_k+\frac{\lambda_j(\bar{m}_k+\bar{\omega}_1)}{\bar{m}_j}}\bigg(|x_j(t)|^{\frac{\bar{m}_k+\bar{\omega}_1}{\bar{m}_j}}\\
&\quad+|x_j(t-\tau_j(t))|^{\frac{\bar{m}_k+\bar{\omega}_1}{\bar{m}_j}}\bigg)+L^{-\lambda_k+\frac{\lambda_j(\bar{r}_k+\bar{\omega}_2)}{\bar{r}_j}}\bigg(|x_j(t)|^{\frac{\bar{r}_k+\bar{\omega}_2}{\bar{r}_j}}\\
&\quad+|x_j(t-\tau_j(t))|^{\frac{\bar{r}_k+\bar{\omega}_2}{\bar{r}_j}}\bigg)\bigg)\\
&\le\alpha L^{1-\nu}\bigg(\sum_{j=1}^{k}\bigg(|\xi_j|^{m_{k+1}p_k}+|\xi_j|^{\frac{m_jr_{k+1}p_k}{r_j}}\\
&\quad+|\xi_j(t-\tau_j(t))|^{m_{k+1}p_k}+|\xi_j(t-\tau_j(t))|^{\frac{m_jr_{k+1}p_k}{r_j}}\bigg)\\
&\quad+\sum_{j=1}^{k-1}\bigg(|\xi_j(t-\tau_{j+1}(t))|^{m_{k+1}p_k}+|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_jr_{k+1}p_k}{r_j}}\bigg)\bigg),
\end{aligned}\tag{55}$$

where $\nu=\min\{1-\lambda_j\frac{\bar{r}_k+\bar{\omega}_1}{\bar{r}_j}+\lambda_k, 1\le j\le k, 1\le k\le n\}>0$, we use α to exemplify a finite positive constant that may be implicitly changed in various places and is independent on L. By (55), Lemmas 4,5, we get

$$\begin{aligned}
&\bigg(\xi_k^{2-m_{k+1}p_k}+\xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\bigg)f_k\\
&\le\frac{L}{2}\sum_{j=1}^{k-2}\bigg(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\bigg)+\frac{L}{2}\sum_{j=1}^{k-1}\bigg(\xi_j^2(t-\tau_j(t))\\
&\quad+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t))+\xi_j^2(t-\tau_{j+1}(t))\\
&\quad+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\bigg)+Lg_{k,2}\bigg(\xi_k^2+\xi_k^{\frac{2\mu m_k}{r_k}}\bigg)\\
&\quad+L\bigg(\xi_k^2(t-\tau_k(t))+\xi_k^{\frac{2\mu m_k}{r_k}}(t-\tau_k(t))\bigg)
\end{aligned}$$

$$+\frac{L}{3}\left(\xi_{k-1}^2+\xi_{k-1}^{\frac{2\mu m_{k-1}}{r_{k-1}}}\right), \tag{56}$$

where $g_{k,2}>0$ is a constant independent on L.

We estimate the last term of (50). From (13), (20), (54) and (55), Lemmas 1,4,5, it follows that

$$\begin{aligned}
&\left|\frac{\partial x_k^{*\frac{1}{m_k}}}{\partial x_l}\right||\dot{x}_l| \\
&\le \prod_{j=l}^{k-1}\left(1+|\xi_j|^{\frac{m_j r_{j+1}}{r_j m_{j+1}}-1}\right)|x_l|^{\frac{1}{m_l}-1}(L|x_{l+1}|^{p_l}+f_l) \\
&\le \alpha L\prod_{j=l}^{k-1}\left(1+|\xi_j|^{\frac{m_j r_{j+1}}{r_j m_{j+1}}-1}\right)\left(|\xi_l|+|\xi_{l-1}|+|\xi_{l-1}|^{\frac{m_{l-1}r_l}{r_{l-1}m_l}}\right)^{1-m_l} \\
&\quad\Bigg(|\xi_{l+1}|^{m_{l+1}p_l}+\sum_{j=1}^{l}\Bigg(|\xi_j|^{m_{l+1}p_l}+|\xi_j|^{\frac{m_j r_{l+1}p_l}{r_j}} \\
&\quad+|\xi_j(t-\tau_j(t))|^{m_{l+1}p_l}+|\xi_j(t-\tau_j(t))|^{\frac{m_j r_{l+1}p_l}{r_j}}\Bigg) \\
&\quad+\sum_{j=1}^{l-1}\left(|\xi_j(t-\tau_{j+1}(t))|^{m_{l+1}p_l}+|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_j r_{l+1}p_l}{r_j}}\right)\Bigg) \\
&\overset{(1)}{\le} \alpha L\prod_{j=l+1}^{k-1}\left(1+|\xi_j|^{\frac{m_j r_{j+1}}{r_j m_{j+1}}-1}\right)\Bigg(\sum_{j=1}^{l+1}\Bigg(|\xi_j|^{m_2p_1} \\
&\quad+|\xi_j|^{\frac{m_j}{r_j}\left(\frac{r_{l+1}}{m_{l+1}}+\omega_2\right)}\Bigg)+\sum_{j=1}^{l}\Bigg(|\xi_j(t-\tau_j(t))|^{m_2p_1} \\
&\quad+|\xi_j(t-\tau_j(t))|^{\frac{m_j}{r_j}\left(\frac{r_{l+1}}{m_{l+1}}+\omega_2\right)}\Bigg)+\sum_{j=1}^{l-1}\Bigg(|\xi_j(t-\tau_{j+1}(t))|^{m_2p_1} \\
&\quad+|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_j}{r_j}\left(\frac{r_{l+1}}{m_{l+1}}+\omega_2\right)}\Bigg)\Bigg) \\
&\vdots \\
&\overset{(i)}{\le} \alpha L\prod_{j=l+i}^{k-1}\left(1+|\xi_j|^{\frac{m_j r_{j+1}}{r_j m_{j+1}}-1}\right)\Bigg(\sum_{j=1}^{l+i}\Bigg(|\xi_j|^{m_2p_1} \\
&\quad+|\xi_j|^{\frac{m_j}{r_j}\left(\frac{r_{l+i}}{m_{l+i}}+\omega_2\right)}\Bigg)+\sum_{j=1}^{l}\Bigg(|\xi_j(t-\tau_j(t))|^{m_2p_1} \\
&\quad+|\xi_j(t-\tau_j(t))|^{\frac{m_j}{r_j}\left(\frac{r_{l+i}}{m_{l+i}}+\omega_2\right)}\Bigg)+\sum_{j=1}^{l-1}\Bigg(|\xi_j(t-\tau_{j+1}(t))|^{m_2p_1} \\
&\quad+|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_j}{r_j}\left(\frac{r_{l+i}}{m_{l+i}}+\omega_2\right)}\Bigg)\Bigg) \\
&\vdots \\
&\overset{(k-1)}{\le} \alpha L\Bigg(\sum_{j=1}^{k}\left(|\xi_j|^{m_2p_1}+|\xi_j|^{\frac{m_j}{r_j}\left(\frac{r_k}{m_k}+\omega_2\right)}\right) \\
&\quad+\sum_{j=1}^{l}\left(|\xi_j(t-\tau_j(t))|^{m_2p_1}+|\xi_j(t-\tau_j(t))|^{\frac{m_j}{r_j}\left(\frac{r_k}{m_k}+\omega_2\right)}\right) \\
&\quad+\sum_{j=1}^{l-1}\Bigg(|\xi_j(t-\tau_{j+1}(t))|^{m_2p_1} \\
&\quad+|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_j}{r_j}\left(\frac{r_k}{m_k}+\omega_2\right)}\Bigg)\Bigg),
\end{aligned}\tag{57}$$

where $\prod_{j=k}^{k-1}(1+|\xi_j|^{\frac{m_j r_{j+1}}{r_j m_{j+1}}-1})=1$, for $i=1,\cdots,k-1$, if $l=k-i$, (57) stops at inequality (i). By (57), Lemmas 2,4,5, we have

$$\begin{aligned}
&\sum_{l=1}^{k-1}\left(\frac{\partial W_{Hk}}{\partial x_l}+\frac{\partial W_{Dk}}{\partial x_l}\right)\dot{x}_l \\
&\le \alpha\left(|\xi_k|^{2-m_{k+1}p_k-1}+|\xi_k|^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}-1}\right) \\
&\quad\cdot|x_k-x_k^*|\cdot\sum_{l=1}^{k-1}\left|\frac{\partial x_k^{*\frac{1}{m_k}}}{\partial x_l}\right||\dot{x}_l| \\
&\le \alpha L\left(|\xi_k|^{2-m_2p_1}+|\xi_k|^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}+m_k-1}\right) \\
&\quad\cdot\sum_{l=1}^{k-1}\Bigg(\sum_{j=1}^{k}\left(|\xi_j|^{m_2p_1}+|\xi_j|^{\frac{m_j}{r_j}\left(\frac{r_k}{m_k}+\omega_2\right)}\right) \\
&\quad+\sum_{j=1}^{l}\left(|\xi_j(t-\tau_j(t))|^{m_2p_1}+|\xi_j(t-\tau_j(t))|^{\frac{m_j}{r_j}\left(\frac{r_k}{m_k}+\omega_2\right)}\right) \\
&\quad+\sum_{j=1}^{l-1}\Bigg(|\xi_j(t-\tau_{j+1}(t))|^{m_2p_1} \\
&\quad+|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_j}{r_j}\left(\frac{r_k}{m_k}+\omega_2\right)}\Bigg)\Bigg) \\
&\le \frac{L}{2}\sum_{j=1}^{k-2}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)+\frac{L}{2}\sum_{j=1}^{k-1}\Bigg(\xi_j^2(t-\tau_j(t)) \\
&\quad+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t))+\xi_j^2(t-\tau_{j+1}(t)) \\
&\quad+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Bigg)+Lg_{k,3}\left(\xi_k^2+\xi_k^{\frac{2\mu m_k}{r_k}}\right) \\
&\quad+\frac{L}{3}\left(\xi_{k-1}^2+\xi_{k-1}^{\frac{2\mu m_{k-1}}{r_{k-1}}}\right),
\end{aligned}\tag{58}$$

where $g_{k,3}>0$ is a positive constant independent on L.

Substituting (51), (56) and (58) into (50), one has

$$\begin{aligned}
&\dot{V}_k(t,\bar{\xi}_{k,t}) \\
&\le -L(n-k+1)\sum_{j=1}^{k-1}\Bigg(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}+\xi_j^2(t-\tau_j(t)) \\
&\quad+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t))+\xi_j^2(t-\tau_{j+1}(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Bigg) \\
&\quad+L\left(\xi_k^{2-m_{k+1}p_k}+\xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)(x_{k+1}^{p_k}-x_{k+1}^{*p_k}) \\
&\quad-(n-k+1)L\Bigg(\xi_k^2(t-\tau_k)+\xi_k^{\frac{2\mu m_k}{r_k}}(t-\tau_k(t)) \\
&\quad+\xi_k^2(t-\tau_{k+1}(t))+\xi_k^{\frac{2\mu m_k}{r_k}}(t-\tau_{k+1}(t))\Bigg) \\
&\quad+L\left(\xi_k^{2-m_{k+1}p_k}+\xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)x_{k+1}^{*p_k}+L\Bigg(\frac{n-k+2}{1-\gamma_k} \\
&\quad+\frac{n-k+1}{1-\gamma_{k+1}}+g_{k,1}+g_{k,2}+g_{k,3}\Bigg)\left(\xi_k^2+\xi_k^{\frac{2\mu m_k}{r_k}}\right).
\end{aligned}$$

Using Lemmas 1,4,5 and choosing $x_{k+1}^*=-2^{\frac{1-m_{k+1}p_k}{p_k}}$ $\cdot\left(\frac{n-k+2}{1-\gamma_k}+\frac{n-k+1}{1-\gamma_{k+1}}+g_{k,1}+g_{k,2}+g_{k,3}+n-k+1\right)^{\frac{1}{p_k}}$ $\cdot\left(\xi_k+\xi_k^{\frac{m_k r_{k+1}}{r_k m_{k+1}}}\right)^{m_{k+1}}\triangleq-\beta_k\left(\xi_k+\xi_k^{\frac{m_k r_{k+1}}{r_k m_{k+1}}}\right)^{m_{k+1}}$ lead to

$$\begin{aligned}
&\dot{V}_k(t,\bar{\xi}_{k,t}) \\
&\le -L(n-k+1)\sum_{j=1}^{k}\Bigg(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}+\xi_j^2(t-\tau_j(t))
\end{aligned}$$

$$+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t))+\xi_j^2(t-\tau_{j+1}(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big) +L\Big(\xi_k^{2-m_{k+1}p_k}+\xi_k^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\Big)(x_{k+1}^{p_k}-x_{k+1}^{*p_k}).$$

APPENDIX B
PROOF OF PROPOSITION 2

By (26), then

$$|z_i+l_{i-1}x_{i-1}|\le \min\Big\{|x_i|^{\frac{m_{i-1}}{m_i}},|x_i|^{\frac{r_{i-1}}{r_i}}\Big\}. \tag{59}$$

Using (20), (54), (55), (59), Lemmas 1,4,5, one leads to

$$\begin{aligned}
&-l_{i-1}\Big(e_i^{\frac{2-m_ip_{i-1}}{m_{i-1}}}+(z_i+l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}} -(\hat z_i+l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\Big)f_{i-1}\\
&\le \alpha l_{i-1}L^{1-\nu}\Big(|e_i|^{\frac{2-m_ip_{i-1}}{m_{i-1}}}+|e_i|^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}} +|\xi_i|^{\frac{(2\mu-r_ip_{i-1})m_i}{r_i}}+|\xi_{i-1}|^{\frac{(2\mu-r_ip_{i-1})m_i}{r_i}} +|\xi_{i-1}|^{\frac{(2\mu-r_ip_{i-1})m_{i-1}}{r_{i-1}}}\Big)\cdot\sum_{j=1}^{i-1}\Big(|\xi_j|^{m_ip_{i-1}}+|\xi_j|^{\frac{m_jr_ip_{i-1}}{r_j}}\\
&\quad +|\xi_j(t-\tau_j(t))|^{m_ip_{i-1}}+|\xi_j(t-\tau_j(t))|^{\frac{m_jr_ip_{i-1}}{r_j}} +|\xi_j(t-\tau_{j+1}(t))|^{m_ip_{i-1}}+|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_jr_ip_{i-1}}{r_j}}\Big)\\
&\le c_1h_{i-1,i-1}L^{1-\nu}\Big(e_i^{\frac{2}{m_{i-1}}}+e_i^{\frac{2\mu}{r_{i-1}}}+\sum_{j=1}^{i}\Big(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\Big)\Big) +\frac{L}{3n}\sum_{j=1}^{i}\Big(\xi_j^2(t-\tau_j(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t)) +\xi_j^2(t-\tau_{j+1}(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big).
\end{aligned}$$

APPENDIX C
PROOF OF PROPOSITION 3

By (20), (54), (55), (59), Lemmas 1,4,5, one knows that for $i=2,\cdots,n-1$,

$$\begin{aligned}
&\Big(e_i^{\frac{2-m_ip_{i-1}}{m_{i-1}}}+\frac{2\mu-r_ip_{i-1}}{r_{i-1}}e_i(z_i+l_{i-1}x_{i-})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}-1}\Big) \mathcal{I}_{1,i}^{-1}p_{i-1}x_i^{p_{i-1}-1}(Lx_{i+1}^{p_i}+f_i)(z_i+l_{i-1}x_{i-1})^{-\frac{\omega_1}{m_{i-1}}}\\
&\le\alpha\Big(|e_i|^{\frac{2-m_ip_{i-1}}{m_{i-1}}}|z_i+l_{i-1}x_{i-1}|^{\frac{m_{i-1}-m_ip_{i-1}}{m_{i-1}}} +|e_i||z_i+l_{i-1}x_{i-1}|^{\frac{2\mu-2r_ip_{i-1}}{r_{i-1}}}\Big)\Big(|\xi_i|^{m_i(p_{i-1}-1)} +|\xi_{i-1}|^{m_i(p_{i-1}-1)}+|\xi_{i-1}|^{\frac{m_{i-1}r_i(p_{i-1}-1)}{r_{i-1}}}\Big)\\
&\quad\cdot\sum_{j=1}^{i+1}\Big(|\xi_j|^{m_{i+1}p_i}+|\xi_j|^{\frac{m_jr_{i+1}p_i}{r_j}}+|\xi_j(t-\tau_j(t))|^{m_{i+1}p_i} +|\xi_j(t-\tau_j(t))|^{r_{i+1}p_i}+|\xi_j(t-\tau_{j+1}(t))|^{m_{i+1}p_i} +|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_jr_{i+1}p_i}{r_j}}\Big)\\
&\le\alpha L\Big(|e_i|^{\frac{2-m_ip_{i-1}}{m_{i-1}}}\Big(|\xi_i|^{m_{i-1}-m_ip_{i-1}}+|\xi_{i-1}|^{m_{i-1}-m_ip_{i-1}} +|\xi_{i-1}|^{\frac{m_{i-1}r_i(m_{i-1}-m_ip_{i-1})}{r_{i-1}m_i}}\Big)+|e_i|\Big(|\xi_i|^{\frac{(2\mu-2r_ip_{i-1})m_i}{r_i}} +|\xi_{i-1}|^{\frac{(2\mu-2r_ip_{i-1})m_i}{r_i}}+|\xi_{i-1}|^{\frac{(2\mu-2r_ip_{i-1})m_{i-1}}{r_{i-1}}}\Big)\Big)\\
&\quad\sum_{j=1}^{i+1}\Big(|\xi_j|^{m_ip_{i-1}+\omega_1}+|\xi_j|^{\frac{m_j(r_ip_{i-1}+\omega_2)}{r_j}} +|\xi_j(t-\tau_j(t))|^{m_ip_{i-1}+\omega_1}+|\xi_j(t-\tau_j(t))|^{\frac{m_j(r_ip_{i-1}+\omega_2)}{r_j}} +|\xi_j(t-\tau_{j+1}(t))|^{m_ip_{i-1}+\omega_1} +|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_j(r_ip_{i-1}+\omega_2)}{r_j}}\Big)\\
&\le Lc_2\Big(e_i^{\frac{2}{m_{i-1}}}+e_i^{\frac{2\mu}{r_{i-1}}}\Big)+\frac{L}{8n}\sum_{j=1}^{i+1}\Big(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\Big) +\frac{L}{3n}\sum_{j=1}^{i+1}\Big(\xi_j^2(t-\tau_j(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t)) +\xi_j^2(t-\tau_{j+1}(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big).
\end{aligned}$$

APPENDIX D
PROOF OF PROPOSITION 4

Firstly, by Lemma 2,

$$\begin{aligned}
&-l_{i-1}L\Big(e_i^{\frac{2-m_ip_{i-1}}{m_{i-1}}}+(z_i+l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}} -(\hat z_i+l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\Big)\Big((z_i+l_{i-1}x_{i-1})^{\frac{r_ip_{i-1}}{r_{i-1}}} -(\hat z_i+l_{i-1}x_{i-1})^{\frac{r_ip_{i-1}}{r_{i-1}}}\Big)\\
&\le -l_{i-1}L\Big(|e_i|^{\frac{2-m_ip_{i-1}}{m_{i-1}}}+2^{1-\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}|e_i|^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\Big) \cdot 2^{1-\frac{r_ip_{i-1}}{r_{i-1}}}|e_i|^{\frac{r_ip_{i-1}}{r_{i-1}}}\\
&\le -2^{2-\frac{2\mu}{r_{i-1}}}l_{i-1}Le_i^{\frac{2\mu}{r_{i-1}}}.
\end{aligned}\tag{60}$$

Secondly, from the definition of e_i,

$$\begin{aligned}
&-l_{i-1}L\Big(e_i^{\frac{2-m_ip_{i-1}}{m_{i-1}}}+(z_i+l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}} -(\hat z_i+l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\Big)\Big((z_i+l_{i-1}x_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}} -(\hat z_i+l_{i-1}x_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}}\Big)\\
&\le -l_{i-1}Le_i^{\frac{2-m_ip_{i-1}}{m_{i-1}}}\Big((z_i+l_{i-1}x_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}} -(z_i+l_{i-1}x_{i-1}-e_i)^{\frac{m_ip_{i-1}}{m_{i-1}}}\Big).
\end{aligned}\tag{61}$$

When $e_i\neq 0$, by Lemma 6, letting $x=\frac{z_i+l_{i-1}x_{i-1}}{e_i}$, $\varepsilon=l_{i-1}^{\frac{-m_{i-1}}{m_{i-1}+m_{i-1}p_{i-1}}}$ and $r=\frac{m_ip_{i-1}}{m_{i-1}}$, we have

$$\begin{aligned}
&-l_{i-1}Le_i^{\frac{2-m_ip_{i-1}}{m_{i-1}}}\Big((z_i+l_{i-1}x_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}} -(z_i+l_{i-1}x_{i-1}-e_i)^{\frac{m_ip_{i-1}}{m_{i-1}}}\Big)\\
&\le -L\Big(2^{\frac{m_ip_{i-1}}{m_{i-1}}}-1\Big)l_{i-1}^{\frac{2m_ip_{i-1}}{m_{i-1}+m_ip_{i-1}}}e_i^{\frac{2}{m_{i-1}}} +Ll_{i-1}^{\frac{\omega_1}{m_{i-1}+m_ip_{i-1}}}(z_i+l_{i-1}x_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}+1}e_i^{\frac{2-m_ip_{i-1}}{m_{i-1}}-1}
\end{aligned}\tag{62}$$

When $e_i = 0$, (62) holds automatically. By (25), (54), (59), Lemmas 1,4, one gets

$$\begin{aligned}
&L l_{i-1}^{\frac{\omega_1}{m_{i-1}+m_i p_{i-1}}} (z_i + l_{i-1}x_{i-1})^{\frac{m_i p_{i-1}}{m_{i-1}}+1} e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}-1}\\
&\le L\alpha\bigg(|\xi_i|^{m_i p_{i-1}+m_{i-1}} + |\xi_{i-1}|^{m_i p_{i-1}+m_{i-1}}\\
&\quad +|\xi_{i-1}|^{\frac{r_i m_{i-1}(m_i p_{i-1}+m_{i-1})}{r_{i-1} m_i}}\bigg)|e_i|^{\frac{2-m_i p_{i-1}}{m_{i-1}}-1}\\
&\le \frac{L}{8n}\sum_{j=i-1}^{i}\Big(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\Big)+Lc_3\Big(e_i^{\frac{2}{m_{i-1}}}+e_i^{\frac{2\mu}{r_{i-1}}}\Big). \qquad (63)
\end{aligned}$$

Combining (61), (62) and (63), we have

$$\begin{aligned}
&-l_{i-1}L\bigg(e_i^{\frac{2-m_i p_{i-1}}{m_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}}\\
&-(\hat{z}_i + l_{i-1}x_{i-1})^{\frac{2\mu - r_i p_{i-1}}{r_{i-1}}}\bigg)\bigg((z_i + l_{i-1}x_{i-1})^{\frac{m_i p_{i-1}}{m_{i-1}}}\\
&-(\hat{z}_i + l_{i-1}x_{i-1})^{\frac{m_i p_{i-1}}{m_{i-1}}}\bigg)\\
&\le -L\Big(2^{\frac{m_i p_{i-1}}{m_{i-1}}}-1\Big) l_{i-1}^{\frac{2m_i p_{i-1}}{m_{i-1}+m_i p_{i-1}}} e_i^{\frac{2}{m_{i-1}}}\\
&\quad +\frac{L}{8n}\sum_{j=i-1}^{i}\Big(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\Big)+Lc_3\Big(e_i^{\frac{2}{m_{i-1}}}+e_i^{\frac{2\mu}{r_{i-1}}}\Big). \qquad (64)
\end{aligned}$$

Put (60), (64) together and notice $2^{2-\frac{2\mu}{r_{i-1}}}\big(2^{\frac{m_i p_{i-1}}{m_{i-1}}}-1\big) l_{i-1}^{\frac{2m_i p_{i-1}}{m_{i-1}+m_i p_{i-1}}} \le 2^{2-\frac{2\mu}{r_{i-1}}}$, $2^{2-\frac{2\mu}{r_{i-1}}}\big(2^{\frac{m_i p_{i-1}}{m_{i-1}}}-1\big)\cdot l_{i-1}^{\frac{2m_i p_{i-1}}{m_{i-1}+m_i p_{i-1}}} \le \big(2^{\frac{m_i p_{i-1}}{m_{i-1}}}-1\big) l_{i-1}^{\frac{2m_i p_{i-1}}{m_{i-1}+m_i p_{i-1}}}$, Proposition 4 holds.

APPENDIX E
PROOF OF PROPOSITION 5

By the inductive argument, we firstly claim that for $i = 2, \cdots, n$,

$$\begin{aligned}
&|x_i - \hat{x}_i|\\
&\le \alpha|e_i|^{\frac{m_i}{m_{i-1}}}\bigg(1+|e_i|^{\frac{r_i}{r_{i-1}}-\frac{m_i}{m_{i-1}}}+\sum_{j=i-1}^{i}\bigg(1\\
&\quad +|\xi_j|^{\frac{m_j r_{i-1}}{r_j}\left(\frac{r_i}{r_{i-1}}-\frac{m_i}{m_{i-1}}\right)}\bigg)\bigg)+\sum_{j=2}^{i-1} h_{i-1,j}|e_j|^{\frac{m_i}{m_{j-1}}}\\
&\quad \cdot\bigg(\sum_{l=2}^{i-1} h_{i-1,l}\Big(1+|e_l|^{\frac{r_{j-1}}{r_{l-1}}\left(\frac{r_i}{r_{j-1}}-\frac{m_i}{m_{j-1}}\right)}\Big)\\
&\quad +\sum_{l=1}^{i}\Big(1+|\xi_l|^{\frac{m_l r_{j-1}}{r_l}\left(\frac{r_i}{r_{j-1}}-\frac{m_i}{m_{j-1}}\right)}\Big)\bigg). \qquad (65)
\end{aligned}$$

Using (26), (54), (59), Lemmas 1-5, one has

$$\begin{aligned}
&|x_2 - \hat{x}_2|\\
&=\bigg|\Big((z_2+l_1x_1)^{\frac{r_2p_1}{r_1}}+(z_2+l_1x_1)^{\frac{m_2p_1}{m_1}}\Big)^{\frac{1}{p_1}}\\
&\quad -\Big((\hat{z}_2+l_1x_1)^{\frac{r_2p_1}{r_1}}+(\hat{z}_2+l_1x_1)^{\frac{m_2p_1}{m_1}}\Big)^{\frac{1}{p_1}}\bigg|\\
&\le \alpha\bigg|(z_2+l_1x_1)^{\frac{r_2p_1}{r_1}}-(\hat{z}_2+l_1x_1)^{\frac{r_2p_1}{r_1}}\\
&\quad +(z_2+l_1x_1)^{\frac{m_2p_1}{m_1}}-(\hat{z}_2+l_1x_1)^{\frac{m_2p_1}{m_1}}\bigg|^{\frac{1}{p_1}}\\
&\le \alpha|e_2|^{\frac{m_2}{m_1}}\Big(1+|e_2|^{\frac{r_2}{r_1}-\frac{m_2}{m_1}}+|x_2|^{\frac{r_1}{r_2}\left(\frac{r_2}{r_1}-\frac{m_2}{m_1}\right)}\Big)\\
&\le \alpha|e_2|^{\frac{m_2}{m_1}}\bigg(1+|e_2|^{\frac{r_2}{r_1}-\frac{m_2}{m_1}}\\
&\quad +\sum_{j=1}^{2}\Big(1+|\xi_j|^{\frac{m_j r_1}{r_j}\left(\frac{r_2}{r_1}-\frac{m_2}{m_1}\right)}\Big)\bigg),
\end{aligned}$$

which implies that (65) is true for $i = 2$.

Suppose that (65) holds for $i = k-1$. Then at step $i = k$, by (26), (54), (59), Lemmas 1-5, it can be deduced that

$$\begin{aligned}
&|x_k - \hat{x}_k|\\
&=\bigg|\Big((z_k+l_{k-1}x_{k-1})^{\frac{r_kp_{k-1}}{r_{k-1}}}+(z_k+l_{k-1}x_{k-1})^{\frac{m_kp_{k-1}}{m_{k-1}}}\Big)^{\frac{1}{p_{k-1}}}\\
&\quad -\Big((\hat{z}_k+l_{k-1}\hat{x}_{k-1})^{\frac{r_kp_{k-1}}{r_{k-1}}}+(\hat{z}_k+l_{k-1}\hat{x}_{k-1})^{\frac{m_kp_{k-1}}{m_{k-1}}}\Big)^{\frac{1}{p_{k-1}}}\bigg|\\
&\le \alpha|e_k|^{\frac{m_k}{m_{k-1}}}\Big(1+|e_k|^{\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}}+|z_k+\\
&\quad l_{k-1}x_{k-1}|^{\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}}\Big)+\alpha|l_{k-1}(x_{k-1}-\hat{x}_{k-1})|^{\frac{m_k}{m_{k-1}}}\\
&\quad \cdot\Big(1+|e_k|^{\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}}+|z_k+l_{k-1}x_{k-1}|^{\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}}\\
&\quad +|l_{k-1}(x_{k-1}-\hat{x}_{k-1})|^{\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}}\Big)\\
&\le \alpha|e_k|^{\frac{m_k}{m_{k-1}}}\bigg(1+|e_k|^{\frac{r_k}{r_{k-1}}\frac{m_k}{m_{k-1}}}+|\xi_k|^{\frac{m_k r_{k-1}}{r_k}\left(\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}\right)}\\
&\quad +|\xi_{k-1}|^{m_{k-1}\left(\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}\right)}\bigg)+\sum_{j=2}^{k-1}h_{k-1,j}|e_j|^{\frac{m_k}{m_{j-1}}}\\
&\quad \cdot\bigg(\sum_{j=l}^{k-1}h_{k-1,l}\Big(1+|e_l|^{\frac{r_{j-1}m_k}{r_{l-1}m_{k-1}}\left(\frac{r_{k-1}}{r_{j-1}}-\frac{m_{k-1}}{m_{j-1}}\right)}\Big)\\
&\quad +\sum_{l=1}^{k-1}\Big(1+|\xi_l|^{\frac{m_l r_{j-1}m_k}{r_l m_{k-1}}\left(\frac{r_{k-1}}{r_{j-1}}-\frac{m_{k-1}}{m_{j-1}}\right)}\Big)\bigg)\\
&\quad \cdot\bigg(\sum_{l=2}^{k-1}h_{k-1,l}\Big(1+|e_l|^{\frac{r_{k-1}}{r_{l-1}}\left(\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}\right)}\Big)\\
&\quad +\sum_{l=1}^{k}\Big(1+|\xi_l|^{\frac{m_l r_{k-1}}{r_l}\left(\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}\right)}\Big)\bigg)\\
&\le \alpha|e_k|^{\frac{m_k}{m_{k-1}}}\bigg(1+|e_k|^{\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}}\\
&\quad +\sum_{j=k-1}^{k}\Big(1+|\xi_j|^{\frac{m_j r_{k-1}}{r_j}\left(\frac{r_k}{r_{k-1}}-\frac{m_k}{m_{k-1}}\right)}\Big)\bigg)+\sum_{j=2}^{k-1}h_{k-1,j}|e_j|^{\frac{m_k}{m_{j-1}}}\\
&\quad \cdot\bigg(\sum_{l=2}^{k-1}h_{k-1,l}\Big(1+|e_l|^{\frac{r_{j-1}}{r_{l-1}}\left(\frac{r_k}{r_{j-1}}-\frac{m_k}{m_{j-1}}\right)}\Big)\\
&\quad +\sum_{l=1}^{k}\Big(1+|\xi_l|^{\frac{m_l r_{j-1}}{r_l}\left(\frac{r_k}{r_{j-1}}-\frac{m_k}{m_{j-1}}\right)}\Big)\bigg). \qquad (66)
\end{aligned}$$

Clearly, (66) implies that (65) holds for $i = k$.

Using (54), (66) and Lemmas 1−5, we have

$$\begin{aligned}
&-l_{i-1}L\bigg(e_i^{\frac{2-m_ip_{i-1}}{m_{i-1}}}+(z_i+l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\\
&-(\hat{z}_i+l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\bigg)\mathcal{I}_{3,i}
\end{aligned}$$

$$
\begin{aligned}
&\le l_{i-1}L|l_{i-1}(x_{i-1}-\hat{x}_{i-1})|^{\frac{m_ip_{i-1}}{m_{i-1}}}\\
&\quad\cdot\Big(1+|l_{i-1}(x_{i-1}-\hat{x}_{i-1})|^{\frac{r_ip_{i-1}}{r_{i-1}}-\frac{m_ip_{i-1}}{m_{i-1}}}\\
&\quad+|e_i|^{\frac{r_ip_{i-1}}{r_{i-1}}-\frac{m_ip_{i-1}}{m_{i-1}}}+|\xi_i|^{\frac{m_ir_{i-1}}{r_i}\left(\frac{r_ip_{i-1}}{r_{i-1}}-\frac{m_ip_{i-1}}{m_{i-1}}\right)}\\
&\quad+|\xi_{i-1}|^{m_{i-1}\left(\frac{r_ip_{i-1}}{r_{i-1}}-\frac{m_ip_{i-1}}{m_{i-1}}\right)}\Big)\\
&\quad\cdot\Big(|e_i|^{\frac{2-m_ip_{i-1}}{m_{i-1}}}+|e_i|^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\\
&\quad+\sum_{j=i-1}^{i}\Big(|\xi_j|^{2-m_ip_{i-1}}+|\xi_j|^{\frac{(2\mu-r_ip_{i-1})m_j}{r_j}}\Big)\Big)\\
&\le Lc_4\left(e_i^{\frac{2}{m_{i-1}}}+e_i^{\frac{2\mu}{r_{i-1}}}\right)+\frac{L}{8n}\sum_{j=1}^{i}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right)\\
&\quad+L\sum_{j=2}^{i-1}h_{i-1,j}\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right).
\end{aligned}
$$

APPENDIX F
PROOF OF PROPOSITION 6

We claim that for $i=2,\cdots,n+1$,

$$
\begin{aligned}
|\hat{x}_i^*|^{\frac{1}{m_i}} \le\ & \alpha\sum_{j=1}^{i-1}\left(|\xi_j|+|\xi_j|^{\frac{m_jr_i}{r_jm_i}}\right)\\
&+\alpha\left(|e_{i-1}|^{\frac{1}{m_{i-2}}}+|e_{i-1}|^{\frac{r_i}{m_ir_{i-2}}}\right)\\
&+\sum_{j=2}^{i-2}h_{i-2,j}\left(|e_j|^{\frac{1}{m_{j-1}}}+|e_j|^{\frac{r_i}{m_ir_{j-1}}}\right). \quad (67)
\end{aligned}
$$

Obviously, (67) holds for $i=2$ due to

$$
|\hat{x}_2^*|^{\frac{1}{m_2}}=\left|-\beta_1^{\frac{1}{m_2}}\left(\hat{\xi}_1+\hat{\xi}_1^{\frac{m_1r_2}{r_1m_2}}\right)\right|\le\alpha\left(|\xi_1|+|\xi_1|^{\frac{m_1r_2}{r_1m_2}}\right).
$$

Using (20), (28), (59), (65) and Lemmas 1−5, it is easy to get

$$
\begin{aligned}
|\hat{x}_{k-1}|^{\frac{1}{m_{k-1}}} &=\Big|(\hat{z}_{k-1}+l_{k-2}\hat{x}_{k-2})^{\frac{r_{k-1}p_{k-2}}{r_{k-2}}}\\
&\quad+(\hat{z}_{k-1}+l_{k-2}\hat{x}_{k-2})^{\frac{m_{k-1}p_{k-2}}{m_{k-2}}}\Big|^{\frac{1}{m_{k-1}p_{k-2}}}\\
&\le\Big(\Big|(\hat{z}_{k-1}+l_{k-2}\hat{x}_{k-2})^{\frac{m_{k-1}p_{k-2}}{m_{k-2}}}\\
&\quad-(z_{k-1}+l_{k-2}x_{k-2})^{\frac{m_{k-1}p_{k-2}}{m_{k-2}}}\Big|\\
&\quad+\Big|(\hat{z}_{k-1}+l_{k-2}\hat{x}_{k-2})^{\frac{r_{k-1}p_{k-2}}{r_{k-2}}}\\
&\quad-(z_{k-1}+l_{k-2}x_{k-2})^{\frac{r_{k-1}p_{k-2}}{r_{k-2}}}\Big|\\
&\quad+|z_{k-1}+l_{k-2}x_{k-2}|^{\frac{m_{k-1}p_{k-2}}{m_{k-2}}}\\
&\quad+|z_{k-1}+l_{k-2}x_{k-2}|^{\frac{r_{k-1}p_{k-2}}{r_{k-2}}}\Big)^{\frac{1}{m_{k-1}p_{k-2}}}\\
&\le\alpha\sum_{j=1}^{k-1}\left(|\xi_j|+|\xi_j|^{\frac{m_jr_{k-1}}{r_jm_{k-1}}}\right)\\
&\quad+\alpha\left(|e_{k-1}|^{\frac{1}{m_{k-2}}}+|e_{k-1}|^{\frac{r_{k-1}}{m_{k-1}r_{k-2}}}\right)\\
&\quad+\sum_{j=2}^{k-2}h_{k-2,j}\left(|e_j|^{\frac{1}{m_{j-1}}}+|e_j|^{\frac{r_{k-1}}{m_{k-1}r_{j-1}}}\right). \quad (68)
\end{aligned}
$$

Suppose (67) holds for $i=k-1$. At step $i=k$, by (67), (68), Lemmas 1,5, it can be shown that

$$
\begin{aligned}
|\hat{x}_k^*|^{\frac{1}{m_k}} &=\Big|-\beta_{k-1}^{\frac{1}{m_k}}\Big(\Big(\hat{x}_{k-1}^{\frac{1}{m_{k-1}}}-\hat{x}_{k-1}^{*\frac{1}{m_{k-1}}}\Big)\\
&\quad+\Big(\hat{x}_{k-1}^{\frac{1}{m_{k-1}}}-\hat{x}_{k-1}^{*\frac{1}{m_{k-1}}}\Big)^{\frac{m_{k-1}r_k}{r_{k-1}m_k}}\Big)\Big|\\
&\le\alpha\Big(|\hat{x}_{k-1}|^{\frac{1}{m_{k-1}}}+|\hat{x}_{k-1}|^{\frac{r_k}{r_{k-1}m_k}}\\
&\quad+\sum_{j=1}^{k-1}\Big(|\xi_j|+|\xi_j|^{\frac{m_jr_k}{r_jm_k}}\Big)+|e_{k-2}|^{\frac{1}{m_{k-3}}}+|e_{k-2}|^{\frac{r_k}{m_kr_{k-3}}}\\
&\quad+\sum_{j=2}^{k-3}h_{k-3,j}\Big(|e_j|^{\frac{1}{m_{j-1}}}+|e_j|^{\frac{r_k}{m_kr_{j-1}}}\Big)\Big)\\
&\le\alpha\sum_{j=1}^{k-1}\left(|\xi_j|+|\xi_j|^{\frac{m_jr_k}{r_jm_k}}\right)\\
&\quad+\alpha\left(|e_{k-1}|^{\frac{1}{m_{k-2}}}+|e_{k-1}|^{\frac{r_k}{m_kr_{k-2}}}\right)\\
&\quad+\sum_{j=2}^{k-2}h_{i-2,j}\left(|e_j|^{\frac{1}{m_{j-1}}}+|e_j|^{\frac{r_k}{m_kr_{j-1}}}\right),
\end{aligned}
$$

which implies that (67) holds for $i=k$.

Using (29), (67) and (68), we have

$$
\begin{aligned}
v^{p_n} &=-\beta_n^{\frac{1}{m_{n+1}}}\left(\hat{\xi}_n+\hat{\xi}_n^{\frac{m_nr_{n+1}}{r_nm_{n+1}}}\right)^{m_{n+1}p_n}\\
&=-\beta_n^{\frac{1}{m_{n+1}}}\Big(\Big(\hat{x}_n^{\frac{1}{m_n}}-\hat{x}_n^{*\frac{1}{m_n}}\Big)\\
&\quad+\Big(\hat{x}_n^{\frac{1}{m_n}}-\hat{x}_n^{*\frac{1}{m_n}}\Big)^{\frac{m_nr_{n+1}}{r_nm_{n+1}}}\Big)^{m_{n+1}p_n}\\
&\le\alpha\Big(\Big(|\hat{x}_n|^{\frac{1}{m_n}}+|\hat{x}_n|^{\frac{r_{n+1}}{r_nm_{n+1}}}\Big)^{m_{n+1}p_n}\\
&\quad+|\hat{x}_n^*|^{\frac{m_{n+1}p_n}{m_n}}+|\hat{x}_n^*|^{\frac{r_{n+1}p_n}{r_n}}\Big)\\
&\le\alpha\sum_{j=1}^{n}\left(|\xi_j|^{m_{n+1}p_n}+|\xi_j|^{\frac{m_jr_{n+1}p_n}{r_j}}\right)\\
&\quad+\alpha\left(|e_n|^{\frac{m_{n+1}p_n}{m_{n-1}}}+|e_n|^{\frac{r_{n+1}p_n}{r_{n-1}}}\right)\\
&\quad+\sum_{j=2}^{n-1}h_{n-1,j}\left(|e_j|^{\frac{m_{n+1}p_n}{m_{j-1}}}+|e_j|^{\frac{r_{n+1}p_n}{r_{j-1}}}\right). \quad (69)
\end{aligned}
$$

With the aid of (25), (55), (59), (69) and Lemmas 1,4,5,

$$
\begin{aligned}
&\left(e_n^{\frac{2-m_np_{n-1}}{m_{n-1}}}+\frac{2\mu-r_{n-1}}{r_{n-1}}e_n(z_n+l_{n-1}x_{n-1})^{\frac{2\mu-r_{n-1}}{r_{n-1}}-1}\right)\\
&\mathcal{I}_{1,n}^{-1}p_{n-1}x_n^{p_{n-1}-1}(Lv^{p_n}(\hat{x})+f_n)(z_n+l_{n-1}x_{n-1})^{-\frac{\omega_1}{m_{n-1}}}\\
&\le\alpha L\Big(|e_n|^{\frac{2-m_np_{n-1}}{m_{n-1}}}\Big(|\xi_n|^{m_{n-1}-m_np_{n-1}}+|\xi_{n-1}|^{m_{n-1}-m_np_{n-1}}\\
&\quad+|\xi_{n-1}|^{\frac{m_{n-1}r_n(m_{n-1}-m_np_{n-1})}{r_{n-1}m_i}}\Big)+|e_n|\Big(|\xi_n|^{\frac{(2\mu-2r_np_{n-1})m_n}{r_n}}\\
&\quad+|\xi_{n-1}|^{\frac{(2\mu-2r_np_{n-1})m_n}{r_n}}+|\xi_{n-1}|^{\frac{(2\mu-2r_np_{n-1})m_{n-1}}{r_{n-1}}}\Big)\Big)\\
&\quad\cdot\Big(|\xi_n|^{m_n(p_{n-1}-1)}+|\xi_{n-1}|^{m_n(p_{n-1}-1)}+|\xi_{n-1}|^{\frac{m_{n-1}r_n(p_{n-1}-1)}{r_{n-1}}}\Big)\\
&\quad\cdot\Big(\alpha\sum_{j=1}^{n}\Big(|\xi_j|^{m_{n+1}p_n}+|\xi_j|^{\frac{m_jr_{n+1}p_n}{r_j}}+|\xi_j(t-\tau_j(t))|^{m_{n+1}p_n}
\end{aligned}
$$

$$
\begin{aligned}
&+|\xi_j(t-\tau_j(t))|^{\frac{m_j r_{n+1}p_n}{r_j}}+|\xi_j(t-\tau_{j+1}(t))|^{m_{n+1}p_n}\\
&+|\xi_j(t-\tau_{j+1}(t))|^{\frac{m_j r_{n+1}p_n}{r_j}}\Big)+\alpha\Big(|e_n|^{\frac{m_{n+1}p_n}{m_{n-1}}}+|e_n|^{\frac{r_{n+1}p_n}{r_{n-1}}}\Big)\\
&+\sum_{j=2}^{n-1}h_{n-1,j}\Big(|e_j|^{\frac{m_{n+1}p_n}{m_{j-1}}}+|e_j|^{\frac{r_{n+1}p_n}{r_{j-1}}}\Big)\Big)\\
\le{}& Lc_5\Big(e_n^{\frac{2}{m_{n-1}}}+e_n^{\frac{2\mu}{r_{n-1}}}\Big)+\frac{L}{8}\sum_{j=1}^{n}\Big(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\Big)\\
&+\frac{L}{3}\sum_{j=1}^{n}\Big(\xi_j^2(t-\tau_j(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t))\\
&+\xi_j^2(t-\tau_{j+1}(t))+\xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big)\\
&+L\sum_{j=2}^{n-1}h_{n-1,j}\Big(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\Big).
\end{aligned}
$$

APPENDIX G
PROOF OF PROPOSITION 7

We firstly prove that for $i=2,\cdots,n$,

$$
\begin{aligned}
&|\xi_i-\hat{\xi}_i|\\
\le{}&\sum_{j=2}^{i}\Big[\alpha|e_j|^{\frac{m_j}{m_{j-1}}}\Big(|e_j|^{\frac{1}{m_{j-1}}-\frac{m_j}{m_{j-1}}}+|e_j|^{\frac{r_i}{r_{j-1}m_i}-\frac{m_j}{m_{j-1}}}\\
&+\sum_{l=2}^{j-1}h_{j-1,l}\Big(|e_l|^{\frac{m_{j-1}}{m_{l-1}}\left(\frac{1}{m_{j-1}}-\frac{m_j}{m_{j-1}}\right)}\\
&+|e_l|^{\frac{r_{j-1}}{r_{l-1}}\left(\frac{r_i}{r_{j-1}m_i}-\frac{m_j}{m_{j-1}}\right)}\Big)+\sum_{l=1}^{j}\Big(|\xi_l|^{m_{j-1}\left(\frac{1}{m_{j-1}}-\frac{m_j}{m_{j-1}}\right)}\\
&+|\xi_l|^{\frac{m_l r_{j-1}}{r_l}\left(\frac{r_i}{r_{j-1}m_i}-\frac{m_j}{m_{j-1}}\right)}\Big)\Big)\\
&+\sum_{q=2}^{j-1}h_{j-1,q}|e_q|^{\frac{m_j}{m_{q-1}}}\Big(|e_j|^{\frac{m_{q-1}}{m_{j-1}}\left(\frac{1}{m_{q-1}}-\frac{m_j}{m_{q-1}}\right)}\\
&+|e_j|^{\frac{r_{q-1}}{r_{j-1}}\left(\frac{r_i}{r_{q-1}m_i}-\frac{m_j}{m_{q-1}}\right)}+\sum_{l=2}^{j-1}h_{j-1,l}\Big(|e_l|^{\frac{m_{q-1}}{m_{l-1}}\left(\frac{1}{m_{q-1}}-\frac{m_j}{m_{q-1}}\right)}\\
&+|e_l|^{\frac{r_{q-1}}{r_{l-1}}\left(\frac{r_i}{r_{q-1}m_i}-\frac{m_j}{m_{q-1}}\right)}\Big)+\sum_{l=1}^{j}\Big(|\xi_l|^{m_{q-1}\left(\frac{1}{m_{q-1}}-\frac{m_j}{m_{q-1}}\right)}\\
&+|\xi_l|^{\frac{m_l r_{q-1}}{r_l}\left(\frac{r_i}{r_{q-1}m_i}-\frac{m_j}{m_{q-1}}\right)}\Big)\Big)\Big)\Big].
\end{aligned}
\tag{70}
$$

By (20), (26), (28), (29), (54), (59), and Lemmas 1-5, we have

$$
\begin{aligned}
&|\xi_2-\hat{\xi}_2|\\
\le{}&\alpha|x_2-\hat{x}_2|\Big(|x_2-\hat{x}_2|^{\frac{1}{m_2}-1}+|x_2|^{\frac{1}{m_2}-1}\Big)\\
\le{}&\alpha\Big(\Big(\Big|(z_2+l_1x_1)^{\frac{m_2p_1}{m_1}}-(\hat{z}_2+l_1x_1)^{\frac{m_2p_1}{m_1}}\Big|\\
&+\Big|(z_2+l_1x_1)^{\frac{r_2p_1}{r_1}}-(\hat{z}_2+l_1x_1)^{\frac{r_2p_1}{r_1}}\Big|\Big)^{\frac{1}{p_1}}\\
&\cdot\Big(\Big(\Big|(z_2+l_1x_1)^{\frac{m_2p_1}{m_1}}-(\hat{z}_2+l_1x_1)^{\frac{m_2p_1}{m_1}}\Big|\\
&+\Big|(z_2+l_1x_1)^{\frac{r_2p_1}{r_1}}-(\hat{z}_2+l_1x_1)^{\frac{r_2p_1}{r_1}}\Big|\Big)^{\frac{1}{m_2p_1}-\frac{1}{p_1}}\\
&+|\xi_2|^{1-m_2}+|\xi_1|^{1-m_2}+|\xi_1|^{\frac{m_1r_2(1-m_2)}{r_1m_2}}\Big)\\
\le{}&\alpha|e_2|^{\frac{m_2}{m_1}}\Big(1+|e_2|^{\frac{r_2}{r_1}-\frac{m_2}{m_1}}+|\xi_2|^{\frac{m_2r_1}{r_2}\left(\frac{r_2}{r_1}-\frac{m_2}{m_1}\right)}\\
&+|\xi_1|^{m_1\left(\frac{r_2}{r_1}-\frac{m_2}{m_1}\right)}\Big)\Big(|e_2|^{\frac{1-m_2}{m_1}}+|e_2|^{\frac{r_2(1-m_2)}{m_2r_1}}\\
&+|\xi_2|^{1-m_2}+|\xi_1|^{\frac{m_1r_2(1-r_2)}{r_1m_2}}\Big)\\
\le{}&\alpha|e_2|^{\frac{m_2}{m_1}}\Big(|e_2|^{\frac{1}{m_1}-\frac{m_2}{m_1}}+|e_2|^{\frac{r_2}{m_2r_1}-\frac{m_2}{m_1}}\\
&+\sum_{l=1}^{2}\Big(|\xi_l|^{m_1\left(\frac{1}{m_1}-\frac{m_2}{m_1}\right)}+|\xi_l|^{\frac{m_lr_1}{r_l}\left(\frac{r_2}{m_2r_1}-\frac{m_2}{m_1}\right)}\Big)\Big),
\end{aligned}
$$

which implies that (70) is true for $i=2$.

Suppose that (70) holds for $i=k-1$, by (25), Lemma 3, then

$$
\begin{aligned}
&|\xi_k-\hat{\xi}_k|\\
\le{}&\Big|x_k^{\frac{1}{m_k}}-x_k^{*\frac{1}{m_k}}-\Big(\hat{x}_k^{\frac{1}{m_k}}-\hat{x}_k^{*\frac{1}{m_k}}\Big)\Big|\\
\le{}&\alpha|x_k-\hat{x}_k|\Big(|x_k-\hat{x}_k|^{\frac{1}{m_k}-1}+|x_k|^{\frac{1}{m_k}-1}\Big)+\alpha|\xi_{k-1}-\hat{\xi}_{k-1}|\\
&\cdot\Big(1+|\xi_{k-1}-\hat{\xi}_{k-1}|^{\frac{m_{k-1}r_k}{r_{k-1}m_k}-1}+|\xi_{k-1}|^{\frac{m_{k-1}r_k}{r_{k-1}m_k}-1}\Big).
\end{aligned}
\tag{71}
$$

From (26), (28), (54), (59), (65), and Lemmas 1,4,5, it follows that

$$
\begin{aligned}
&\alpha|x_k-\hat{x}_k|\Big(|x_k-\hat{x}_k|^{\frac{1}{m_k}-1}+|x_k|^{\frac{1}{m_k}-1}\Big)\\
\le{}&\alpha|e_k|^{\frac{m_k}{m_{k-1}}}\Big(|e_k|^{\frac{1}{m_{k-1}}-\frac{m_k}{m_{k-1}}}+|e_k|^{\frac{r_k}{r_{k-1}m_k}-\frac{m_k}{m_{k-1}}}\\
&+\sum_{l=2}^{k-1}h_{k-1,l}\Big(|e_l|^{\frac{m_{k-1}}{m_{l-1}}\left(\frac{1}{m_{k-1}}-\frac{m_k}{m_{k-1}}\right)}\\
&+|e_l|^{\frac{r_{k-1}}{r_{l-1}}\left(\frac{r_k}{r_{k-1}m_k}-\frac{m_k}{m_{k-1}}\right)}\Big)+\sum_{l=1}^{k}\Big(|\xi_l|^{m_{k-1}\left(\frac{1}{m_{k-1}}-\frac{m_k}{m_{k-1}}\right)}\\
&+|\xi_l|^{\frac{m_lr_{k-1}}{r_l}\left(\frac{r_k}{r_{k-1}m_k}-\frac{m_k}{m_{k-1}}\right)}\Big)\Big)\\
&+\sum_{j=2}^{k-1}h_{k-1,j}|e_j|^{\frac{m_k}{m_{j-1}}}\Big(|e_k|^{\frac{m_{j-1}}{m_{k-1}}\left(\frac{1}{m_{j-1}}-\frac{m_k}{m_{j-1}}\right)}\\
&+|e_k|^{\frac{r_{j-1}}{r_{k-1}}\left(\frac{r_k}{r_{j-1}m_k}-\frac{m_k}{m_{j-1}}\right)}\\
&+\sum_{l=2}^{k-1}h_{k-1,l}\Big(|e_l|^{\frac{m_{j-1}}{m_{l-1}}\left(\frac{1}{m_{j-1}}-\frac{m_k}{m_{j-1}}\right)}\\
&+|e_l|^{\frac{r_{j-1}}{r_{l-1}}\left(\frac{r_k}{r_{j-1}m_k}-\frac{m_k}{m_{j-1}}\right)}\Big)+\sum_{l=1}^{k}\Big(|\xi_l|^{m_{j-1}\left(\frac{1}{m_{j-1}}-\frac{m_k}{m_{j-1}}\right)}\\
&+|\xi_l|^{\frac{m_lr_{j-1}}{r_l}\left(\frac{r_k}{r_{j-1}m_k}-\frac{m_k}{m_{j-1}}\right)}\Big)\Big).
\end{aligned}
\tag{72}
$$

By (54), (70) and Lemmas 1,4,5, we have

$$
\begin{aligned}
&\alpha|\xi_{k-1}-\hat{\xi}_{k-1}|\Big(1+|\xi_{k-1}-\hat{\xi}_{k-1}|^{\frac{m_{k-1}r_k}{r_{k-1}m_k}-1}\\
&+|\xi_{k-1}|^{\frac{m_{k-1}r_k}{r_{k-1}m_k}-1}\Big)\\
\le{}&\sum_{j=2}^{k-1}\Big[\alpha|e_j|^{\frac{m_j}{m_{j-1}}}\Big(|e_j|^{\frac{1}{m_{j-1}}-\frac{m_j}{m_{j-1}}}+|e_j|^{\frac{r_k}{r_{j-1}m_k}-\frac{m_j}{m_{j-1}}}\\
&+\sum_{l=2}^{j-1}h_{j-1,l}\Big(|e_l|^{\frac{m_{j-1}}{m_{l-1}}\left(\frac{1}{m_{j-1}}-\frac{m_j}{m_{j-1}}\right)}\\
&+|e_l|^{\frac{r_{j-1}}{r_{l-1}}\left(\frac{r_k}{r_{j-1}m_k}-\frac{m_j}{m_{j-1}}\right)}\Big)+\sum_{l=1}^{j}\Big(|\xi_l|^{m_{j-1}\left(\frac{1}{m_{j-1}}-\frac{m_j}{m_{j-1}}\right)}
\end{aligned}
$$

$$+|\xi_l|^{\frac{m_l r_{j-1}}{r_l}\left(\frac{r_k}{r_{j-1}m_k}-\frac{m_j}{m_{j-1}}\right)}\Big)\Big)$$
$$+\sum_{q=2}^{j-1}h_{j-1,q}|e_q|^{\frac{m_j}{m_{q-1}}}\Big(|e_j|^{\frac{m_{q-1}}{m_{j-1}}\left(\frac{1}{m_{q-1}}-\frac{m_j}{m_{q-1}}\right)}$$
$$+|e_j|^{\frac{r_{q-1}}{r_{j-1}}\left(\frac{r_k}{r_{q-1}m_k}-\frac{m_j}{m_{q-1}}\right)}\Big)+\sum_{l=2}^{j-1}h_{j-1,l}\Big(|e_l|^{\frac{m_{q-1}}{m_{l-1}}\left(\frac{1}{m_{q-1}}-\frac{m_j}{m_{q-1}}\right)}$$
$$+|e_l|^{\frac{r_{q-1}}{r_{l-1}}\left(\frac{r_k}{r_{q-1}m_k}-\frac{m_j}{m_{q-1}}\right)}\Big)+\sum_{l=1}^{j}\Big(|\xi_l|^{m_{q-1}\left(\frac{1}{m_{q-1}}-\frac{m_j}{m_{q-1}}\right)}$$
$$+|\xi_l|^{\frac{m_l r_{q-1}}{r_l}\left(\frac{r_k}{r_{q-1}m_k}-\frac{m_j}{m_{q-1}}\right)}\Big)\Big)\Big]. \quad (73)$$

Substituting (72), (73) into (71) yields (70) with $i = k$. Applying Lemmas 4,5 to (70), one has

$$\begin{aligned}&|\xi_n-\hat{\xi}_n|\\ &\le\alpha\left(|e_n|^{\frac{1}{m_{n-1}}}+|e_n|^{\frac{r_n}{m_n r_{n-1}}}\right)+\sum_{j=2}^{n-1}h_{n-1,j}\Big(|e_j|^{\frac{1}{m_{j-1}}}\\ &\quad+|e_j|^{\frac{r_n}{m_n r_{j-1}}}\Big)+\sum_{j=2}^{n}\left(|\xi_j|+|\xi_j|^{\frac{m_j r_n}{r_j m_n}}\right).\end{aligned} \quad (74)$$

By (54), (74), Lemmas 1,3,4,5, then

$$\begin{aligned}&L\left(\xi_n^{2-m_{n+1}p_n}+\xi_n^{\frac{(2\mu-r_{n+1}p_n)m_n}{r_n}}\right)(v^{p_n}-x_{n+1}^{*p_n})\\ &\le\alpha L\left(\xi_n^{2-m_{n+1}p_n}+\xi_n^{\frac{(2\mu-r_{n+1}p_n)m_n}{r_n}}\right)\Big(|\xi_n-\hat{\xi}_n|\\ &\quad+\left|\xi_n^{\frac{m_n r_{n+1}}{r_n m_{n+1}}}-\hat{\xi}_n^{\frac{m_n r_{n+1}}{r_n m_{n+1}}}\right|\Big)^{m_{n+1}p_n}\\ &\le\alpha L|\xi_n-\hat{\xi}_n|^{m_{n+1}p_n}\left(|\xi_n|^{2-m_{n+1}p_n}+|\xi_n|^{\frac{(2\mu-r_{n+1}p_n)m_n}{r_n}}\right)\\ &\quad\cdot\Big(1+|e_n|^{\frac{r_n m_{n+1}p_n}{r_{n-1}m_n}\left(\frac{m_n r_{n+1}}{r_n m_{n+1}}-1\right)}\\ &\quad+\sum_{j=2}^{n-1}h_{n-1,j}\left(1+|e_j|^{\frac{r_n m_{n+1}p_n}{r_{j-1}m_n}\left(\frac{m_n r_{n+1}}{r_n m_{n+1}}-1\right)}\right)\\ &\quad+\sum_{l=1}^{n}\left(1+|\xi_l|^{\frac{r_n m_j m_{n+1}p_n}{r_j m_n}\left(\frac{m_n r_{n+1}}{r_n m_{n+1}}-1\right)}\right)\Big)\\ &\le Lc_6\left(e_n^{\frac{2}{m_{n-1}}}+e_n^{\frac{2\mu}{r_{n-1}}}\right)+L\sum_{j=2}^{n-1}h_{n-1,j}\left(e_j^{\frac{2}{m_{j-1}}}+e_j^{\frac{2\mu}{r_{j-1}}}\right)\\ &\quad+\frac{L}{4}\sum_{j=1}^{n}\left(\xi_j^2+\xi_j^{\frac{2\mu m_j}{r_j}}\right).\end{aligned}$$

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